Viscous flow of vortices in type-II superconducting alloys

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General kinetic equations for viscous flow of separate filaments are obtained on basis of the microscopic theory of superconductivity[8] for a low vortex velocity $u/\xi \ll \Delta$ and not very large filament density ($H \ll H_{c2}$). The relation between the transport current and filament velocity is derived. The general equations are solved numerically for $T \ll \Delta$; the conductivity of a "dirty" superconducting alloy with a vortex structure is $\sigma_{\rm eff} = 0.9\,\sigma H_{c2}/B_0$.

1. INTRODUCTION

Various phenomenological (see, e.g.,[1]) and semi-phenomenological[2] theories were proposed to explain the phenomena that occur when vortex filaments move in the mixed state. Naturally, by virtue of their nature, these theories did not provide a consistent description of the kinetics in superconductors. The microscopic approach to the motion of a system of vortices was developed mainly for type-II superconductors with large vortex-filament density ($H \approx H_{c2}$)[3], owing to the relative simplicity of the theory, since the order parameter is small in this region. The microscopic theory of Éliashberg and one of the authors[8] was used recently[4-7] to find the viscosity in the mixed state in the region of intermediate fields $H_{c1} < H < H_{c2}$. Alloys with paramagnetic impurities were discussed by the present authors[4] as well as by Kupriyanov and Likharev[5] and by Hu and Thompson[6]. In[7] we considered the case of ordinary alloys at a temperature close to critical. The recent experimental data of Gubankov[9] and Fogel'[10] are in qualitative agreement with the theory. In particular, the viscosity coefficient increases as the temperature approaches the critical value. This effect was predicted in[7].

In the present paper, which is a continuation of[7], we derive equations that make it possible, in principle, to study the motion of a vortex structure in a superconducting alloy in the entire temperature range. The calculation of the mixed-state resistance is carried through to conclusion only for very low temperatures, owing to the large computational difficulties that arise at finite values of the temperature. We consider also dirty alloys, in which the electron mean free path $l = v_0\tau$ is much smaller than the correlation radius ξ_0. The Ginzburg-Landau parameter for such alloys is $\kappa \gg 1$. This means that there exists a wide field range $H_{c1} < H < H_{c2}$ in which the cores of the vortex filaments do not overlap. This, as already indicated[4,7], greatly simplifies the problem and makes it possible to reduce it to the problem of determining the motion of a single vortex filament in a current (velocity) field that is homogeneous at large from the filament center and is governed by the average transport current flowing through the sample.

2. DERIVATION OF THE FUNDAMENTAL EQUATIONS

In the derivation of the nonstationary equations for superconductors[8] it was shown that when the expressions for the order parameter and for the current are analytically continued to the real-frequency axis one obtains terms having different analytic structures with respect to the complex variable ϵ, namely terms that are regular either in the upper or to the lower half of the ϵ plane, and so-called "anomalous" terms that are kinetic in nature and do not have simple analytic properties with respect to ϵ. We write down the expressions derived in[8], choosing for Δ a gauge such that the expressions contain only the gauge-invariant combinations

$$|\Delta|,\quad \mathbf{Q} = \mathbf{A} - \frac{c}{2e}\nabla\chi,\quad \mu = \varphi + \frac{1}{2e}\frac{\partial\chi}{\partial t},$$

where χ is the phase of the order parameter. We shall henceforth omit the absolute-value sign of Δ. We have

$$\frac{\Delta_\omega(\mathbf{r})}{|g|} = \int\frac{d\varepsilon}{4\pi i}\left\{\operatorname{th}\frac{\varepsilon_-}{2T}F^R_{\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r}) - \operatorname{th}\frac{\varepsilon_+}{2T}F^A_{\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r}) + F^{(a)}_{\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r})\right\},$$

$$\mathbf{j}_\omega(\mathbf{r}) = -\frac{e}{m}\left\{(\hat{\mathbf{p}}-\hat{\mathbf{p}}')\int\frac{d\varepsilon}{4\pi i}\left[\operatorname{th}\frac{\varepsilon_-}{2T}G^R_{\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r}') - \operatorname{th}\frac{\varepsilon_+}{2T}G^A_{\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r}') + G^{(a)}_{\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r}')\right]\right\}_{\mathbf{r}'=\mathbf{r}} - \frac{Ne^2}{mc}\mathbf{Q}_\omega(\mathbf{r}), \tag{1}$$

where $\epsilon_\pm = \epsilon \pm \omega/2$, $\hat{p} = -i\nabla$, p' acts on r', while $G^{R,A,(a)}$ and $F^{R,A,(a)}$ are respectively the retarded, advanced, and anomalous Green functions. The latter satisfy the integral equations shown schematically in Fig. 1.

It is convenient to use the matrix notation:

$$\mathcal{G} = \begin{pmatrix} G & F \\ -F^+ & \bar{G}\end{pmatrix}.$$

We shall need also the Green functions integrated with respect to $\zeta = v_0(p - p_0)$ (p_0 is the Fermi momentum); they are designated as follows:

$$\hat{g}^{R(A)}_{\varepsilon_+,\varepsilon_-}(\mathbf{v}_0,\mathbf{k}) = \int\hat{\mathcal{G}}^{R(A)}_{\varepsilon_+,\varepsilon_-}(\mathbf{p},\mathbf{p}-\mathbf{k})\frac{d\zeta}{\pi i},$$

$$\hat{g}^{(a)}_{\varepsilon_+,\varepsilon_-}(\mathbf{v}_0,\mathbf{k}) = \hat{\mathcal{G}}^{(a)}_{\varepsilon_+,\varepsilon_-}(\mathbf{p},\mathbf{p}-\mathbf{k})\,d\zeta.$$

Quantities of this type were first introduced by Eilenberger[11] and, in kinetics, by Éliashberg[12] (see also the paper by Larkin and Ovchinnikov[13]). The notation is that introduced in[11,12].

As noted by Usadel[14], in dirty alloys the functions $\hat{g}(v_0, r)$ are isotropic in first approximation with respect to the directions of the vector v_0, so that we can write

$$\hat{g}^{R(A)}(\mathbf{v}_0,\mathbf{r}) = \hat{g}_0^{R(A)}(\mathbf{r}) + \frac{\mathbf{v}_0}{v_0}\hat{\mathbf{g}}^{R(A)}(\mathbf{r}),\quad \hat{g}^{(a)}(\mathbf{v}_0,\mathbf{r}) = \hat{\gamma}(\mathbf{r}) + 3\frac{\mathbf{v}_0}{v_0}\hat{\boldsymbol{\gamma}}(\mathbf{r}), \tag{2}$$

with $|\mathbf{g}|, |\boldsymbol{\gamma}| \sim (l/\xi)g_0, \gamma$, where $\xi \sim (\xi_0 l)^{1/2}$ is the characteristic scale of variation of the functions $\hat{g}^{R(A)}$ in the alloy. The functions

$$\hat{\gamma} = \begin{pmatrix}\gamma_1 & \gamma_2 \\ -\gamma_2^+ & \bar{\gamma}_1\end{pmatrix},\quad \hat{\boldsymbol{\gamma}} = \begin{pmatrix}\boldsymbol{\gamma}_1 & \boldsymbol{\gamma}_2 \\ -\boldsymbol{\gamma}_2^+ & \bar{\boldsymbol{\gamma}}_1\end{pmatrix}$$

introduced in this manner are the same as in[7].

It was indicated in[7] that to derive the general equations it is necessary to expand the kernels in the ladder

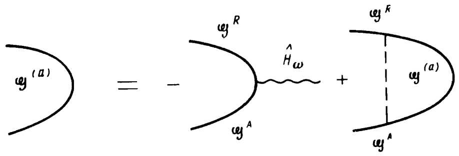

FIG. 1

equation of Fig. 1 up to terms of order $(l/\xi)^2$ inclusive. The same result can be obtained with somewhat less labor, by starting from Éliashberg's kinetic equations[12], which are generalizations of the Eilenberger equations[11] to the nonstationary case:

$$-iv_0\nabla\hat{g}^{R(A)}_{\varepsilon,\varepsilon'} + \hat{\varepsilon}\hat{g}^{R(A)}_{\varepsilon,\varepsilon'} - \hat{g}^{R(A)}_{\varepsilon,\varepsilon'}\hat{\varepsilon}' + \{\hat{H}\hat{g}^{R(A)} - \hat{g}^{R(A)}\hat{H}\}_{\varepsilon,\varepsilon'} - \frac{i}{2\tau}\{\hat{g}_0^{R(A)}\hat{g}^{R(A)} - \hat{g}^{R(A)}\hat{g}_0^{R(A)}\}_{\varepsilon\varepsilon'} = 0, \quad (3)$$

$$- iv_0\nabla\hat{g}^{(a)}_{\varepsilon,\varepsilon'} + \hat{\varepsilon}\hat{g}^{(a)}_{\varepsilon,\varepsilon'} - \hat{g}^{(a)}_{\varepsilon,\varepsilon'}\hat{\varepsilon}' + \{\hat{H}\hat{g}^{(a)} - \hat{g}^{(a)}\hat{H}\}_{\varepsilon,\varepsilon'} - \frac{i}{2\tau}\{\hat{g}_0^R\hat{g}^{(a)} - \hat{g}^{(a)}\hat{g}_0^A\}_{\varepsilon,\varepsilon'} + \frac{i}{2\tau}\{\hat{g}^R\hat{\gamma} - \hat{\gamma}\hat{g}^A\}_{\varepsilon,\varepsilon'} \quad (4)$$
$$= \pi i\int\frac{d\omega}{2\pi}\left[\hat{g}^R_{\varepsilon,\varepsilon'+\omega}\hat{H}_\omega\left(\mathrm{th}\frac{\varepsilon'+\omega}{2T} - \mathrm{th}\frac{\varepsilon'}{2T}\right) - \hat{H}_\omega\hat{g}^A_{\varepsilon-\omega,\varepsilon'}\left(\mathrm{th}\frac{\varepsilon}{2T} - \mathrm{th}\frac{\varepsilon-\omega}{2T}\right)\right],$$

$$\hat{\varepsilon} = \begin{pmatrix}-\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}, \quad \hat{H}_\omega = \begin{pmatrix}-\frac{e}{c}v_0Q_\omega + e\mu_\omega & -\Delta_\omega\\ \Delta_\omega & \frac{e}{c}v_0Q_\omega + e\mu_\omega\end{pmatrix},$$

where the symbol $\{\hat{H}\hat{g}\}_{\epsilon\epsilon'}$ denotes integration with respect to the internal frequency:

$$\{\hat{H}\hat{g}\}_{\varepsilon,\varepsilon'} = \int\hat{H}_\omega\hat{g}_{\varepsilon-\omega,\varepsilon'}\frac{d\omega}{2\pi}.$$

We introduce the function

$$\hat{K}^{R(A)}_{\varepsilon,\varepsilon'} = \int\hat{g}^{R(A)}_{\varepsilon,\varepsilon_1}(v_0,r)\hat{g}^{R(A)}_{\varepsilon_1,\varepsilon'}(v_0,r)\frac{d\varepsilon_1}{2\pi}.$$

It is easily seen that it also satisfies Eq. (3). The initial conditions are set at $t \to -\infty$, when Δ and $\mathbf{Q}$, do not depend on the time. In this case the equilibrium functions

$$\hat{g}^{R(A)}_{\varepsilon,\varepsilon'} = 2\pi\delta(\varepsilon-\varepsilon')\hat{g}^{R(A)}_\varepsilon$$

satisfy the normalization condition $(\hat{g}_\epsilon)^2 = \hat{1}$[11]. Equation (3) with these initial conditions has a solution

$$\hat{K}^{R(A)}_{\varepsilon,\varepsilon'} = 2\pi\delta(\varepsilon-\varepsilon')\hat{1}. \quad (5)$$

which is an analog of the normalization condition $(\hat{g}_\epsilon)^2 = \hat{1}$ in the stationary case[1)].

Next, multiplying (4) from the left by $\hat{g}^R_{\epsilon,\epsilon'}$ (with integration with respect to the internal frequency) and from the right by $\hat{g}^A_{\epsilon,\epsilon'}$, and accordingly Eq. (3) for g^A from the left by $\hat{g}^{(a)}_{\epsilon,\epsilon'}$ and the equation for $\hat{g}^R$ from the right by $\hat{g}^{(a)}_{\epsilon,\epsilon'}$, and then adding the resultant four equations, we obtain with the aid of (5) for the function

$$\hat{K}^{(a)}_{\varepsilon,\varepsilon'} = \int\{\hat{g}^R_{\varepsilon,\varepsilon_1}\hat{g}^{(a)}_{\varepsilon_1\varepsilon'} + \hat{g}^{(a)}_{\varepsilon,\varepsilon_1}\hat{g}^A_{\varepsilon_1,\varepsilon'}\}\frac{d\varepsilon_1}{2\pi}$$

the homogeneous equation

$$-iv_0\nabla\hat{K}^{(a)}_{\varepsilon,\varepsilon'} + \hat{\varepsilon}\hat{K}^{(a)}_{\varepsilon,\varepsilon'} - \hat{K}^{(a)}_{\varepsilon,\varepsilon'}\hat{\varepsilon}' + \{\hat{H}\hat{K}^{(a)} - \hat{K}^{(a)}\hat{H}\}_{\varepsilon,\varepsilon'} - \frac{i}{2\tau}\{\hat{g}_0^R\hat{K}^{(a)} - \hat{K}^{(a)}\hat{g}_0^A\}_{\varepsilon,\varepsilon'} = 0.$$

Since $\hat{g}^{(a)} = 0$ at $t \to -\infty$, we can conclude that

$$\hat{K}^{(a)}_{\varepsilon,\varepsilon'} = 0. \quad (6)$$

In terms of the complete functions[12]

$$\hat{g}_{\varepsilon,\varepsilon'} = \pi i\,\mathrm{th}\frac{\varepsilon'}{2T}\hat{g}^R_{\varepsilon,\varepsilon'} - \pi i\,\mathrm{th}\frac{\varepsilon}{2T}\hat{g}^A_{\varepsilon,\varepsilon'} + \hat{g}^{(a)}_{\varepsilon,\varepsilon'}$$

the conditions (5) and (6) yield

$$\int\{\hat{g}^R_{\varepsilon,\varepsilon_1}\hat{g}_{\varepsilon_1,\varepsilon'} + \hat{g}_{\varepsilon,\varepsilon_1}\hat{g}^A_{\varepsilon_1\varepsilon'}\}\frac{d\varepsilon_1}{2\pi} = 0;$$

We confine ourselves below to low velocities of the vortex structure $u/\xi \ll \Delta$ ($\omega \ll \Delta$). Since the right-hand side of (4) together with $g^{(a)}$ vanishes as u, $\omega \to 0$, the regular functions in (4), as well as $\hat{H}_\omega$ in the left-hand side, can be regarded to the equilibrium functions:

$$\hat{g}^{R(A)}_{\varepsilon,\varepsilon'} = 2\pi\delta(\varepsilon-\varepsilon')\hat{g}^{R(A)}_\varepsilon, \quad \hat{H}_\omega = \left[-\Delta i\sigma_y - \frac{e}{c}v_0Q\sigma_z\right]2\pi\delta(\omega),$$

where σ_y and σ_z are Pauli matrices. For the same reasons, it is necessary to omit the term with $\mu \sim \omega$ from the right-hand side of H_ω.

We write out first the equations for the regular functions. In the stationary case we have

$$g_\varepsilon^{R(A)} + \bar{g}_\varepsilon^{R(A)} = 0, \quad (g_\varepsilon^{R(A)})^2 - f_\varepsilon^{R(A)}f_\varepsilon^{+R(A)} = 1.$$

Since $(\hat{g}^{R(A)}_{0\epsilon})^2 = \hat{1}$ in the approximation (2), we obtain from (3)[14] (see also[7])

$$\hat{g}^{R(A)}_\varepsilon = -l\hat{g}^{R(A)}_{0\varepsilon}\cdot\partial\hat{g}^{R(A)}_{0\varepsilon} = l\partial\hat{g}^{R(A)}_{0\varepsilon}\cdot\hat{g}^{R(A)}_{0\varepsilon};$$
$$-2\varepsilon f_{0\varepsilon}^{R(A)} + 2\Delta g_{0\varepsilon}^{R(A)} + iD\left(\nabla - i\frac{2e}{c}\mathbf{Q}\right) \quad (7')$$
$$\times\left[g_{0\varepsilon}^{R(A)}\left(\nabla - i\frac{2e}{c}\mathbf{Q}\right)f_{0\varepsilon}^{R(A)} - f_{0\varepsilon}^{R(A)}\nabla g_{0\varepsilon}^{R(A)}\right] = 0,$$

$$-2\varepsilon f_{0\varepsilon}^{+R(A)} + 2\Delta g_{0\varepsilon}^{R(A)} + iD\left(\nabla + i\frac{2e}{c}\mathbf{Q}\right) \quad (7'')$$
$$\times\left[g_{0\varepsilon}^{R(A)}\left(\nabla + i\frac{2e}{c}\mathbf{Q}\right)f_{0\varepsilon}^{+R(A)} - f_{0\varepsilon}^{+R(A)}\nabla g_{0\varepsilon}^{R(A)}\right] = 0.$$

The matrix $\partial\hat{g}$ denotes here gauge-invariant differentiations:

$$\begin{pmatrix}\nabla g, & \left(\nabla - i\frac{2e}{c}\mathbf{Q}\right)f\\ -\left(\nabla + i\frac{2e}{c}\mathbf{Q}\right)f^+, & -\nabla g\end{pmatrix}$$

Replacing in (4) and (6) the regular functions by the equilibrium functions and averaging (4) and (6) over the directions of the vector $\mathbf{v}_0$ we obtain

$$-iv_0\partial\hat{\gamma} + (\hat{\lambda}_+\hat{\gamma} - \hat{\gamma}\hat{\lambda}_-)$$
$$= \alpha\omega\left\{-\Delta_\omega(\hat{g}^R_{0\varepsilon_+}i\sigma_y - i\sigma_y\hat{g}^A_{0\varepsilon_-}) - \frac{ev_0}{3c}\mathbf{Q}_\omega(\hat{g}^R_{\varepsilon_+}\sigma_z - \sigma_z\hat{g}^A_{\varepsilon_-})\right\}, \quad (8)$$

$$\hat{g}^R_{0\varepsilon_+}\hat{\gamma} + \hat{\gamma}\hat{g}^A_{0\varepsilon_-} = 0. \quad (9)$$

We have retained in (9) only the principal terms in l/ξ.

Multiplying (4) and (6) by $\mathbf{v}_0$ and averaging over the directions of $\mathbf{v}_0$, we obtain, with the same accuracy:

$$-\frac{iv_0}{3}\partial\hat{\gamma} - \frac{i}{2\tau}(\hat{g}^R_{0\varepsilon_+}\hat{\gamma} - \hat{\gamma}\hat{g}^A_{0\varepsilon_-}) + \frac{i}{6\tau}(\hat{g}^R_{\varepsilon_+}\hat{\gamma} - \hat{\gamma}\hat{g}^A_{\varepsilon_-}) \quad (10)$$

$$= \alpha\omega\left\{-\frac{ev_0}{3c}\mathbf{Q}_\omega(\hat{g}^R_{0\varepsilon_+}\sigma_z - \sigma_z\hat{g}^A_{0\varepsilon_-})\right\},$$
$$\hat{g}^R_{0\varepsilon_+}\hat{\gamma} + \hat{\gamma}\hat{g}^A_{0\varepsilon_-} + \frac{1}{3}(\hat{g}^R_{\varepsilon_+}\hat{\gamma} + \hat{\gamma}\hat{g}^A_{\varepsilon_-}) = 0. \quad (11)$$

In (8)–(11) we have used the notation

$$\alpha = \frac{\pi i}{2T}ch^{-2}\frac{\varepsilon}{2T}, \quad \lambda_\pm = -\varepsilon_\pm\sigma_z - \Delta i\sigma_y.$$

We shall henceforth drop the frequency labels of $\hat{g}^{R(A)}$ for the sake of brevity.

As noted earlier[7], Eqs. (8) and (10) alone do not make it possible to determine $\hat{\gamma}$ and $\hat{\gamma}$ uniquely. The complete system of equations includes the relations (9) and (11). The latter arise when the equations of Fig. 1 are solved or, much more simply, as corollaries of (5) and (6). We consider the matrices

$$\hat{g}_0^R\hat{\gamma}-\hat{\gamma}\hat{g}_0^A, \quad \hat{g}_0^R\hat{\gamma}+\hat{\gamma}\hat{g}_0^A$$

in (10) and (11). It is easily seen that $\hat{\gamma} = \hat{\gamma}_- + \hat{\gamma}_+$, where $\hat{\gamma}_-$ and $\hat{\gamma}_+$ cause the first matrix and second matrices, respectively, to vanish identically. Thus, a matrix in the form

$$\hat{A}_\pm^{(i)}=\hat{g}_0^R\hat{a}_i\mp\hat{a}_i\hat{g}_0^A,$$

where $\hat{a}$ is an arbitrary matrix, has precisely this property. Taking four linearly independent matrices $\hat{A}_\pm^{(i)}$ (i = 1, 2), we expand $\hat{\gamma}$ and γ in their terms. From (10) and (11) we obtain immediately, taking (9) into account,

$$\hat{\gamma}=-\frac{l}{3}\left\{\hat{g}_0^R\partial\hat{\gamma}+\hat{\gamma}\partial\hat{g}_0^A+\alpha i\omega\frac{e}{c}Q_\omega(\sigma_z-\hat{g}_0^R\sigma_z\hat{g}_0^A)\right\}. \tag{12}$$

It is convenient to choose the matrices $\hat{A}$ in the form

$$\hat{A}_+^{(1)}=\hat{g}_0^R-\hat{g}_0^A, \quad \hat{A}_+^{(2)}=\hat{g}_0^R\sigma_z-\sigma_z\hat{g}_0^A,$$
$$\hat{A}_-^{(1)}=\hat{g}_0^R\sigma_x+\sigma_x\hat{g}_0^A, \quad \hat{A}_-^{(2)}=\hat{g}_0^R i\sigma_y+i\sigma_y\hat{g}_0^A.$$

The equations with which to determine $\hat{\gamma}$ are obtained by multiplying (8) and (9) respectively by the "conjugate" matrices $\hat{A}_+^{(i)}$ and $\hat{A}_-^{(i)}$, given by

$$\hat{\bar{A}}^{(1)}=\hat{g}_0^R-\hat{g}_0^A, \quad \hat{\bar{A}}_+^{(2)}=\sigma_z\hat{g}_0^R-\hat{g}_0^A\sigma_z,$$
$$\hat{\bar{A}}_-^{(1)}=\sigma_x\hat{g}_0^R+\hat{g}_0^R\sigma_x, \quad \hat{\bar{A}}_-^{(2)}=i\sigma_y\hat{g}_0^R+\hat{g}_0^A i\sigma_y. \tag{13}$$

and taking the traces. We thus obtain from (9)

$$\mathrm{Sp}\,\{\hat{\bar{A}}_-^{(i)}\hat{\gamma}\}=0. \tag{14}$$

From (8) we obtain after rather prolonged transformations

$$D\left\{\mathrm{Sp}\,(\hat{\bar{A}}_+^{(i)}\partial^2\hat{\gamma})-\frac{1}{4}\mathrm{Sp}\,(\hat{\bar{A}}_+^{(i)}\hat{\gamma})\,\mathrm{Sp}\,(\hat{g}_0^R\partial^2\hat{g}_0^R+\hat{g}_0^A\partial^2\hat{g}_0^A)\right\}$$
$$-i\,\mathrm{Sp}\,\{\hat{\bar{A}}_+^{(i)}\hat{g}_0^R(\hat{\lambda}_+\hat{\gamma}-\hat{\gamma}\hat{\lambda}_-)\}=-2i\alpha\omega\,\mathrm{Sp}\,\{\hat{\bar{A}}_+^{(i)}\Big[-\Delta_\omega i\sigma_y \tag{15}$$
$$+\frac{e}{c}DQ_\omega(\partial\hat{g}_0^R\sigma_z-\sigma_z\partial\hat{g}_0^A)+\frac{e}{c}D\,\mathrm{div}\,Q_\omega\hat{g}_0^R\sigma_z\Big]\Big\}.$$

3. EQUATIONS FOR THE ANOMALOUS GREEN FUNCTIONS IN THE CASE OF A SINGLE VORTEX

In the case of an immobile vortex, the functions $f_0^{R(A)}$ and $g_0^{R(A)}$ depend only on the distance ρ from the center of the vortex filament in the gauge of Δ chosen by us. It is seen from (7) that $f_0^{+R(A)} = f_0^{R(A)}$. The rather complicated equations of (15) become much simpler in two limiting cases:

a) Slow variation of g_0 and f_0. This means that (see (7″)) the characteristic distance $\xi(T)$ over which g_0 and f_0 vary, should be much larger than $(D/\Delta(T))^{1/2}$. This takes place in the vicinity of the critical temperature $T \gg \Delta(T)$, where

$$\xi(T)\sim(D/\Delta(T))^{1/2}(T/\Delta(T))^{1/2},$$

and far from the center of the filament, $\rho \gg \xi$, at $T \lesssim T_c$.

b) Case of low temperatures, $T \ll \Delta$.

The first case was considered in detail earlier[7]. We shall show here only how to reduce (15) to the equations obtained in[7]. As shown in[7], assuming slow variation of g_0 and f_0 we obtain

$$g_0^{R(A)}=\varepsilon/\xi^{R(A)}, \quad f_0^{R(A)}=\Delta/\xi^{R(A)}, \tag{16}$$

where $\xi^{R(A)}$ are the values of the root $(\epsilon^2-\Delta^2)^{1/2}$ so defined that $\xi^{R(A)} = \pm(\epsilon_0^2-\Delta^2)^{1/2}$ at $\epsilon = \epsilon_0 + i\delta$, $\epsilon_0 > \Delta(\rho)$, and $\delta \to +0$. At $|\epsilon| > \Delta(\rho)$ we therefore have

$$f_0^R=-f_0^A\equiv f, \quad g_0^R=-g_0^A\equiv g, \tag{16'}$$

and at $|\epsilon| < \Delta(\rho)$

$$f_0^R=f_0^A, \quad g_0^R=g_0^A. \tag{16''}$$

(We have neglected above the dependences of $g_0^{R(A)}$ and of $f_0^{R(A)}$ on the frequency ω.)

At $|\epsilon| > \Delta$, using (16) and (16′), we obtain from (14)

$$\gamma_2-\gamma_2^+=0, \quad g(\gamma_2+\gamma_2^+)=f(\gamma_1-\bar{\gamma}_1).$$

Equations (15) yield

$$D\nabla^2\left(\frac{\gamma_1-\bar{\gamma}_1}{g}\right)=4\alpha f\frac{\partial\Delta}{\partial t};$$
$$\nabla^2(\gamma_1+\bar{\gamma}_1)-\left[\left(\frac{\nabla f}{g}\right)^2+\left(\frac{2e}{c}Qf\right)^2\right](\gamma_1+\bar{\gamma}_1)=\frac{2\alpha}{g}\left\{\frac{2e}{c}\,\mathrm{div}\left(g^2\frac{\partial Q}{\partial t}\right)\right\}.$$

These agree with the corresponding equations of[7]. We note that the last equation means that at large distances from the vortex-filament center we have $\gamma_1 + \bar{\gamma}_1 \sim \rho^{-q}$, where $q = |\epsilon|/(\epsilon^2-\Delta^2)^{1/2}$.

To obtain the equations at $|\epsilon| < \Delta$ from (15) we must use the identity

$$(f_0^R-f_0^A)(f_0^R+f_0^A)=(g_0^R-g_0^A)(g_0^R+g_0^A).$$

We present by way of example one of the resultant equations:

$$D\left\{\nabla^2\left(\frac{\gamma_2+\gamma_2^+}{g}\right)-\frac{4e^2}{c^2}Q^2(g^2+f^2)\left(\frac{\gamma_2+\gamma_2^+}{g}\right)+i\frac{4e}{c}Q\nabla(\gamma_2-\gamma_2^+)\right\}$$
$$-2(\Delta^2-\varepsilon^2)^{1/2}\left(\frac{\gamma_2+\gamma_2^+}{g}\right)=-2\alpha\left\{2g\frac{\partial\Delta}{\partial t}+iD\left(\frac{2e}{c}\right)^2Q\frac{\partial Q}{\partial t}fg\right\}.$$

We see thus that in first order in Δ/T we have at $|\epsilon| < \Delta$ (cf.[7])

$$\gamma_2+\gamma_2^+=2\alpha\frac{g^2}{(\Delta^2-\varepsilon^2)^{1/2}}\frac{\partial\Delta}{\partial t}.$$

We turn now to the case of low temperatures. As seen from (15), $\hat{\gamma}$ is proportional to $\cosh^{-2}(\epsilon/2T)$, so that the main contribution to the integral with respect to ϵ is made by the frequencies $\epsilon \sim T \ll \Delta$. Confining ourselves to zeroth order in T/Δ, we can put $\epsilon = 0$ in $\hat{g}_0^{R(A)}$ in Eqs. (14) and (15) for $\hat{\gamma}$. According to (16) we have as $\rho \to \infty$ and at $\epsilon = 0$:

$$f_0^{R(A)}=-i \text{ and } g_0^{R(A)}=0.$$

At $\rho = 0$ we have for the gap $\Delta = 0$, so that we can use as the boundary conditions in (7″)

$$f_0^{R(A)}=0, \quad g_0^{R(A)}=\pm1.$$

It is convenient to introduce the notation

$$f_0^{R(A)}=-i\cos[\theta^{R(A)}]; \quad g_0^{R(A)}=\sin[\theta^{R(A)}].$$

From (7″) at $\epsilon = 0$ and the boundary conditions for $f_0^{R(A)}$ and $g_0^{R(A)}$ it is clear that

$$f_0^R=f_0^A, \quad g_0^R=-g_0^A. \tag{17}$$

In other words, if $\theta^{R(A)} \to 0$ as $\rho \to \infty$ and $\theta^{R(A)} = \pm/2$ as $\rho \to 0$, Then $\theta^R = -\theta^A \equiv \theta$.

The equation for θ takes the form (cf.[16])

$$D\left[\nabla^2\theta+\frac{\sin\theta\cos\theta}{\rho^2}\right]-2\Delta\sin\theta=0. \quad (18)$$

As seen from (18), $\theta(\rho)$ attenuates exponentially at large distances, and as $\rho\to0$ it tends to zero linearly like $\theta=\pi/2-C\rho$.

Using the conditions (17), we can readily get from (14) and (15)

$$\gamma_2+\gamma_2^+=0,\quad \gamma_1=\bar{\gamma}_1,\quad \sin\theta\gamma_2-i\cos\theta\gamma_1=0,$$

and the function γ_2 is determined from

$$D\nabla^2\left(\frac{\gamma_2}{\cos\theta}\right)-2\cos\theta\Delta\left(\frac{\gamma_2}{\cos\theta}\right)=2i\alpha\frac{e}{c}D\operatorname{div}\frac{\partial\mathbf{Q}}{\partial t}. \quad (19)$$

For $\gamma_1-\bar{\gamma}_1$ we obtain in turn from (12)

$$\gamma_1-\bar{\gamma}_1=\frac{2l}{3}\left[i\nabla\left(\frac{\gamma_2}{\cos\theta}\right)+2\alpha\frac{e}{c}\frac{\partial\mathbf{Q}}{\partial t}\right]. \quad (20)$$

The expression for the anomalous part of the current contains precisely the combination $\gamma_1-\bar{\gamma}_1$:

$$\mathbf{j}^{(a)}=-\frac{ep_0^2}{2\pi^2}\int\frac{d\varepsilon}{4\pi i}(\gamma_1-\bar{\gamma}_1),$$

so that Eq. (19) should be solved under the condition that expression (20) is finite at the center of the filament[5].

4. MOTION OF VORTEX FILAMENT IN THE TRANSPORT-CURRENT FIELD

Assume that the transport current j_{tr} flows through the sample in a direction perpendicular to the external magnetic field H_0. We recall that we have assumed $H_0 \ll H_{c2}$. This means that the vortex-filament density B_0/Φ_0 is small (B_0 is the induction and Φ_0 the quantum flux), so that the distance between neighboring filaments is $d\gg\xi$. When the current j_{tr} flows, the vortex lattice is set in motion and is deformed both by the pinning forces and by the current field. This effect, however, will be neglected. and we assume the current and the pinning forces to be small enough.

If the velocity u is low, we can assume in first order that each individual filament moves as a unit

$$\Delta=\Delta_0(\mathbf{r}-\mathbf{u}t),\quad \mathbf{Q}=\mathbf{Q}_0(\mathbf{r}-\mathbf{u}t),$$

where Δ_0 and $\mathbf{Q}_0$ are quantities characterizing the immobile filaments. The effects connected with "slowing down," i.e., with distortion of the filament shape, constitute small corrections proportional to the velocity. We can thus write

$$\Delta=\Delta_0(\mathbf{r}-\mathbf{u}t)+\Delta_1,\quad \mathbf{Q}=\mathbf{Q}_0(\mathbf{r}-\mathbf{u}t)+\mathbf{Q}_1. \quad (21)$$

In exactly the same manner we have for the current $\mathbf{j}=\mathbf{j}_0(\mathbf{r}-\mathbf{u}t)+\mathbf{J}_1$. The value of $\mathbf{j}_1$ far from the filament centers ($\xi\ll\rho\ll d$) is determined by the structure of the entire vortex lattice; this structure is determined in turn by the average macroscopic current j_{tz}. At distances $\rho\gg\xi$ from an individual filament (see[4]) we have $j_{1\infty}=j_{tr}$.[2)]

Thus, at $\kappa\gg1$ and $H\ll H_{c2}$ it suffices to solve the problem of an individual filament. Namely, we determine the current $j_{1\infty}$ from (1), equate it to j_{tr}, and establish thereby the connection between the transport current and the vortex velocity u. The formula

$$\operatorname{rot}\overline{\mathbf{E}}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$$

is then used to determine the average electric field

$$\overline{\mathbf{E}}=\frac{B_0}{c}[\mathbf{n}_H,\mathbf{u}], \quad (22)^*$$

where $\mathbf{n}_H$ is a unit vector in the direction of the magnetic field and consequently we obtain the resistance of the sample.

The condition connecting u with j_{tr} and valid at arbitrary temperatures takes the form (cf.[7])

$$[\mathbf{n}_H,\mathbf{d}]\mathbf{j}_{tr}=\frac{mp_0e}{2\pi^3}\int d^2\mathbf{r}\left\{(\mathbf{d}\nabla)\Delta_0\left[\int\frac{d\varepsilon}{4T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}\frac{i(\mathbf{u}\nabla)}{2}(f_{0\varepsilon}^R+f_{0\varepsilon}^A)+\int\frac{d\varepsilon}{4\pi i}(\gamma_2+\gamma_2^+)\right]-(\mathbf{d}\nabla)\varphi\int\frac{d\varepsilon}{4\pi i}\frac{v_0}{2}\operatorname{div}(\gamma_1-\bar{\gamma}_1)\right\}, \quad (23)$$

where d is an arbitrary constant vector.

Owing to its complexity, we defer the derivation of (23) to the Appendix. We note here only that the reason why a relation in closed form can be obtained is the spatial homogeneity of the problem. Indeed, the system (1) linearized with respect to Δ_1 and $\mathbf{Q}_1$ has solutions of the "displacement" type, $\tilde{\Delta}_1=(\mathbf{d}\nabla)\Delta_0$ and $\mathbf{Q}_1=(\mathbf{d}\nabla)\mathbf{Q}_0$. Equation (23) therefore a sort of orthogonality relation. As seen from the Appendix, Eq. (23) holds true also for a pure type-II superconductor.

In the case of low temperatures of interest to us, we have $\gamma_2+\gamma_2^+=0$, and we obtain from (19) and (20)

$$[\mathbf{n}_H,\mathbf{d}]\mathbf{j}_{tr}=\frac{mp_0e}{2\pi^3}\int d^2\mathbf{r}\{[(\mathbf{d}\nabla)\Delta_0](\mathbf{u}\nabla)\cos\theta-[(\mathbf{d}\nabla)\varphi]2\Delta\cos\theta y\},$$

where

$$\frac{\gamma_2}{\cos\theta}=\frac{\pi}{T}y\operatorname{ch}^{-2}\frac{\varepsilon}{2T}.$$

The function y is obtained from (19). Since $\partial/\partial t=-\mathbf{u}\nabla$ if the approximation (21) is used, the right-hand side of (19) vanishes by virtue of $\operatorname{div}\mathbf{Q}_0=0$. Thus,

$$D\nabla^2y-2\Delta\cos\theta y=0. \quad (24)$$

The boundary condition that determines the solution y is the requirement that the current remain finite (expression (20)) as $\rho\to0$. In a cylindrical coordinate system (ρ, φ, z) with the z axis along the vortex axis and with the x axis parallel to the velocity u, this condition yields

$$y\to-{}^1/_2(\mathbf{u}\nabla)\varphi={}^1/_2u\rho^{-1}\sin\varphi \text{ and } \rho\to0.$$

Introducing the dimensionless quantities

$$\rho=\xi\rho',\quad \Delta_0=\Delta_\infty\Delta',\quad y=\frac{u}{2}\sin\varphi\xi^{-1}y',$$

where $\xi=(D/2\Delta_\infty)^{1/2}$, we obtain

$$[\mathbf{n}_H,\mathbf{d}]\mathbf{j}_{tr}=\frac{mp_0e\Delta_\infty}{2\pi^2}\mathbf{ud}\beta(0);$$

$$\beta(0)=\beta_1+\beta_2,$$

$$\beta_1=\int_0^\infty\rho'\frac{d\Delta'}{d\rho'}\frac{d\cos\theta}{d\rho'}d\rho',\quad \beta_2=\int_0^\infty\Delta'\cos\theta y'\,d\rho'.$$

The value of the upper critical current at T = 0, as calculated by Maki[17], is

$$H_{c2}(0)=\Delta_\infty c/2De,$$

therefore the final answer can be written in the form

$$\mathbf{j}_{tr}=\beta(0)\frac{\sigma H_{c2}(0)}{c}[\mathbf{n}_H\mathbf{u}].$$

For the conductivity of the sample we obtain with the aid of (22)

$$\sigma_{\text{eff}}=\beta(0)\sigma H_{c2}(0)/B_0.$$

The viscosity coefficient is $\eta=\beta(0)\sigma\Phi_0H_{c2}(0)/c^2$.

To determine β_1 and β_2 it is necessary to solve Eq. (24), which includes the functions $\Delta_0(\rho)$ and $\cos\theta$. We have used for these quantities the results of the numerical calculations of the equilibrium structure of the vortex at low temperatures, performed by Watts-Tobin and Waterworth[16]. Figure 2 shows these results for the functions Δ_0 and $g = \sin\theta$. According to[16], Δ' is well approximated by the function

$$\Delta'(\rho')=\frac{16.27\rho'+(\rho')^3}{(16+(\rho')^2)(2.12+(\rho')^2)^{1/2}}$$

A numerical calculation yields

$$\beta_1=0.261,\quad \beta_2=0.639,\quad \beta(0)=0.9.$$

We recall that as $T \to T_c$ we obtain[7]

$$\sigma_{\text{eff}}=\beta(T)\sigma H_{c2}(T)/B_0,\quad \beta(T)=1.1(1-T/T_c)^{-1/2}. \tag{25}$$

For comparison, Fig. 3 shows the measured values of $\beta(T)$ given in[9,10]. The solid line is a plot of the function $1.1\,(1 - T/T_c)^{-1/2}$. The circles represent the data of Gubankov[9] and the points the data of N. Fogel'[10]. The following remark should be made concerning Gubankov's results. Expression (25) has been derived for a bulky sample, while Gubankov's experiments[9] were performed on thin films. Figure 3 shows data for a film 780 Å thick. According to[9], $\beta(T)$ increases quite rapidly with increasing film thickness. Since the screening of the superfluid currents ceases to be exponential in the vicinity of a single vortex in films, this means effectively that the vortex density increases, corresponding to a decrease of σ_{eff} with decreasing film thickness.

We note that the empirical formula of Kim et al.[8], in which $\beta(T) = H_{c2}(0)/H_{c2}(T)$, agrees well with the obtained value $\beta(0) = 0.9$ as $T \to 0$. As already noted by other workers, however (see, e.g.,[10]), it does not describe the temperature dependence of σ_{eff} satisfactorily at $T \to T_c$.

In conclusion, the authors thank A. I. Larkin and Yu. N. Ovchinnikov for useful discussions and V. V. Avilov for the numerical calculations. We are also grateful to R. Watts-Tobin and Mrs. G. M. Waterworth for communicating their results prior to publication.

APPENDIX

We proceed to derive the condition that relates $\mathbf{u}$ with $j_{tr} = j_{1\infty}$. For the small corrections Δ_1 and $\mathbf{Q}_1$ we obtain from (1)

$$\frac{\Delta_{1\omega}(\mathbf{r})}{|g|}=\int\frac{d\varepsilon}{4\pi i}\Big\{\operatorname{th}\frac{\varepsilon}{2T}[F^R_{1\varepsilon_+,\varepsilon_-}-F^A_{1\varepsilon_+,\varepsilon_-}] -\frac{\omega}{4T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}[F^R_{0\varepsilon_+,\varepsilon_-}+F^A_{0\varepsilon_+,\varepsilon_-}]+\frac{mp_0}{2\pi^2}\gamma_2\Big\}, \tag{A.1}$$

$$\mathbf{j}_{1\omega}(\mathbf{r})=-\frac{e}{m}\Big\{(\hat{\mathbf{p}}-\hat{\mathbf{p}}')\int\frac{d\varepsilon}{4\pi i}\Big[\operatorname{th}\frac{\varepsilon}{2T}(G^R_{1\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r}')-G^A_{1\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r}')) -\frac{\omega}{4T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}(G^R_{0\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r}')+G^A_{0\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r}'))\Big]\Big\}_{\mathbf{r}'=\mathbf{r}}-\frac{Ne^2}{mc}\mathbf{Q}_{1\omega}(\mathbf{r})-\frac{ep_0^2}{\pi^2}\int\frac{d\varepsilon}{4\pi i}\gamma_1. \tag{A.2}$$

Here $G^{R(A)}_{0\epsilon_+,\epsilon_-}$ are the equilibrium Green functions,

$$G^{R(A)}_{0\varepsilon_+,\varepsilon_-}=2\pi\delta(\omega)G_\varepsilon^{R(A)},$$

and $F_1^{R(A)}$ and $G_1^{R(A)}$ are the corrections to $F^{R(A)}$ and $G^{R(A)}$, due to Δ_1 and $\mathbf{Q}_1$.[3)] Since we seek corrections to quantities that describe the motion of the vortex system as a whole, the frequency ω in the right-hand sides

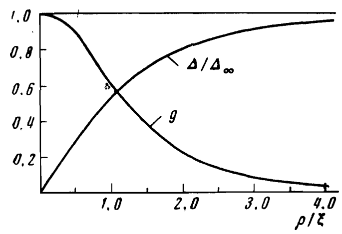

FIG. 2

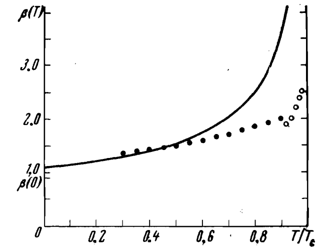

FIG. 3

FIG. 2. The functions Δ_0/Δ_∞ and $g = \sin\theta$ according to the results of [16].
FIG. 3. Measured values of $\beta(T)$ [9,10].

of (A.1) and (A.2) has the meaning of differentiation with respect to the coordinate, with the correspondence

$$-i\omega\to\partial/\partial t=-\mathbf{u}\nabla$$

The matrix $\mathscr{G}^{R(A)}_{1\epsilon_+,\epsilon_-}$ satisfies an equation analogous to Fig. 1, except that the right-hand side, at first glance, should contain terms stemming from differentiating with respect to time the "displacement" solutions $\Delta_0(\mathbf{r} - \mathbf{u}t)$, $\mathbf{Q}_0(\mathbf{r} - \mathbf{u}t)$, and $g(\mathbf{r} - \mathbf{u}t)$. The structure of these expressions is shown in Fig. 4. The existence of terms of this type is easiest to discern with the aid of Eq. (3), since, for example, the expression $\hat{\epsilon}_+\hat{g}_{\epsilon^+,\epsilon^-} - \hat{g}_{\epsilon^+,\epsilon^-}\hat{\epsilon}$ contains the term $(1/2)\,\omega(\sigma_z\hat{g} + \hat{g}\sigma_z)$, etc. Proceeding as in the derivation of (15), we can easily show, however, that all the terms containing differentiation of the "displacement" solutions with respect to time drop out from the equations for g_{10} and from the expressions for the vectors $\mathbf{g}_1$. Thus, the right-hand side of the equation for $\mathscr{G}_1^{R(A)}$ contains only the matrix $\hat{H}_1$. We present, for example, the expression for $\mathbf{g}_1^{R(A)}$:

$$\hat{\mathbf{g}}_1^{R(A)}=-l\Big\{\hat{g}_0^{R(A)}\partial\hat{g}_{10}^{R(A)}+\hat{g}_{10}^{R(A)}\partial g_0^{R(A)}+i\frac{e}{c}\mathbf{Q}_1(\sigma_z-\hat{g}_0^{R(A)}\sigma_z\hat{g}_0^{R(A)})\Big\}. \tag{A.3}$$

On the other hand, owing to the translation invariance, the static equations (1) are satisfied by the displacement solutions $\Delta_d = (\mathbf{d}\nabla)\Delta_0$ and $\mathbf{Q}_d = (\mathbf{d}\nabla)\mathbf{Q}_0$. For example, for Δ_d we have

$$\frac{\Delta_d}{|g|}=\int\frac{d\varepsilon}{4\pi i}\operatorname{th}\frac{\varepsilon}{2T}[F_{d\varepsilon}{}^R(\mathbf{r},\mathbf{r})-F_{d\varepsilon}{}^A(\mathbf{r},\mathbf{r})]$$

(and an analogous expression for the current). The functions $F_d^{R(A)}$ and $G_d^{R(A)}$ satisfy the sake equations as $F^{R(A)}$ and $G^{R(A)}$, with Δ_1 and $\mathbf{Q}_1$ replaced by Δ_d and $\mathbf{Q}_d$, respectively.

We multiply (A.1) by Δ_d, (A.2) by $\mathbf{Q}_d/c$, and the equations for Δ_d and j_d by Δ_1 and $\mathbf{Q}_1/c$, respectively. Proceeding in the same manner with the equations expressing Δ in terms of F^+ and $\mathbf{j}$ in terms of $\overline{G}$, and combining them, we obtain after integrating over the volume of a cylinder of radius R around the filament ($\xi \ll R \ll d$, δ, where δ is the penetration depth)

$$\frac{1}{c}\int_{V_R}(\mathbf{j}_1\mathbf{Q}_d-\mathbf{j}_d\mathbf{Q}_1)\,d^3\mathbf{r}$$

$$=\int\frac{d\varepsilon}{4\pi i}\operatorname{th}\frac{\varepsilon}{2T}\operatorname{Sp}\Big\{\int_{V_R}d^3r[\hat{\mathscr{G}}_1{}^R(\mathbf{r},\mathbf{r})\hat{H}_d(\mathbf{r})-\hat{\mathscr{G}}_d{}^R(\mathbf{r},\mathbf{r})\hat{H}_1(\mathbf{r}) +\hat{\mathscr{G}}_1{}^A(\mathbf{r},\mathbf{r})\hat{H}_d(\mathbf{r})-\hat{\mathscr{G}}_d{}^A(\mathbf{r},\mathbf{r})\hat{H}_1(\mathbf{r})]\Big\}$$

$$+\frac{mp_0}{2\pi^2}\int_{V_R}d^3\mathbf{r}\,\Delta_d\Big\{\frac{i\mathbf{u}\nabla}{4}\int\frac{d\varepsilon}{2T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}(f_{0\varepsilon}{}^R+f_{0\varepsilon}{}^A)$$

$$
+\int\frac{d\varepsilon}{4\pi i}(\gamma_2+\gamma_2^{+})\Big\}+\frac{mp_0}{2\pi^2}\int_{V_R} d^3r\,\frac{e}{c}\,v_0\mathbf{Q}_d\cdot
$$
$$
\times\Big\{-\frac{i\mathbf{u}\nabla}{12}\int\frac{d\varepsilon}{2T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}(\mathbf{g}^R+\mathbf{g}^A)-\int\frac{d\varepsilon}{4\pi i}(\boldsymbol{\gamma}_1-\overline{\boldsymbol{\gamma}}_1)\Big\}. \tag{A.4}
$$

The integrals in the right-hand side converge over distances on the order of ξ. Indeed, as seen from (A.3), the "dangerous" terms containing the product $\mathbf{Q}_1\mathbf{Q}_d$ cancel each other, and the remaining terms decrease rapidly at $\rho \gg \xi$. In addition, an uncertainty can arise in the integration near the origin. Let us consider, for example, the integral of the first term in the right-hand side of (A.4). After integrating over the volume of a cylinder with small radius $R_1 \ll \xi$ we obtain

$$
\frac{1}{c}\int_{V_{R_1}}\mathbf{j}_1\mathbf{Q}_d\,d^3\mathbf{r}=-\frac{1}{c}\int_{V_{R_1}}\mathbf{j}_1\frac{c}{2e}(\mathbf{d}\nabla)\nabla\varphi\,d^3\mathbf{r}=-\frac{1}{2e}\int_{S_{R_1}}\mathbf{j}_1(0)\,(\mathbf{d}\nabla)\varphi\,d\mathbf{S},
$$

since $\operatorname{div}\mathbf{j}=0$. Analogous terms appear also in the right-hand side; thus, the integral of $\boldsymbol{\gamma}_1-\overline{\boldsymbol{\gamma}}_1$ yields

$$
\frac{e}{c}\int_{V_R}\mathbf{Q}_d(\boldsymbol{\gamma}_1-\overline{\boldsymbol{\gamma}}_1)\,d^3\mathbf{r}=\frac{1}{2}\int_{S_{R_1}}(\mathbf{d}\nabla)\varphi[(\boldsymbol{\gamma}_1-\overline{\boldsymbol{\gamma}}_1)|_{\rho=0}]\,d\mathbf{S}
$$
$$
+\frac{1}{2}\int_{V_R}(\mathbf{d}\nabla)\varphi\operatorname{div}(\boldsymbol{\gamma}_1-\overline{\boldsymbol{\gamma}}_1)\,d^3\mathbf{r}.
$$

Gathering in the right-hand side all the integrals over the surface of infinitesimally small radius, we obtain also on the right hand side the term

$$
-\frac{1}{2e}\int_{S_{R_1}}\mathbf{j}_1(0)\,(\mathbf{d}\nabla)\varphi\,d\mathbf{S},
$$

which cancels a like term in the left-hand side. Thus, the contribution from integration near the origin need not be taken into account in (A.4).

We see thus that the integration in the right-hand of (A.4) can be extended over all of space. It is easy to verify that the expression in (A.4), which contains the trace operation, is equal to zero. Indeed, we write down $\hat{\mathscr{G}}_1^{R(A)}$ in the form of a series of ladder diagrams

$$
\hat{\mathscr{G}}_1=-\int\hat{\mathscr{G}}_\varepsilon(\mathbf{r},\mathbf{r}_1)\hat{H}_1(\mathbf{r}_1)\hat{\mathscr{G}}_\varepsilon(\mathbf{r}_1,\mathbf{r})\,d^3\mathbf{r}_1
$$
$$
-(2\pi\tau)^{-1}\frac{2\pi^2}{mp_0}\int\hat{\mathscr{G}}_\varepsilon(\mathbf{r},\mathbf{r}_1)\hat{\mathscr{G}}_\varepsilon(\mathbf{r}_1,\mathbf{r}_2)\hat{H}_1(\mathbf{r}_2)\hat{\mathscr{G}}_\varepsilon(\mathbf{r}_2,\mathbf{r}_1)\hat{\mathscr{G}}_\varepsilon(\mathbf{r}_1,\mathbf{r})\,d^3\mathbf{r}_1d^3\mathbf{r}_2-\ldots
$$

Permuting cyclically the matrices under the trace sign, we see readily that

$$
\operatorname{Sp}\Big\{\int d^3\mathbf{r}\,\hat{\mathscr{G}}_1^{R(A)}(\mathbf{r},\mathbf{r})\hat{H}_d(\mathbf{r})\Big\}=\operatorname{Sp}\Big\{\int d^3\mathbf{r}\,\hat{\mathscr{G}}_d^{R(A)}(\mathbf{r},\mathbf{r})\hat{H}_1(\mathbf{r})\Big\}.
$$

Next, using Maxwell's equation

$$
\operatorname{rot}\operatorname{rot}\mathbf{Q}=\frac{4\pi}{c}\mathbf{j}
$$

we transform the integral in the left-hand side of (A.4) into an integral over the surface of a cylinder of radius R, $\xi \ll R \ll \delta$:

$$
\frac{1}{c}\int_{V_R}(\mathbf{j}_1\mathbf{Q}_d-\mathbf{j}_d\mathbf{Q}_1)\,d^3\mathbf{r}=-\frac{L}{4\pi}\int_{S_R}[(\nabla_iQ_{1k})Q_{dk}+Q_{1k}(\nabla_iQ_{dk})]\,dS_i,
$$

where L is the length of the vortex. We have chosen for simplicity the gauge $\operatorname{div}\mathbf{Q}_1=0$. It is easy to verify, however, that this does not alter the result. To calculate the surface integral accurate to the principal terms in ξ/δ, it is necessary to know $\mathbf{Q}_1$ accurate to terms of order ρ^2. Solving the equation

$$
\operatorname{rot}\operatorname{rot}\mathbf{Q}_1=\frac{4\pi}{c}\mathbf{j}_{1\infty},
$$

in the region $\rho \ll \delta$ we obtain

$$
Q_{1y}=-\frac{4\pi}{c}j_{1y}\Big(\delta^2+\frac{1}{2}\rho^2\cos^2\varphi\Big),
$$

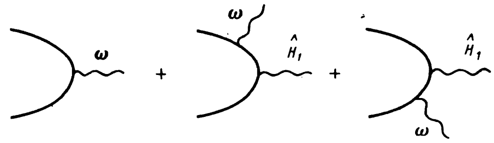

FIG. 4

where the constant term is determined from the Londons' equation

$$
\mathbf{j}_{1\infty}=-\frac{c}{4\pi\delta^2}\mathbf{Q}_{1\infty}.
$$

Substituting in the surface integral $Q_0\varphi = -(c/2e\delta)K_1(\rho/\delta)$ and confining ourselves to the first two terms of the expansion of $K_1(\rho/\delta)$ in powers of ρ/δ, we obtain

$$
\frac{1}{c}\int_{V_R}(\mathbf{j}_1\mathbf{Q}_d-\mathbf{j}_d\mathbf{Q}_1)\,d^3\mathbf{r}=\frac{L\pi}{e}[\mathbf{n}_H\mathbf{d}]\mathbf{j}_{1\infty}.
$$

Finally,

$$
[\mathbf{n}_H\mathbf{d}]\mathbf{j}_{1\infty}=\frac{mp_0e}{2\pi^3}\int d^2\mathbf{r}\,\Big\{(\mathbf{d}\nabla)\Delta_0\Big[\int\frac{d\varepsilon}{4T}\operatorname{ch}^{-2}\frac{\varepsilon}{2T}\frac{i(\mathbf{u}\nabla)}{2}(f_{0\varepsilon}^R+f_{0\varepsilon}^A)
$$
$$
+\int\frac{d\varepsilon}{4\pi i}(\gamma_2+\gamma_2^{+})\Big]-(\mathbf{d}\nabla)\varphi\int\frac{d\varepsilon}{4\pi i}\frac{v_0}{2}\operatorname{div}(\boldsymbol{\gamma}_1-\overline{\boldsymbol{\gamma}}_1)\Big\}. \tag{A.5}
$$

We have used here the fact that $\operatorname{div}\mathbf{g}^{R(A)}=0$ according to (7′). The expression (A.5) is derived for arbitrary temperatures.

*$[\mathbf{n}_H,\mathbf{u}] \equiv \mathbf{n}_H \times \mathbf{u}$.

1)The authors thank A. I. Larkin and Yu. N. Ovchinnikov for calling their attention to the existence of the identity (5) [15].

2)The fact that in [4] we considered a superconductor with paramagnetic impurities does not change the situation, since we have used there only the Londons' equations at $\rho \ll \xi$.

3)In the expression for the current we have used the condition of electroneutrality of the superconductor as a metal, N = const, which is equivalent in the isotropic model to the condition $\operatorname{div}\mathbf{j}=0$. The latter, according to [7], is equivalent to the condition that Δ be real in our gauge.

1 W. F. Vinen and P. Nozieres, Phil. Mag., 14, 667 (1966).

2 J. Bardeen and M. J. Stephen, Phys. Rev., 140, A1197 (1965).

3 C. Caroli and K. Maki, Phys. Rev., 164, 591 (1967). K. Maki, J. Low.Temp. Phys., 1, 54 (1969). Y. Baba, and K. Maki, Progr. Theor. Phys., 44, 1432 (1970).

4 L. P. Gor'kov and N. B. Kopnin, Zh. Eksp. Teor. Fiz. 60, 2331 (1971) [Sov. Phys.-JETP 33, 1251 (1971)].

5 M. Yu. Kupriyanov and K. K. Likharev, ZhETF Pis. Red. 15, 349 (1972) [JETP Lett. 15, 247 (1972)].

6 C. R. Hu and R. S. Thompson, Phys. Rev., B6, 110 (1972).

7 L. P. Gor'kov and N. B. Kopnin, Zh. Eksp. Teor. Fiz. 64, 356 (1973) [Sov. Phys.-JETP 37, 185 (1973)].

8 L. P. Gor'kov and G. M. Éliashberg, ibid. 54, 612 (1968) [27, 328 (1968)].

9 V. N. Gubankov, Fiz. Tverd. Tela 14, 2618 (1972) [Sov. Phys.-Solid State 14, 2264 (1973)].

10 N. Ya. Fogel', Zh. Eksp. Teor. Fiz. 63, 1371 (1972) [Sov. Phys.-JETP 36, 725 (1973)].

11 G. Eilenberger, Zs. Phys., 214, 195 (1968).

12 G. M. Éliashberg, Zh. Eksp. Teor. Fiz. 61, 1254 (1971) [Sov. Phys.-JETP 34, 668 (1972)].

13 A. I. Larkin and Yu. N. Ovchinnikov, ibid. 55, 2262

(1968) [28, 1200 (1969)].
[14] K. D. Usadel, Phys. Rev. Lett., 25, 507 (1970).
[15] A. I. Larkin and Yu. N. Ovchinnikov, J. Low Temp. Phys. 10, 407 (1973).
[16] R. J. Watts-Tobin and G. M. Waterworth, Preprint, England, 1972.
[17] K. Maki, Physics, 1, 21 (1964).
[18] Y. B. Kim, C. F. Hepstead, and A. R. Strnad, Phys. Rev., 139, A1163 (1965).

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