Frequency dependence of the conductivity of one-dimensional systems

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The dependence of the conductivity of one-dimensional systems on the frequency of an external electric field, ω, is investigated. It is shown that the static conductivity ($\omega = 0$) is zero. For sufficiently high electron Fermi energies the conductivity tends to zero not slower than $\sim|\omega|$.

1. The conductivity of one-dimensional systems has been the subject of many investigations. Nonetheless, the dependence of the conductivity on the frequency of the external electric field has not been determined so far. All the arguments advanced to date concerning this dependence were exclusively indirect and based on investigations of the energy spectra of one-dimensional systems. These include the researches by Mott and Twoose[1] and Borland[2]. Using purely intuitive considerations (Mott) or investigating the ergodicity properties of the equation for the distribution function of the phase shifts of the particle wave function (Borland), these authors reached the conclusion that the wave function is localized, and indicated by the same token the possibility that one-dimensional systems have no static conductivity. No direct calculation of the conductivity was made, however, and the question remained open.

We show here on the basis of exact equations for the conductivity that it is equal to zero at $\omega = 0$ and that it decreases with frequency at a rate not lower than $\sim|\omega|$ at sufficiently high energies.

The first exact equations for the conductivity were derived by Halperin[3] for the particular case of white noise. Dykhne and I[4] investigated the problem of averaging the product of two Green functions of a particle. The present paper deals with the case when the impurity potential is of the form $U(x) = U_0\sum\limits_i\delta(x - x_i)$ and the distances between impurities have a Poisson distribution.

The plan of the paper is the following. In Sec. 2 are introduced the characteristic functions $f^{(0,1)}(z_1, z_2)$ and the equations they satisfy [Eqs. (12)] are derived. The conductivity σ is expressed in terms of these functions with the aid of formulas (1) and (10). We note here that the quantity $[f(\epsilon + \omega) - f(\epsilon)]/\omega$ is replaced in (1) by the derivative $\partial f/\partial\epsilon$. Equations (12) are investigated in Sec. (12) in the case of high energies, when the collision term in (12) takes the form (13). It is shown that in this approximation the conductivity decreases with frequency no slower than $\sim|\omega|$, and the limiting expression for the function $f^{(0)}(z_1, z_2)$ takes the form of a δ-function, i.e., $f^{(0)}(z_1, z_2) \to \delta(z_1 - z_2)$ as $\omega \to 0$. It is shown in Sec. 4 that $f^{(0)}(z_1, z_2) \propto \delta(z_1 - z_2)$ is an exact solution of (12) at all values of the energy if $\omega = 0$. From this and from formulas (10) and (1) it follows that there is no static conductivity at any energy.

2. We start with the well known expression for the conductivity in an external electric field of frequency ω (we put $\hbar = 1$ throughout):

$$\sigma(\omega) = -\pi e^2\int d\varepsilon\frac{\partial f(\varepsilon)}{\partial\varepsilon}\Phi(\varepsilon, \omega), \tag{1}$$

where ϵ is the particle energy, $f(\epsilon)$ is the particle energy distribution function, and

$$\Phi(\varepsilon, \omega) = \frac{1}{L}\sum_{n,m}|v_{nm}|^2\delta(\varepsilon_n - \varepsilon)\delta(\varepsilon_m - \varepsilon'), \tag{2}$$

where v_{nm} is the matrix element of the velocity operator, $\epsilon' = \epsilon + \omega$, and L is the dimension of the system (we are, of course, interested in the limit as $L \to \infty$). In formula (2), the matrix elements are calculated between exact wave functions of a system having a potential energy (x_i are the impurity coordinates)

$$U(x) = U_0\sum_i\delta(x - x_i).$$

In the one-dimensional case, formula (2) can be represented in a different form by introducing the logarithmic derivative of the wave function of the electron, $z(x) = \psi^{-1}(x)d\psi(x)/dx$, which satisfies the obvious equation ($k^2 = 2m\epsilon$):

$$dz(x)/dx + z^2(x) + k^2 = 2mU_0\sum_i\delta(x - x_i). \tag{3}$$

The energy levels of a system with dimensions L are determined from the boundary conditions (the energy ϵ is a parameter in Eq. (3))

$$z(x = 0, \varepsilon) = z_0, \quad z(x = L, \varepsilon) = z_L.$$

It is now easy to rewrite the expression for $\Phi(\epsilon, \omega)$ in the form

$$L\Phi(\varepsilon, \omega) = \frac{1}{(2m)^2}\Big\langle\Big[\int_0^L\psi^2(x, \varepsilon)dx\int_0^L\psi^2(x', \varepsilon')dx'\Big]^{-1}$$
$$\times\Big\{\int_0^L\Big[\psi(x, \varepsilon)\frac{d\psi(x, \varepsilon')}{dx} - \psi(x, \varepsilon')\frac{d\psi(x, \varepsilon)}{dx}\Big]dx\Big\}^2$$
$$\times\Big|\frac{\partial z(L, \varepsilon)}{\partial\varepsilon}\frac{\partial z(L, \varepsilon')}{\partial\varepsilon'}\Big|\delta[z_L - z(L, \varepsilon)]\delta[z_L - z(L, \varepsilon')]\Big\rangle.$$

The angle brackets denote averaging over the impurity coordinates x_i. From (3) we find that

$$\frac{\partial z(x, \varepsilon)}{\partial\varepsilon} = -2m\psi^{-2}(x, \varepsilon)\int_0\psi^2(x', \varepsilon)dx',$$

and finally, we have for $\Phi(\epsilon, \omega)$

$$L\Phi[\varepsilon, \omega] = \Big\langle\psi^{-2}(L, \varepsilon)\psi^{-2}(L, \varepsilon')\delta[z_L - z(L, \varepsilon)]\delta[z_L - z(L, \varepsilon')]$$
$$\times\Big\{\int_0^L\Big[\psi(x, \varepsilon)\frac{d\psi(x, \varepsilon')}{dx} - \psi(x, \varepsilon')\frac{d\psi(x, \varepsilon)}{dx}\Big]dx\Big\}^2\Big\rangle. \tag{5}$$

We note here that in the derivation of the expression for $\Phi(\epsilon, \omega)$ in (4) we have used the definition (2) of Φ, and then have written down in explicit form the square of the matrix element of the current and have taken into account the fact that

$$\delta(\varepsilon-\varepsilon_n)=\delta[z_L-z(L,\varepsilon)]\left|\frac{\partial z(L,\varepsilon)}{\partial\varepsilon}\right|.$$

The corresponding transformations are considered in greater detail in Halperin's paper[3].

We now introduce the functions

$$F^{(l)}(z_1,z_2;L)=\left\langle\psi^{-l}(L,\varepsilon)\psi^{-l}(L,\varepsilon')\delta[z_1-z(L,\varepsilon)]\delta[z_2-z(L,\varepsilon')]\times\left\{\int_0^L\left[\psi(x,\varepsilon)\frac{d\psi(x,\varepsilon')}{dx}-\psi(x,\varepsilon')\frac{d\psi(x,\varepsilon)}{dx}\right]dx\right\}^l\right\rangle. \tag{6}$$

We confine ourselves to the case when the distribution of the distances between the impurities obeys a Poisson law (the impurity concentration is equal to n). The quantities $F^{(l)}$ are functions of the variables z_1 and z_2, of the system dimension L, and of the parameters ϵ and ϵ'. When regarded as functions of z_1, z_2, and L, they satisfy rather simple continuity equations, which can be derived by recognizing that

$$F^{(l)}(z_1,z_2;L)=\langle W^{(l)}(L)\delta[z_1-z(L,\varepsilon)]\times\delta[z_2-z(L,\varepsilon')]\rangle.$$

It is obvious that

$$\partial W^{(l)}/\partial L=l[z(L,\varepsilon')-z(L,\varepsilon)]W^{(l-1)}-l[z(L,\varepsilon)+z(L,\varepsilon')]W^{(l)}.$$

On the other hand, if we use the equations for the logarithmic derivatives (3), we can show (for details see the paper of Frisch and Lloyd[5]) that

$$\frac{\partial}{\partial L}\{\delta[z_1-z(L,\varepsilon)]\delta[z_2-z(L,\varepsilon')]\}\equiv\frac{\partial V(z_1,z_2;L)}{\partial L}=\frac{\partial}{\partial z_1}[(z_1^2+k_1^2)V]+\frac{\partial}{\partial z_2}[(z_2^2+k_2^2)V]+n[V(z_1-2k_0,z_2-2k_0;L)-V(z_1,z_2;L)],$$

where

$$k_1^2=2m\varepsilon,\quad k_2^2=2m\varepsilon',\quad k_0=mU_0.$$

We obtain ultimately the continuity equations for the functions $F^{(l)}$

$$\frac{\partial}{\partial L}F^{(l)}(z_1,z_2;L)=\frac{\partial}{\partial z_1}[(z_1^2+k_1^2)F^{(l)}]+\frac{\partial}{\partial z_2}[(z_2^2+k_2^2)F^{(l)}]+n[F^{(l)}(z_1-2k_0,z_2-2k_0;L)-F^{(l)}(z_1,z_2;L)]+l(z_2-z_1)F^{(l-1)}(z_1,z_2;L)-l(z_1+z_2)F^{(l)}(z_1,z_2;L). \tag{7}$$

It is easy to prove (see[3]) that the functions $F^{(0)}$ and $F^{(1)}$ are independent of L as $L\to\infty$:

$$F^{(0,1)}(z_1,z_2;L)\to f^{(0,1)}(z_1,z_2),$$

and

$$F^{(2)}(z_1,z_2;L)\to L\Phi(\varepsilon,\omega)+f^{(2)}(z_1,z_2). \tag{8}$$

The function Φ does not depend on z_1 and z_2. The function $f^{(2)}$ can be easily eliminated from Eq. (7) for $F^{(2)}$ by multiplying its left and right sides by $f^{(0)}(-z_1,-z_2)$ and integrating with respect to z_1 and z_2. It follows then from the definition of $f^{(0)}(z_1,z_2)$ that

$$\iint_{-\infty}^{\infty}f^{(0)}(z_1,z_2)\,dz_1\,dz_2=1. \tag{9}$$

From this we obtain immediately

$$\Phi(\varepsilon,\omega)=2\iint_{-\infty}^{\infty}f^{(0)}(-z_1,-z_2)(z_2-z_1)f^{(1)}(z_1,z_2)\,dz_1\,dz_2. \tag{10}$$

We note here that the definition (6) leads to boundary conditions for $f^{(l)}$ (they can be easily obtained if it is recognized that $z\to\infty$ corresponds to the zeroes of the wave functions, so that as $z\to\infty$ we have $\delta[z-z(L,\epsilon)]\to1/z^2$)::

$$f^{(l)}(z_1,z_2)z_i^{2-l}\Big|_{z_i\to+\infty}=f^{(l)}(z_1,z_2)z_i^{2-l}\Big|_{z_i\to-\infty}\quad(l=0,1;\ i=1,2), \tag{11}$$

and the limiting equations ($L\to\infty$) for them are

$$0=\frac{\partial}{\partial z_1}[(z_1^2+k_1^2)f^{(0)}]+\frac{\partial}{\partial z_2}[(z_2^2+k_2^2)f^{(0)}]+n[f^{(0)}(z_1-2k_0,\ z_2-2k_0)-f^{(0)}(z_1,z_2)],$$
$$(z_1-z_2)f^{(0)}(z_1,z_2)=\frac{\partial}{\partial z_1}[(z_1^2+k_1^2)f^{(1)}]+\frac{\partial}{\partial z_2}[(z_2^2+k_2^2)f^{(1)}]+n[f^{(1)}(z_1-2k_0,\ z_2-2k_0)-f^{(1)}(z_1,z_2)]-(z_1+z_2)f^{(1)}(z_1,z_2). \tag{12}$$

It is thus necessary to solve Eqs. (12) with conditions (11) and to substitute the obtained solutions in formula (10) for $\Phi(\epsilon,\omega)$, which is the formula for the conductivity.

We now proceed to investigate Eqs. (12) and begin with the case when the particle energy ϵ is high enough (k_1, $k_2\gg k_0$).

In this case it is natural to restrict the analysis to the Born approximation in scattering by a single impurity, i.e., to expand $f(z-2k_0)$ in terms of k_0 and terminate the expansion with the second-order term. The terms of first order in k_0 then lead to an energy renormalization (nk_0), and the collision term in (12) is

$$n[f(z_1-2k_0,\ z_2-2k_0)-f(z_1,z_2)]\to D(\partial/\partial z_1+\partial/\partial z_2)^2f(z_1,z_2), \tag{13}$$

where $D=2nk_0^2=km/\tau$ and τ is the free-path time. We call attention first to the following very important circumstance: Eq. (13) contains a derivative with respect to the variable z_1+z_2, and the dependence of the functions f on this variable is in practice always weak (this will be demonstrated later on).

It is more convenient to investigate Eqs. (12) with the collision term in the form (13) by changing over to other variables. We introduce the angles θ_1 and θ_2 ($-\pi/2\le\theta_i\le\pi/2$):

$$z_1=k_1\operatorname{tg}\theta_1,\quad z_2=k_2\operatorname{tg}\theta_2, \tag{14}$$

and the new functions $\varphi^{(0)}(\theta_1,\theta_2)$ and $\varphi^{(1)}(\theta_1,\theta_2)$:

$$f^{(0)}(z_1,z_2)=[(z_1^2+k_1^2)(z_2^2+k_2^2)]^{-1}k_1k_2\varphi^{(0)}(\theta_1,\theta_2),$$
$$f^{(1)}(z_1,z_2)=[(z_1^2+k_1^2)(z_2^2+k_2^2)]^{-1/2}\varphi^{(1)}(\theta_1,\theta_2). \tag{15}$$

The physical meaning of the new variables is that they are the phases of the wave functions. They are more convenient because we need no longer worry about satisfaction of the conditions (11), since they are taken into account immediately, and in addition they arise automatically if an attempt is made to solve (12) by iteration with respect to the impurity concentration n.

It follows from (11) that ($i=1,2$)

$$\varphi^{(0)}(\theta_i=-\pi/2)=\varphi^{(0)}(\theta_i=\pi/2),$$
$$\varphi^{(1)}(\theta_i=-\pi/2)=-\varphi^{(1)}(\theta_i=\pi/2), \tag{16}$$

and from (9) and from the definition (15) of $\varphi^{(0)}$ it follows that

$$\iint_{-\pi/2}^{\pi/2}\varphi^{(0)}(\theta_1,\theta_2)\,d\theta_1\,d\theta_2=1. \tag{17}$$

In terms of the new variables, the expression (10) for $\Phi(\epsilon,\omega)$ takes the form

$$\Phi(\varepsilon,\omega)=\frac{2}{k_1k_2}\iint_{-\pi/2}^{\pi/2}\varphi^{(0)}(-\theta_1,-\theta_2)[k_2\sin\theta_2\cos\theta_1-k_1\sin\theta_1\cos\theta_2]\varphi^{(1)}(\theta_1,\theta_2)\,d\theta_1\,d\theta_2. \tag{18}$$

From the conditions (16) we get

$$\varphi^{(0)}(\theta_1,\theta_2)=\sum_{l,m}\exp(2il\theta_1+2im\theta_2)\varphi^{(0)}_{l,m},$$
$$\varphi^{(1)}(\theta_1,\theta_2)=e^{i(\theta_1+\theta_2)}\sum_{l,m}\exp(2il\theta_1+2im\theta_2)\varphi^{(1)}_{l,m}. \tag{19}$$

We omit the very cumbersome transformation involved in the transition form the variables z to the angles θ, and present directly the equations for $\varphi^{(0)}(\theta_1, \theta_2)$:

$$\begin{aligned}0={}&k_1\,\partial\varphi^{(0)}/\partial\theta_1+k_2\,\partial\varphi^{(0)}/\partial\theta_2\\&-D\varphi^{(0)}\Big\{\frac{1}{k_1^2}(\cos 2\theta_1+\cos 4\theta_1)+\frac{1}{k_2^2}(\cos 2\theta_2+\cos 4\theta_2)\\&+\frac{1}{k_1k_2}[\cos 2(\theta_1+\theta_2)-\cos 2(\theta_2-\theta_1)]\Big\}\\&-D\frac{\partial\varphi^{(0)}}{\partial\theta_1}\Big\{\frac{3}{2k_1^2}\Big(\sin 2\theta_1+\frac{1}{2}\sin 4\theta_1\Big)+\frac{1}{k_1k_2}\Big[\sin 2\theta_2\\&+\frac{1}{2}\sin 2(\theta_2+\theta_1)+\frac{1}{2}\sin 2(\theta_2-\theta_1)\Big]\Big\}-D\frac{\partial\varphi^{(0)}}{\partial\theta_2}\{k_1\leftrightarrow k_2,\theta_2\leftrightarrow\theta_1\}\\&+\frac{D}{k_1^2}\cos^4\theta_1\frac{\partial^2\varphi^{(0)}}{\partial\theta_1^2}+\frac{2D}{k_1k_2}\cos^2\theta_1\cos^2\theta_2\\&\times\frac{\partial^2\varphi^{(0)}}{\partial\theta_1\partial\theta_2}+\frac{D}{k_2^2}\cos^4\theta_2\frac{\partial^2\varphi^{(0)}}{\partial\theta_2^2},\end{aligned}\tag{20$'$}$$

and the function $\varphi^{(1)}(\theta_1, \theta_2)$

$$\begin{aligned}&-\varphi^{(0)}(\theta_1,\theta_2)\Big[\frac{k_2+k_1}{2}\sin(\theta_2-\theta_1)+\frac{k_2-k_1}{2}\sin(\theta_2+\theta_1)\Big]\\&=k_1\frac{\partial\varphi^{(1)}}{\partial\theta_1}+k_2\frac{\partial\varphi^{(1)}}{\partial\theta_2}-D\varphi^{(1)}(\theta_1,\theta_2)\Big\{\frac{1}{8k_1^2}(1\\&+4\cos 2\theta_1+3\cos 4\theta_1)+\frac{1}{8k_2^2}(\theta_1\leftrightarrow\theta_2)+\frac{1}{4k_1k_2}[\cos 2(\theta_2+\theta_1)\\&-\cos 2(\theta_2-\theta_1)]\Big\}-D\frac{\partial\varphi^{(1)}}{\partial\theta_1}\Big\{\frac{1}{2k_1^2}(2\cos 2\theta_1+\cos 4\theta_1)\\&+\frac{1}{4k_1k_2}[2\sin 2\theta_2+\sin 2(\theta_2+\theta_1)+\sin 2(\theta_2-\theta_1)]\Big\}\\&-D\frac{\partial\varphi^{(1)}}{\partial\theta_2}\{\theta_1\leftrightarrow\theta_2,\ k_1\leftrightarrow k_2\}+\frac{D}{k_1^2}\cos^4\theta_1\frac{\partial^2\varphi^{(1)}}{\partial\theta_1^2}\\&+\frac{2D}{k_1k_2}\cos^2\theta_1\cos^2\theta_2\frac{\partial^2\varphi^{(1)}}{\partial\theta_1\partial\theta_2}+\frac{D}{k_2^2}\cos^4\theta_2\frac{\partial^2\varphi^{(1)}}{\partial\theta_2^2}\end{aligned}\tag{20$''$}$$

From (20′) and (20″) we can obtain relations for their Fourier components. We write out only the equation for $\varphi^{(0)}_{p,q}$, where $p = l + m$ and $q = l - m$:

$$\begin{aligned}&[ip(k_1+k_2)+iq(k_1-k_2)]\varphi^{(0)}_{p,q}\\&=D\Big[p^2\Big(\frac{3}{8k_1^2}+\frac{1}{2k_1k_2}+\frac{3}{8k_2^2}\Big)+q^2\Big(\frac{3}{8k_1^2}-\frac{1}{2k_1k_2}\\&+\frac{3}{8k_2^2}\Big)+\frac{3pq(k_1^2-k_2^2)}{4k_1k_2}\Big]\varphi^{(0)}_{p,q}\\&+A_{p,q}(k_1,k_2)\varphi^{(0)}_{p-1,q-1}+A_{-p,-q}(k_1,k_2)\varphi^{(0)}_{p+1,q+1}\\&+A_{p,-q}(k_2,k_1)\varphi^{(0)}_{p-1,q+1}+A_{-p,q}(k_2,k_1)\varphi^{(0)}_{p+1,q-1}\\&+B_{p,q}(k_1)\varphi^{(0)}_{p-2,q-2}+B_{-p,-q}(k_1)\varphi^{(0)}_{p+2,q+2}\\&+B_{p,-q}(k_2)\varphi^{(0)}_{p-2,q+2}+B_{-p,q}(k_2)\varphi^{(0)}_{p+2,q-2}\\&+\frac{D(p^2-q^2)}{8k_1k_2}[\varphi^{(0)}_{p-2,q}+\varphi^{(0)}_{p+2,q}+\varphi^{(0)}_{p,q-2}+\varphi^{(0)}_{p,q+2}].\end{aligned}\tag{20$'''$}$$

Here

$$A_{p,q}(k_1,k_2)=\frac{D}{4k_1^2k_2}[p^2(k_1+k_2)+q^2(k_2-k_1)+2pqk_2-k_2(p+q)],$$
$$B_{p,q}(k)=\frac{D}{16k^2}(p+q)(p+q-2).$$

We shall show now that we can confine ourselves to Fourier components with $p = 0$ ($l = -m$), and that the components $\varphi^{(0)}_{p,q}$ with $p \neq 0$ are small in comparison with $\varphi^{(0)}_{0,q}$. We could attempt to solve (20) by iteration, using the fact that the terms containing D are of the order of $D/k^2 \sim k/\epsilon\tau \ll k$. But we can see immediately that if, for example, $p = 0$ ($l = -m$), then a quantity on the order of $q(k_2 - k_1) \sim qk\omega/\epsilon$ appears in the left-hand side of (20), i.e., there can be no expansion in terms of $1/\epsilon\tau \ll 1$ for these components. We therefore sum first all the terms with $p = 0$, and we show subsequently that they are indeed the most important terms at $\omega \to 0$.

Retaining in (20) only $\varphi^{(0)}_{0,q} \equiv \varphi^{(0)}_{q}$, we obtain

$$\begin{aligned}&\Big\{i(k_1-k_2)-Dq\Big[\frac{3(k_2-k_1)^2}{8k_1^2k_2^2}+\frac{1}{4k_1k_2}\Big]\Big\}\varphi^{(0)}_q\\&+\frac{Dq}{8k_1k_2}[\varphi^{(0)}_{q+2}+\varphi^{(0)}_{q-2}]=i(k_1-k_2)\delta_{0,q}.\end{aligned}\tag{21}$$

In the derivation of (21) we divided (20‴) by q, and this is the cause of the normalization term in the right-hand side of (21) ($\delta_{0,q}$ is a δ-function).

Changing over now from the recurrence relation for the Fourier components (21) to the function ($\theta = \theta_2 - \theta_1$)

$$\varphi^{(0)}(\theta)=\sum_q e^{-iq\theta}\varphi^{(0)}_q,$$

we readily obtain an equation for $\varphi^{(0)}(\theta)$:

$$\alpha\varphi^{(0)}(\theta)+\frac{d}{d\theta}[(a-\cos 2\theta)\varphi^{(0)}(\theta)]=\frac{\alpha}{\pi^2},\tag{22}$$

and with the aid of (20′) we obtain for $\varphi^{(1)}(\theta)$

$$\begin{aligned}&(a-\cos 2\theta)\frac{d^2\varphi^{(1)}(\theta)}{d\theta^2}+(\alpha+2\sin 2\theta)\frac{d\varphi^{(1)}(\theta)}{d\theta}\\&-(b-\cos 2\theta)\varphi^{(1)}(\theta)=-\frac{2k_1k_2}{D}(k_1+k_2)\sin\theta\varphi^{(0)}(\theta).\end{aligned}\tag{22$'$}$$

Here

$$\begin{aligned}\alpha&=\frac{4(k_2-k_1)}{D}k_1k_2\xrightarrow[\omega\to 0]{}4\omega\tau,\\a&=1+3(k_2-k_1)^2/2k_1k_2\xrightarrow[\omega\to 0]{}1+{}^3/_8(\omega/\varepsilon)^2;\\b&=1+(k_2-k_1)^2/2k_1k_2\longrightarrow 1+{}^1/_8(\omega/\varepsilon)^2,\end{aligned}\tag{23}$$

and the right-hand side in (22) is chosen in accord with the normalization condition (17).

Equation (22) can be easily solved, and the answer is

$$\varphi^{(0)}(\theta)=\frac{\alpha}{\pi^2}\frac{\psi^{(0)}(\theta)}{a-\cos 2\theta},\tag{24}$$

where $\psi^{(0)}(\theta)$ is given by

$$\psi^{(0)}(\theta)=\frac{1}{u(\theta)}\frac{1}{e^\gamma-1}\Big[\int_\theta^{\pi/2}u(\theta')d\theta'+e^\gamma\int_{-\pi/2}^{\theta}u(\theta')d\theta'\Big]\tag{25}$$

and

$$u(\theta)=\exp\Big\{\frac{\alpha}{\sqrt{a^2-1}}\operatorname{arctg}\Big(\sqrt{\frac{a+1}{a-1}}\operatorname{tg}\theta\Big)\Big\},\tag{26}$$

while

$$\gamma=\alpha\pi/\sqrt{a^2-1}\sim\varepsilon\tau\gg 1.\tag{27}$$

We now investigate the behavior of $\psi^{(0)}(\theta)$ as a function of the angle θ. It is easily seen that since $a - 1 \sim (\omega/\epsilon)^2 \ll 1$ even when θ takes on values

$$\theta \gtrsim \alpha \sim \omega\tau \ll 1,$$

it follows that the function $u(\theta)$ assumes the constant value $\exp(\pm\gamma/2)$, and the sign coincides with that of the angle θ. It follows therefore, since $\gamma \gg 1$, that the function $\psi^{(0)}(\theta)$, up to $\theta \sim \alpha$, is given by

$$\psi^{(0)}(\theta)=\begin{cases}\theta, & \theta>0\\ \theta+\pi, & \theta<0\end{cases}.\tag{28}$$

At the same time, $\varphi^{(0)} \sim \alpha$ down to small angles. At sufficiently small angles θ, the function $\varphi^{(0)}$ increases in such a way that the normalization condition is satisfied. At small angles, Eq. (22) takes the form ($\theta = \beta\alpha$)

$$\varphi^{(0)}(\beta)+\frac{d}{d\beta}\left\{\left[\frac{3}{2}\left(\frac{1}{8\varepsilon\tau}\right)^2+2\beta^2\right]\varphi^{(0)}(\beta)\right\}=\frac{1}{\pi^2}. \quad (29)$$

We note now that from the exact expression (25) for $\varphi^{(0)}(\theta)$ we can readily find that the contribution made to the integral of $\varphi^{(0)}(\theta)$ over the angle region $\beta \lesssim 1/\epsilon\tau$ is much smaller than unity. Then the letter parameters of Eq. (29) vanish when $\beta \gtrsim 1/\epsilon\tau$. By the same token we find that in the angle region that makes the largest contribution to the integration of $\varphi^{(0)}$ the scale of variation of the function $\varphi^{(0)}(\theta)$ is the quantity α. This leads to the obvious conclusion that $\varphi_q^0 = y^{(0)}(q\alpha)$, with $y^{(0)}(0) = 1$.

We have confined ourselves so far to the Fourier components $\varphi_{p,q}^{(0)}$ with $p = 0$. We show now that the components with $p \neq 0$ are small in comparison with $\varphi_{0,q}^{(0)}$. We estimate the terms with $p \neq 0$ by the iteration method and show that this method is applicable. Let us find the term with $p = 1$. We retain in the right-hand side of (20′), which contains the factor D, only the terms with $p = 0$ and 1, and assume that the dependence of $\varphi_{p,q}^{(0)}$ on q is weak, i.e., we assume that $\varphi_{p,q\pm1}^{(0)} \approx \varphi_{p,q}^{(0)}$. We then obtain

$$\left[i(k_1+k_2)+iq(k_1-k_2)-\frac{3D(k_1+k_2)^2}{8k_1^2k_2^2}-\frac{3Dq^2(k_1-k_2)^2}{8k_1^2k_2^2}-\frac{3Dq(k_1^2-k_2^2)}{4k_1^2k_2^2}\right]\varphi_{1,q}^{(0)}=\frac{D}{4k_1^2k_2^2}[2k_1k_2+q^2(k_1-k_2)^2+q(k_2^2-k_1^2)]\varphi_{0,q}^{(0)}. \quad (30)$$

The functions $\varphi_{0,q}^{(0)}$ differ appreciably from zero at $q \lesssim 1/\alpha \sim 1/\omega\tau$. In this region we have with good accuracy

$$\varphi_{1,q}^{(0)} \approx \frac{D}{4ik^3}\varphi_{0,q}^{(0)} \sim \frac{1}{\varepsilon\tau}\varphi_{0,q}^{(0)},$$

i.e., indeed $\varphi_{1,q}^{(0)} \ll \varphi_{0,q}^{(0)}$. At larger values of q we have $\varphi_{1,q}^{(0)} \sim \varphi_0^{(0)}$, but in this region the $\varphi_{0,q}^{(0)}$ themselves are much smaller than unity. We note here that the result justifies at the same time the approximation (13), since the fast decrease of $\varphi_{p,q}^{(0)}$ with decreasing p means a weak dependence on $z_1 + z_2$.

We proceed now to the function $\varphi^{(1)}(\theta_1, \theta_2)$. For this function, too, we confine ourselves to an approximation in which the dependence on the angles takes the form $\varphi^{(1)}(\theta_1, \theta_2) = \varphi^{(1)}(\theta_2 - \theta_1)$. The analog of Eq. (21) (obtained with the aid of (20″)) is then

$$[-a(1+q)^2+i\alpha(1+q)-b]\varphi_q^{(1)}+[{}^1/_2(q-1)^2+q-{}^1/_2]\varphi_{q-2}^{(1)}+[{}^1/_2(q+3)^2-q-{}^5/_2]\varphi_{q+2}^{(1)}=ik_1k_2[k_1+k_2]D^{-1}(\varphi_q^{(0)}-\varphi_{q+2}^{(0)}). \quad (31)$$

It is impossible to solve (31) in quadratures. We are therefore forced to confine ourselves to an estimate of $\varphi_q^{(1)}$. We note that it is meaningful to solve only the inhomogeneous equation that satisfies the conditions (16). But since the right-hand side of (31) is proportional to $\varphi_q^{(0)}$, and since $\varphi_q^{(0)}$ varies slowly with q (scale $1/\alpha \gg 1$), we assume that

$$\varphi_q^{(0)}-\varphi_{q+2}^{(0)}\approx-2d\varphi_q^{(0)}/dq,\quad \varphi_{q\pm2}^{(1)}\approx\varphi_q^{(1)}\mp2\frac{d\varphi_q^{(1)}}{dq}+2\frac{d^2\varphi_q^{(1)}}{dq^2}\quad a=1,\quad b=1 \quad (32)$$

and obtain approximately

$$iq\alpha\varphi_q^{(1)}+4q\frac{d\varphi_q^{(1)}}{dq}+2q^2\frac{d^2\varphi_q^{(1)}}{dq^2}=-4i\varepsilon\tau\frac{d\varphi_q^{(0)}}{dq}.$$

Thus, $\varphi_q^{(1)} = \alpha y^{(1)}(q\alpha)$. In our approximation, when only the principal dependence on $\theta_2 - \theta_1 = \theta$ is taken into account, we have

$$\Phi(\varepsilon,\omega)\sim\int\varphi^{(0)}(-\theta)\sin\theta\varphi^{(1)}(\theta)d\theta\sim\sim\sum_q\varphi_q^{(1)}\frac{d\varphi_q^{(0)}}{dq}\sim\alpha\sum_q y^{(1)}(q\alpha)\frac{d\varphi_q^{(0)}}{dq}. \quad (33)$$

Using the fact that we know the characteristic scale of variation of $\varphi_q^{(0)} = y^{(0)}(q, \alpha)$, we find from this that $\Phi(\epsilon, \omega) \sim |\omega|$.

4. Thus, the conductivity of a one-dimensional system vanishes as $\omega \to 0$ if the energy is high enough. This leads to the conclusion that there is no static conductivity also in the general case of arbitrary particle energies.

Let us turn to the exact equations (12). We have shown by direct calculation that as $\omega \to 0$ the character of the dependence on the angles $\theta_2 - \theta_1$, and by the same token on $z_2 - z_1$, is delta-like. We can now verify by direct substitution in the first equation of (12) that the exact solution for $f^{(0)}(z_1, z_2)$ at $\omega = 0$, i.e., $k_1 = k_2$, is

$$f^0(z_1,z_2)=\delta(z_1-z_2)f^{(0)}(z_1), \quad (34)$$

where $f^{(0)}(z_1)$ is the solution of the equation

$$\frac{d}{dz}[(z^2+k^2)f^{(0)}(z)]+n[f^{(0)}(z-2k_0)-f^{(0)}(z)]=0. \quad (35)$$

The solution of this equation can be obtained, in particular, by iteration with respect to the impurity concentration at sufficiently small n. It is easily seen that since $f^{(0)}(z_1, z_2)$ takes the form (34), the left-hand side of Eq. (12) for the functions $f^{(1)}(z_1, z_2)$ vanishes, the equation becomes homogeneous, and its formal solution is

$$f^{(1)}(z_1,z_2)=(z_1-z_2)f^{(0)}(z_1,z_2)\equiv0.$$

This result again agrees with the properties of the function $f^{(1)}$ as $\omega \to 0$. By the same token, it follows from (10) that $\Phi = 0$, meaning that there is no static conductivity at any particle energy. It becomes more or less obvious that the function $\Phi(\omega, \epsilon)$ $(\omega \to 0)$ is different at different energies.

5. Unfortunately, it is impossible to obtain any simple physical picture explaining this peculiar absence of static conductivity in the one-dimensional case.

It seems, however, natural to attribute the result to the fact that if an attempt is made to construct the usual diagram technique (see[6]) to find even one Green function of the particle, then it turns out that, unlike the three-dimensional case, even the simplest diagram with overlapping lines from different impurities leads not to a contribution $\sim 1/\epsilon\tau$ (see[6]), but in fact to a new pole of the Green function, at $p^2/18m$, and it becomes necessary to sum all the diagrams. That this is indeed the case can be shown by using other considerations.

Halperin[3] has shown that the quantity $F(p, \epsilon) = \operatorname{Im} G(p, \epsilon)$, where $G(p, \epsilon)$ is the Green function of the particle, p is the momentum, and ϵ is the energy, can be obtained exactly with the aid of the integral

$$F(p,\varepsilon)=4m\operatorname{Re}\int_{-\infty}^{\infty}f^{(0)}(-z)y(z)dz,$$

where $f^{(0)}(z)$ satisfies Eq. (35) (Halperin obtained it in the Born approximation for the collision term), and $y(z)$ can be obtained with the aid of the equation

$$\frac{d}{dz}[(k^2+z^2)y(z)]+n[y(z-2k_0)-y(z)]+(ip-z)y(z)=-f^{(0)}(z),$$

subject to the conditions

$$f^{(0)}(z)z^2\big|_{z\to+\infty}=f^{(0)}(z)z^2\big|_{z\to-\infty},\qquad y(z)z\big|_{z\to+\infty}=y(z)z\big|_{z\to-\infty}.$$

The function $f^{(0)}(z)$ is generally smooth and does not have unpleasant properties, unlike $y(z)$. Indeed, let us introduce new functions and variables θ:

$$f^{(0)}(z)=\frac{k\varphi^{(0)}(\theta)}{k^2+z^2},\qquad y(z)=\frac{\varphi(\theta)}{(z^2+k^2)^{1/2}},$$
$$z=k\,\mathrm{tg}\,\theta.$$

Then the equation for the Fourier components of the function $\varphi(\theta)$ is

$$[ip+ik(2l+1)]\varphi_l+n\sum_m A_{l,m}\varphi_m=\frac{(\varphi_l^{(0)}-\varphi_{l+1}^{(0)})}{2}. \qquad (36)$$

We did not write out in this equation the coefficients $A_{l,m}\sim D$, since they are of no interest to us. It is seen from (36) that for the function φ_l the singular values of the momenta and energies, when one cannot expand in terms of the concentration, are $p^2=k^2(2l+1)^2$, i.e.,

$$\varepsilon=p^2/2m\,(2l+1)^2. \qquad (37)$$

Thus, in addition to the usual branch $\epsilon=p^2/2m$ we obtain an infinite set of branches. In the language of diagram technique this means that new singularities appear in diagrams that become ever more complex topologically. The fact that by specifying the particle energy we do not specify its momentum uniquely seems to point to a localized character of the particle wave function.

Note added in proof (25 April 1973). A more detailed analysis of (33) shows that the corresponding integral decreases with frequency more rapidly than $|\omega|$, as might follow from scale considerations. (I am grateful to V. Berezinskii for pointing this circumstance out to me.)

The reason is that in the angle region where $\varphi^{(0)}(\theta)$ and $\varphi^{(1)}(\theta)$ assume the largest values, the dependence on $\theta=\theta_2-\theta_1$ takes the form $\varphi^{(0)}(\theta)\sim\alpha^{-1}\zeta_0(\theta/\alpha)$ and $\varphi^{(1)}(\theta)\sim\zeta_1(\theta/\alpha)$. This result can be obtained from the explicit expression (24) for $\varphi^{(0)}(\theta)$ and from Eq. (22′) for $\varphi^{(1)}(\theta)$. Both functions differ from zero only if $\theta<0$ ($|\theta|\lesssim\alpha$). A more accurate statement is that when account is taken of formulas (28) and (24) the two functions $\varphi^{(0)}(\theta)$ and $\varphi^{(1)}(\theta)$ are of the order of α outside the region $\theta<0$, $|\theta|\lesssim\alpha$ (see the discussion preceding formula (29)). Therefore if we confine ourselves only to this region of θ, we obtain zero because the functions $\varphi^{(0)}(\theta)$ and $\varphi^{(1)}(\theta)$ enter in (33) with opposite signs of the angles. We did not take into account here, however, the dependence on the variable $\theta_2+\theta_1$. It was shown earlier that the Fourier components $\varphi^{(0)}_{p,q}$ decrease with decreasing p like the parameter $1/\epsilon\tau\gg\alpha$ raised to a certain power. This means that the function $\varphi^{(0)}(\theta_1,\theta_2)$ varies over the scale α like the function $\theta=\theta_2-\theta_1$, and over a much larger scale as a function of the variable $\theta_2+\theta_1$. If we want to retain the dependence on $\theta_2+\theta_1$ (with $\theta_2-\theta_1\lesssim\alpha$) we must therefore use Eqs. (20′) – (20″), in which we can put $k_2=k_1$ everywhere except in the term with the derivative with respect to $\theta=\theta_2-\theta_1$, and the factor of $\varphi^{(0)}(\theta_2,\theta_1)$ in (20″) is

$$\sim\left[k\theta+\frac{k_2-k_1}{2}\sin(\theta_2+\theta_1)\right],$$

since at small θ one of them is approximately equal to θ and the other approximately equal to α. The function Φ must now be found with the aid of formula (18).

Even with all the simplifications, it is impossible to solve (20′) and (20″), since the variables cannot be separated. Moreover, even in this case we cannot state that $\sigma\sim|\omega|$, since it is even impossible to demonstrate that when account is taken of the dependence on the variable $\theta_2+\theta_1$ the functions $\varphi^{(0)}$ and $\varphi^{(1)}$ differ from zero at $\theta>0$ as functions of the variable θ/α, since all the equations of the theory are parabolic and it is therefore impossible to find the lines for one level (where, say, the functions vanish). By the same token we can only state that the conductivity decreases with frequency not slower than $|\omega|$ at high energies.

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