Emission of gravitational waves by an electromagnetic cavity

L. P. Grishchuk and M. V. Sazhin

State Astronomy Institute
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The possibility of gravitational wave emission under terrestrial conditions is considered. A qualitative formula is derived which yields the upper limit for the gravitation radiation flux from an arbitrary source with a small gravitational potential. The formula takes into account the coherence of the source and "focusing" of its gravitational radiation. As a concrete emitter, an electromagnetic cavity is considered. It is shown that an emitter whose parameters seem to be quite reasonable from a technical viewpoint can create a gravitation energy flux of the order of 10^{-7} erg/cm^2 sec over an area of 1 cm^2 at a distance of 10^3 cm from the emitter. The possible parameters of the emitter are as follows: a superconducting resonant system with a total volume of about 10^9 cm^3 in which a standing wave 4 cm long exists, the mean energy density of the electromagnetic field being 10^{10} erg/cm^3.

At the present time the main hope for detecting gravitational radiation is directed at relativistic astrophysical sources. This is understandable, since a large power in gravitational radiation can be produced only by processes which are accompanied by substantial changes of the space-time metric. It follows from Hawkings' results[1] that when two collapsing rotating masses m collide, an amount of energy of the order of mc^2 can be emitted in the form of gravitational radiation. Taking into account the fact that this energy is produced over a characteristic time interval $r_g/c = 2Gm/c^3$, we obtain for the power of the gravitational radiation $W \approx 10^{59}$ erg/s. This magnitude does not depend at all on the mass and represents in a certain sense the maximal possible power of a source of gravitational waves.[2] Assuming that the colliding collapsars are situated in the nucleus of our Galaxy, we obtain for the flux density of gravitational radiation energy on Earth the value 10^{13} erg/s-cm^2, i.e., a colossal quantity, exceeding by 16 orders of magnitude the flux density of electromagnetic energy coming from the nucleus of the Galaxy.

Of course, this optimistic estimate should be treated with utmost caution. One must take into account the hypothetical character of the very existence of collapsars and of their collisions, their rarity (about once a year) and the short duration ($\sim 10^{-5}$ s) of their possible radiation, the indeterminacy in the interpretation of observations, to say nothing of the fact that the radiation process cannot be controlled. Therefore it seems quite reasonable to look for artificial terrestrial sources of gravitational radiation. Such source could, apparently, guarantee a larger flux density than many manifestly existing astronomical systems (e.g., double stars), which have a gravitational potential small compared to c^2. In this connection it is worth remembering that two of the most troubling predictions of general relativity, gravitational waves and collapsing objects (and also black holes), are distinguished by the fact that gravitational waves follow from the relativistic theory of gravitation already in the weak field approximation, whereas black holes can be realized in principle only in extremely strong gravitational fields.

We give an estimate of the maximal flux of gravitational energy on which one can count, in principle, from a source with small gravitational potential. As is well known,[3,4] under the assumption that the gravitational field is weak one can reduce the Einstein equations in a harmonic coordinate system to the following system of wave equations:

$$\frac{1}{2}\Box\psi_i^k = \frac{8\pi G}{c^4}\tau_i^k, \quad \psi_i^k = h_i^k - {}^1/_2\delta_i^k h, \quad \psi_{i,k}^k = 0, \tag{1}$$

where h_{ik} are the deviations from the flat-space metric and τ_i^k consists of the components T_{ik} of the energy-momentum tensor of matter if all of them are of the same order of magnitude, or it also contains terms which are quadratic in ψ_i^k if the components of T_{ik} differ strongly from one another. At time t and at a point defined by the direction **n** and the distance R which is large compared to the characteristic dimensions l of the system, the solution of Eq. (1) has the form of the retarded integral

$$\psi_{ik} = -\frac{4G}{c^4 R}\int [\tau_{ik}]dV, \tag{2}$$

where the integrand is taken at the earlier time

$$\tilde{t} = t - R/c + \mathbf{rn}/c.$$

If the dimensions of the system are small compared to the wavelength of the gravitational waves emitted by it, the retardation at the source is the same for all its points, $\tilde{t} \approx t - R/c$, and in the final count the computation of $\psi_{\alpha\beta}$ reduces to the determination of the (time-dependent) moments of inertia of the system, on account of the equation $\tau_{i;k}^k = 0$. Essential use is being made here of the assumption that the source is isolated. The energy flux density in the wave zone (for $R \gg \lambda$) determined from the energy-momentum pseudotensor has a quadratic expression in terms of $\psi_{\alpha\beta}$, and consequently increases as the characteristic frequency ω of the motion of the source increases (for a fixed amplitude $\tau_{ik}(0)$ in the expression $\tau_{ik} = \tau_{ik}(0)e^{i\omega t}$). The increase of the flux with the growth of the frequency will go on until $\lambda \approx c/\omega$ becomes of the order of l. If all the dimensions of the system and their variations are of the order of λ, and the average energy density of the energy in the system is ϵ, the flux density of gravitational energy at distance R is $G\epsilon^2\lambda^4/c^3R^2$. At the boundary of the wave zone, for $R = \lambda$, we obtain $G\epsilon^2\lambda^2/c^3$. For $\omega l/c \gg 1$ one can no longer neglect retardation inside the source, since the contributions to $\psi_{\alpha\beta}$ from different parts of the system may compensate one another. Under these conditions an increase in ω will generally lead to a decrease of the flux.

In order that all parts of the source with dimensions $l \gg \lambda$ yield a positive contribution to the gravitational field in the wave zone, it is necessary to ensure the coherence of the whole volume of the radiator.[1)] In other words, one must realize at the reception point addition of the amplitudes coming from all the elementary radiators which make up the system. For this the equal-phase surfaces of the oscillations of the elementary emitters must be concentric spheres, centered at the reception point (we call this "focusing"), and the phase shift along the line of sight must correspond to a wave traveling in the direction of the observer with the speed of propagation of gravitational waves in the source material, i.e., practically with the speed of light (coherence along the line of sight). If these conditions are satisfied the radiation is collected in a spot of area of order λ^2. Then the limiting flux density (at distance $R = l$) is determined by

$$\frac{G}{c^3}\varepsilon^2\left(\frac{v}{c}\right)^4\frac{l^4}{\lambda^2}=\frac{G}{c^3 l^2}\mathscr{E}^2\left(\frac{v}{c}\right)^4\frac{1}{\lambda^2}=\frac{1}{l^2}WN^2, \tag{3}$$

where v is the characteristic speed in directions perpendicular to the line of sight, $\mathscr{E} = \epsilon l^3$ is the total energy stored in the source,

$$W=\frac{G}{c^3}\varepsilon^2\left(\frac{v}{c}\right)^4\lambda^4$$

is the power emitted by one elementary emitter of characteristic dimensions λ, $N = l^3/\lambda^3$ is the number of elementary emitters.

The expression (3) establishes an upper limit for the gravitational energy flux density which can be obtained if one takes into account coherence and focussing of the radiation from a source with small gravitational potential. If the characteristic dimensions of the system are different along different axes, the flux density of gravitational energy at the focal point situated at a distance R from the center of the system can be rewritten in the form

$$\frac{dI}{R^2 do}\approx\frac{G}{c^3R^2}\varepsilon^2\left(\frac{v}{c}\right)^4\left(\frac{l_1}{\lambda}\right)^2\left(\frac{l_2}{\lambda}\right)^2\left(\frac{l_3}{\lambda}\right)^2\lambda^4. \tag{4}$$

The equations (3) and (4) are applicable both for mechanical and electromagnetic systems. In the latter case the factor v/c is to be replaced by 1. As can be seen from (3) and (4), other conditions being equal, it is convenient to make use of short waves. This circumstance, together with the relative simplicity of construction seems to give some advantage to systems making use of alternating electromagnetic fields over mechanical systems.

Below we consider as emitters electromagnetic cavity resonators. Since the energy-momentum tensor of the electromagnetic field is quadratic in $\mathbf{E}$ and $\mathbf{H}$ a standing electromagnetic wave of frequency ω will emit a gravitational wave of twice that frequency. It is obvious that two standing electromagnetic waves of frequencies ω_1 and ω_2 will emit, in addition to the harmonics $2\omega_1$ and $2\omega_2$, also gravitational waves of the frequencies $\omega_1-\omega_2$ and $\omega_1+\omega_2$. A wave of frequency ω will in the presence of a constant external field be able to emit gravitational waves of the same frequency ω. In the terminology of quantum theory one can describe these processes as the production of a graviton by a photon pair, or as graviton production by a photon in an external field, etc. (it is clear that on account of the cavity walls the conservation laws are satisfied).

The alternating electromagnetic field of the resonant cavity interacts with the walls of the cavity via the surface currents and forces these walls to oscillate and emit. One may say that in this case the gravitons are produced by phonons. The variable elastic stress tensor $\sigma_{\alpha\beta}$ of the shell of the cavity is of the same order of magnitude as the components $T_{\alpha\beta}$ of the energy-momentun tensor of the electromagnetic field, however the contribution of $\sigma_{\alpha\beta}$ to the gravitational field in the wave zone is small compared to the contribution of $T_{\alpha\beta}$.

In Secs. 1–3 we consider a spherical resonant cavity. In this case the problem of oscillations of the shell can be solved relatively simply. The conclusions obtained in Sec. 2 relative to the radiation of the shell of the resonator are generally applicable to mechanical systems. A spherical cavity does not have a noticeable directivity of radiation, therefore the gravitational energy flux density is small in all directions. The total energy losses of the resonant cavity on account of gravitational radiation are so small that one can hardly separate them indirectly on the background of losses due to the nonideal character of the cavity, i.e., ohmic losses in the walls, its dielectric, etc.

Section 4 considers a flat resonant cavity. The contribution of the walls to the gravitational radiation is not taken into account. It is shown that systems that can apparently be produced within the limits of present technological capabilities are able to generate a flux of gravitational radiation approaching the threshold of sensitivity of detectors which have been proposed recently. The idea of using a gravitational-electromagnetic resonance has been proposed by Braginskiĭ and Menskiĭ[7] and this question has been discussed in detail in [8].

1. THE ELECTROMAGNETIC FIELD IN A SPHERICAL CAVITY AND THE OSCILLATIONS OF ITS SHELL

We assume that the walls of a spherical cavity of radius r_0 are perfect conductors and the medium inside the cavity is a perfect dielectric. For definiteness we consider inside the cavity oscillations of the magnetic type (TE waves) with the condition $E_r \equiv 0$. The Maxwell equations reduce to an eigenvalue problem with the boundary conditions $E_\theta = E_\varphi = 0$ for $r = r_0$. The solutions of this problem can be written (here we have corrected a typographical error in E_θ as given in the book by Koshlyakov et al.[9]):

$$E_r=0,\quad E_\theta=\frac{ik}{c}\frac{1}{\sin\theta}\frac{\partial v}{\partial\varphi},\quad E_\varphi=-\frac{ik}{c}\frac{\partial v}{\partial\theta},$$
$$H_r=\frac{\partial}{\partial r}r\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+k^2rv, \tag{5}$$
$$H_\theta=\frac{\partial}{\partial\theta}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right),\quad H_\varphi=\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right),$$

where v is determined by the equation

$$v_{mn}=\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{r^{1/2}}J_{n+1/2}(kr)P_{nm}(\cos\theta)\cos(m\varphi+\psi_m), \tag{6}$$

and, because of the boundary conditions, k is one of the roots of the equation $J_{n+1/2}(kr_0) = 0$. The time-dependence of the field is given by the exponential $\exp(-i\omega t)$ $(\omega = kc)$.

The interaction of the electromagnetic field inside the cavity with the surface currents leads to the appearance of a force acting on the walls of the cavity. If n^α is the unit vector along the external normal to the sphere $r = \text{const}$, the force is determined by the equation $F_\alpha = T_{\alpha\beta}(r_0)n^\beta$, where $T_{\alpha\beta}(r_0)$ are the components of the energy-momentum tensor of the electromagnetic field at the points $r = r_0$. For a field of the type under consideration $F_\theta = F_\varphi = 0$ and $F_r = (1/8\pi)\mathbf{H}^2|_{r=r_0}$. The solution (5) is written in terms of the so-called "natural components", therefore $H^2 = H_r^2 + H_\varphi^2 + H_\theta^2$. Since F_α contains the square of the vector $\mathbf{H}$ the force consists of two terms, a term which does not depend on time, and a term which is harmonic with frequency 2ω. The first term produces a constant strain, which does not interest us here and the second one produces alternating stresses, which together with the electromagnetic field are the source of the gravitational radiation.

We assume that the shell of the cavity consists of a homogeneous spherical layer of density ρ and external radius r_1. In order to determine the deformation vector u which determines the behavior of the cavity shell it is necessary to solve the equation of motion with the following boundary conditions: $\sigma_{\alpha\beta}n^\beta = 0$ at the outer boundary and $\sigma_{\alpha\beta}n^\beta = -F_\alpha$ at the interior boundary. We consider the stationary regime of oscillations, with $u \sim \exp(-i2\omega t)$. Introducing the wave numbers $k_l = 2\omega/c_l$ and $k_t = 2\omega/c_t$ for the longitudinal and transverse waves, respectively, we write the equations of motion in the form[10]

$$\Delta u_l + k_l^2 u_l = 0, \quad \Delta u_t + k_t^2 u_t = 0. \tag{7}$$

The displacement vector is

$$\mathbf{u} = (\mathbf{u}_l + \mathbf{u}_t)e^{-i2\omega t}; \quad \operatorname{div} \mathbf{u}_t = 0, \quad \operatorname{rot} \mathbf{u}_l = 0.$$

Following the Kelvin method described in [11], the solution of the problem can be expressed in terms of spherical harmonics, considered as functions of the cartesian coordinates. Then the general solution of the equations (7) is

$$u_x = -\frac{1}{k_l^2}\frac{\partial \delta}{\partial x} + \sum_n \Big\{ \Big(y\frac{\partial \chi_n}{\partial z} - z\frac{\partial \chi_n}{\partial y} \Big) [\alpha_n^{(1)} \psi_n(k_t r) + \alpha_n^{(2)} \zeta_n(k_t r)]$$
$$+ \frac{\partial \varphi_{n+1}}{\partial x} [\beta_{n+1}^{(1)} \psi_n(k_t r) + \beta_{n+1}^{(2)} \zeta_n(k_t r)] - \frac{n+1}{n+2} k_t^2 r^{2n+5} \frac{\partial}{\partial x}\Big(\frac{\varphi_{n+1}}{r^{2n+3}} \Big) \tag{8}$$
$$\times [\beta_{n+1}^{(1)} \psi_{n+2}(k_t r) + \beta_{n+1}^{(2)} \zeta_{n+2}(k_t r)] \Big\}.$$

Here we have used the notation

$$\delta = \operatorname{div} \mathbf{u}_l = \sum_n \omega_n [\gamma_n^{(1)} \psi_n(k_l r) + \gamma_n^{(2)} \zeta_n(k_l r)],$$

$\omega_n, \varphi_n, \chi_n$ are the so-called "solid spherical harmonics" of order n, which can be represented in the form $r^n Y_n$, where $r = (x^2 + y^2 + z^2)^{1/2}$, and $Y_n(\theta, \varphi)$ is the usual spherical harmonic; the functions ψ_n and ζ_n are defined by

$$\psi_n(x) = \Big(\frac{1}{x}\frac{d}{dx} \Big)^n \frac{\sin x}{x}, \quad \zeta_n = \Big(\frac{1}{x}\frac{d}{dx} \Big)^n \frac{\cos x}{x}$$

and can be expressed in terms of Bessel functions of order $(n + 1/2)$ and $-(n + 1/2)$, respectively; $\alpha_n, \beta_n, \gamma_n$ with the superscripts (1) and (2) are arbitrary constants which have to be found from the boundary conditions. The expressions for the y and z components of the vector u are obtained from (8) by cyclic permutation of the coordinates x, y, z.

The terms of the vector $\mathbf{u}$ containing χ_n describe the displacements which are orthogonal to the radius. The displacements related to this function can exist only as natural modes of the spherical shell, since they are independent on the external force, which in our case acts only along the radius. We do not consider these oscillations, i.e., we set $\alpha_n^{(1)} = \alpha_n^{(2)} = 0$. The boundary conditions allow one to find also the other arbitrary constants. Thus, the formulas written down here solve the problem of forced vibrations of the shell under the action of an arbitrary electromagnetic field (5), (6). However, the determination of the undetermined coefficients in their general form involves cumbersome calculations. We restrict ourselves to the case of the simplest angular dependence of the field.

Let the potential v be independent of φ, and assume its θ-dependence is determined by a factor $\cos\theta$. This assumption corresponds to the choice $m = 0$, $n = 1$ in the expression (6). Then the field components take the form

$$E_r = 0, \quad E_\theta = 0, \quad E_\varphi = A\frac{k}{r}(kr)^{1/2} J_{3/2}(kr) \sin\theta \sin\omega t,$$
$$H_\varphi = 0, \quad H_r = 2A\frac{1}{r^2}(kr)^{1/2} J_{3/2}(kr) \cos\theta \cos\omega t, \tag{9}$$
$$H_\theta = -A\frac{1}{r}\frac{\partial}{\partial r}((kr)^{1/2} J_{3/2}(kr)) \sin\theta \cos\omega t.$$

The constant A is related to the total energy accumulated in the resonant cavity

$$\mathscr{E} = \frac{1}{8\pi}\int (\mathbf{E}^2 + \mathbf{H}^2)\, dV,$$

by the relation

$$\frac{1}{3\pi} A^2 k^2 r_0 \sin^2 kr_0 = \mathscr{E}.$$

The natural frequencies of the field oscillations in the cavity $\omega = kc$ are determined from the condition $\tan(kr_0) = kr_0$, from which it follows that $kr_0 > 1$. The force F_r is expressed in the following form:

$$F_r = {}^1/_3\varepsilon[1 - P_2(\cos\theta) + \cos 2\omega t - P_2(\cos\theta)\cos 2\omega t]. \tag{10}$$

Here ϵ is the average energy density, $\epsilon = \mathscr{E}/V$, where V is the cavity volume. The first two terms in (10) create constant stresses and the third and fourth are alternating, the third corresponding to spherically-symmetric oscillations of the shell which do not produce gravitational radiation. The term of interest to us is

$$F_r = -{}^1/_3\varepsilon P_2(\cos\theta)\cos 2\omega t \tag{11}$$

and describes quadrupole oscillations of the shell, during which it is alternately elongated along the z axis, contracting in the equatorial plane, and then is contracted along the z axis accompanied by bulging out in the equatorial plane. From the boundary conditions and the explicit form of the force (11) it follows that $\mathbf{u}$ can be expressed in terms of the single function

$$\omega_2 = r^2 P_2(\cos\theta) = z^2 - {}^1/_2 x^2 - {}^1/_2 y^2.$$

In the sequel we shall omit the subscript $n = 2$.

Let us write out the explicit form of the deformation vector. For this purpose we recognize that in the shell, i.e., at $r_0 \le r \le r_1$, the following relations hold:

$$\frac{1}{k_l r} = \frac{1}{2}\Big(\frac{r_0}{r}\Big)\Big(\frac{c_l}{c}\Big)\frac{1}{kr_0} \ll 1, \quad \frac{1}{k_t r} = \frac{1}{2}\Big(\frac{r_0}{r}\Big)\Big(\frac{c_t}{c}\Big)\frac{1}{kr_0} \ll 1.$$

The functions of r which enter into the boundary conditions will be expanded in powers of these parameters. Then in the leading approximation in $1/kr$ and $1/k_t r$, the x component of the vector $\mathbf{u}$ can be written in the form

$$u_x = \frac{1}{3}\varepsilon\frac{k_l r_0}{\rho c_l^2 \sin k_l(r_1 - r_0)}\frac{\cos k_l(r_1 - r)}{(k_l r)^2} x P_2(\cos\theta)\cos 2\omega t. \tag{12}$$

The components of u_y, u_z can be obtained by replacing in (12) x by y and z, respectively.

In spherical coordinates $u_\varphi \equiv 0$, in the leading approximation ($\sim 1/kr$) u_r equals

$$u_r = -\frac{1}{3}\varepsilon\frac{r_0}{\rho c_l^2 \sin k_l(r_1 - r_0)}\frac{\cos k_l(r_1 - r)}{k_l r}P_2(\cos\theta)\cos 2\omega t,$$

and u_θ vanishes in this approximation.

The condition of applicability of the equation of small vibrations (7) can be written in the form $\operatorname{div}\mathbf{u} \ll 1$. In our case

$$\delta = \operatorname{div}\mathbf{u} = \frac{\varepsilon}{3\rho c_l^2 \sin k_l(r_1 - r_0)}\frac{r_0}{r}\sin k_l(r_1 - r)P_2(\cos\theta)\cos 2\omega t,$$

which leads to the restriction

$$\Delta = \frac{1}{3}\varepsilon\frac{1}{\rho c_l^2 \sin k_l(r_1 - r_0)} \ll 1.$$

The resonance factor $\sin^{-1}k_l(r_1 - r_0)$ characterizes the deviation of the frequency of the forced vibrations from the natural frequency.

2. THE GRAVITATIONAL FIELD IN THE WAVE ZONE

We first consider the gravitational field produced in the wave zone by the alternating electromagnetic field in the cavity. Since all components of the energy-momentum tensor T_{ik} of the electromagnetic field have the same order of magnitude, τ_{ik} in (2) consists simply of T_{ik}. It makes sense not to calculate all the components of ψ_{ik}, but only the "physical" ones, i.e., those which cannot be removed by coordinate transformations that preserve the condition of harmonicity. It is just these components (and only these) which enter into the expression of the gravitational energy derived from the energy-momentum pseudotensor of the gravitational field.[2)]

Introducing the quantities

$$\zeta_{\alpha\beta} = \psi_{\alpha\beta} - {}^1\!/_3\delta_{\alpha\beta}(\psi_{11} + \psi_{22} + \psi_{33}),$$

one can express the energy flux density in the direction $\mathbf{n}$ in the form

$$\frac{dI}{R^2\,do} = \frac{c^3}{16\pi G}\left[\frac{1}{4}(\dot\zeta_{\alpha\beta}n^\alpha n^\beta)^2 + \frac{1}{2}\dot\zeta_{\alpha\beta}\dot\zeta^{\alpha\beta} - \dot\zeta_{\alpha\beta}\dot\zeta_\gamma{}^\alpha n^\beta n^\gamma\right]. \quad (13)$$

This quantity depends on the off-diagonal components of the tensor ψ_{ik} and on the differences between its diagonal elements.

The radiating system considered here exhibits axial symmetry and therefore the gravitational field depends only on the coordinates r and θ, and in order to determine it completely it suffices to consider a vector of the form $\mathbf{n} = \{\sin\theta_0, 0, \cos\theta_0\}$. Substituting this vector directly into (13), or (equivalently) rotating the coordinate system so that the direction of $\mathbf{n}$ coincides with the z axis, we find the energy flux density in the direction n:

$$\frac{dI}{R^2\,do} = \frac{c^3}{16\pi G}\left[\frac{1}{4}(\dot{\bar\psi}_{33} - \dot{\bar\psi}_{22})^2 + \dot{\bar\psi}_{23}^2\right]; \quad (14)$$

$$\bar\psi_{33} - \bar\psi_{22} = (\psi_{33} - \psi_{22}) - \cos^2\theta_0(\psi_{33} - \psi_{11}) - 2\sin\theta_0\cos\theta_0\,\psi_{13}, \quad (15)$$

$$\bar\psi_{23} = \sin\theta_0\psi_{23} - \cos\theta_0\psi_{12}.$$

Let us compute, for example, $\psi_{33} - \psi_{22}$; the other functions entering into (15) are computed in a similar manner. We introduce the notation $\xi = kr$. The Cartesian components of the electric and magnetic field intensities will be

$$\begin{aligned} E_x &= -Ak^2\xi^{-1/2}J_{3/2}(\xi)\sin\theta\sin\varphi\sin\omega t,\\ E_y &= Ak^2\xi^{-1/2}J_{3/2}(\xi)\sin\theta\cos\varphi\sin\omega t,\\ H_x &= Ak^2\xi^{-1/2}J_{3/2}(\xi)\sin\theta\cos\theta\cos\varphi\cos\omega t,\\ H_y &= Ak^2\xi^{-1/2}J_{5/2}(\xi)\sin\theta\cos\theta\sin\varphi\cos\omega t,\\ E_z &= 0, \quad H_z = Ak^2\xi^{-1/2}[2J_{3/2}(\xi) - J_{5/2}(\xi)\sin^2\theta]\cos\omega t. \end{aligned} \quad (16)$$

The alternating part of the difference $T_{33} - T_{22} = (4\pi)^{-1}(-H_Z^2 + E_y^2 + H_y^2)$ can be written in the form

$$T_{33} - T_{22} = \frac{A^2k^4}{8\pi}\frac{1}{\xi^4}(M + N\cos 2\varphi)\cos 2\omega t;$$

$$M = -4\xi J_{3/2}^2(\xi) + ({}^1\!/_2\xi^3 J_{5/2}^2(\xi) - {}^1\!/_2\xi^3 J_{3/2}^2(\xi) + 4\xi^2 J_{3/2}(\xi)J_{5/2}(\xi))\sin^2\theta - {}^3\!/_2\xi^3 J_{5/2}^2(\xi)\sin^4\theta,$$

$$N = -{}^1\!/_2\xi^3(J_{3/2}^2(\xi) + J_{5/2}^2(\xi))\sin^2\theta + {}^1\!/_2\xi^3 J_{5/2}^2(\xi)\sin^4\theta.$$

The integrand in the expression (2) equals

$$r^2\sin\theta\,dr\,d\theta\,d\varphi[T_{33} - T_{22}] = \frac{A^2k}{8\pi}\frac{1}{\xi^2}\sin\theta\,d\xi\,d\theta\,d\varphi(M + N\cos 2\varphi) \times\cos[2\omega t' - 2\xi(\sin\theta\cos\varphi\sin\theta_0 + \cos\theta\cos\theta_0)].$$

Here and in the sequel $t' = t - R/c$.

The integration of this expression with respect to φ from 0 to 2π leads to the Bessel functions $J_0(2\xi\sin\theta\sin\theta_0)$ and $J_2(2\xi\sin\theta\sin\theta_0)$. The integration with respect to θ from 0 to π leads finally to integrals that can be found in tables[12]

$$\int_0^{\pi/2}\sin^\nu\theta\cos(2\xi\cos\theta\cos\theta_0)J_\nu(2\xi\sin\theta\sin\theta_0)\,d\theta = \left(\frac{\pi}{2}\right)^{1/2}(2\xi\sin\theta_0)^\nu\frac{J_{\nu+1/2}(2\xi)}{(2\xi)^{\nu+1/2}}.$$

Thus, there remains the integral with respect to ξ from 0 to kr_0 of a relatively complicated combination of Bessel functions of ξ and 2ξ. It is more convenient to integrate with respect to ξ directly for the whole expression $\bar\psi_{33} - \bar\psi_{22}$. As regards $\bar\psi_{23}$, it vanishes identically, since ψ_{12} and ψ_{23} vanish already upon integrating the appropriate terms with respect to φ. We finally obtain the alternating part of the gravitational field in the wave zone

$$\begin{aligned} \bar\psi_{33} - \bar\psi_{22} &= \frac{4G}{c^4R}\frac{A^2k\sqrt{\pi}}{(2)^{3/2}}B\sin^2\theta_0\cos 2\omega t',\\ B &= \frac{1}{\pi\sqrt{2\pi}}\frac{\sin^4 kr_0}{kr_0} + \int_0^{kr_0}\frac{d\xi}{(2\xi)^{3/2}}J_{5/2}^2(\xi)J_{5/2}(2\xi). \end{aligned} \quad (17)$$

In the computation of B it was taken into account that the integration with respect to ξ extended from 0 to kr_0, which is one of the roots of the equation $J_{3/2}(kr_0) = 0$. For large kr_0 we have

$$B \approx \frac{7}{8\pi}\frac{1}{\sqrt{2\pi}}\frac{1}{kr_0}, \quad \sin kr_0 \approx 1.$$

Approximately

$$\bar\psi_{33} - \bar\psi_{22} = \frac{7\pi}{2}\frac{G}{c^4R}\frac{\varepsilon r_0^3}{(kr_0)^2}\sin^2\theta_0\cos 2\omega t'. \quad (18)$$

Substituting (17) into (14), we find the average (over one period) energy flux density

$$\frac{dI}{R^2\,do} = \frac{G}{c^3R^2}\pi^4\frac{\varepsilon^2r_0^4}{\sin^4 kr_0}B^2\sin^4\theta_0 \approx \frac{G}{c^3R^2}\varepsilon^2r_0^2\lambda^2\sin^4\theta_0, \quad (19)$$

where λ is the gravitational wavelength: $\lambda = \pi c/\omega$. The factor $\sin^4\theta_0$ determines the angular distribution of the radiation. The radiation does not depend on the angle φ and vanishes in the direction of the poles. The polarization of the radiation is the same everywhere.

Comparing the expressions (19) and (4) we see that in a spherical resonant cavity only part of the volume radiates effectively: namely a volume with one characteristic dimension r_0 and two characteristic dimensions of the order λ (an annulus of radius r_0 of thickness and width $\sim\lambda$, containing the z axis in its plane). If, for illustration we set $\epsilon = 10^{10}$ erg/cm^3 and $r_0 = 10^3$ cm, we obtain at the upper limit for $\lambda \approx r_0$ and $R \approx r_0$ in the equatorial plane of the emitter a flux density $dI/R^2 do$ $\approx 10^{-13}$ erg/s-cm^2. The total radiated power is $W \approx 10^{-7}$ erg/s.

We now consider the radiation from the shell of the resonator. Since the vibration of the shell are considered small, in the linear approximation with respect to the small parameter—the ratio of the amplitude of vibrations to the wavelength of the acoustic wave—the space components of the energy-momentum tensor of the solid, Θ_{ik}, reduce to the components of the elastic stress tensor

$$\sigma_{\alpha\beta} = \mu\left(\frac{\partial u_\alpha}{\partial x^\beta} + \frac{\partial u_\beta}{\partial x^\alpha}\right) + \left(K - \frac{2}{3}\mu\right) \operatorname{div} u \delta_{\alpha\beta},$$

where μ and K are the elastic moduli. The tensor $\sigma_{\alpha\beta}$ is of the same order of magnitude as the tensor $T_{\alpha\beta}$ of the electromagnetic field (at the boundary $r = r_0$ their normal components coincide). However, the contribution to the gravitational field $\psi_{\alpha\beta}$ is determined by a volume integration, for which the spatial dependence of the tensors is important. We shall show that the contribution of $\sigma_{\alpha\beta}$ is, generally, small compared with that of $T_{\alpha\beta}$ and consequently the gravitational radiation from the shell is small.[3)]

The components $\Theta_{\alpha\beta}$ are small compared with Θ_{00} and $\Theta_{0\alpha}$. In these conditions one must generally take into account in the right-hand side of Eq. (1) for $\psi_{\alpha\beta}$, in addition to $\Theta_{\alpha\beta}$, also terms which are quadratic in the derivatives of ψ_{00} and $\psi_{0\alpha}$ (as happens in gravitationally coupled systems of the type of binary stars). In this case we are entitled to neglect these terms, since the gravitational potential of the cavity shell $\varphi \sim GM/r_1$ where M is its mass, is small compared to the square of the speed of sound.

Let us find the gravitational radiation produced by the elastic stresses. The angular dependence of the radiation is similar to (19) and has no singularities, therefore we consider the flux only in the direction of the x axis. We make use of the components of the displacement vector (12) and determine $\sigma_{zz} - \sigma_{yy}$ in the leading approximation in $1/k_0 r$:

$$\sigma_{zz} - \sigma_{yy} = 2\mu\Delta r_0 \frac{\sin k_l(r_1 - r)}{r}(\cos^2\theta - \sin^2\theta\sin^2\varphi) P_2(\cos\theta)\cos 2\omega t.$$

Integration of this expression, taken at the instant $\tilde{t} = t - kr\sin\theta \times \cos\varphi$, with respect to φ leads to the Bessel functions $J_0(2kr\sin\theta)$ and $J_1(2kr\sin\theta)$. As a result of subsequent integration with respect to θ there appear Bessel functions of half-integer order of the variable 2kr, which can be reduced to sin 2kr, cos 2kr, and powers of kr. Finally, the integral reduces to the form

$$\int [\sigma_{zz} - \sigma_{yy}]\, dV = 2\mu\Delta r_0 \frac{1}{k} \cos 2\omega t' \int_{r_0}^{r_1} F(r)\, dr;$$

$$F(r) = \sin k_l(r_1 - r) \left[\left(-\frac{3}{2(kr)^2} + \frac{9}{8(kr)^4}\right)\sin 2kr + \left(3 + \frac{387}{8(kr)^3}\right)\cos 2kr\right].$$

Since $k_l r_1 \gg 1$ and $k_l r_0 \gg 1$, the integral is approximately equal to $1/k_l$. Then, taking into account that $\mu = \rho c_t^2$ and setting $c_t \approx c_l = c_s$, one can write $\psi_{33} - \psi_{22}$ at large distances in the following form:

$$\psi_{33} - \psi_{22} \approx \frac{G}{c^4 R}\rho c_t^2 \Delta \frac{r_0}{kk_l}\cos 2\omega t' \approx$$
$$\approx \frac{G}{c^4 R}\frac{\varepsilon r_0^3}{(kr_0)^2}\frac{c_s}{c \sin k_l(r_1 - r_0)}\cos 2\omega t'. \quad (20)$$

In a regime far from the resonance, i.e., for $\sin k_l(r_1 - r_0) \approx 1$,

$$\psi_{33} - \psi_{22} \approx \frac{G}{c^4 R}\frac{\varepsilon r_0^3}{(kr_0)^2}\frac{c_s}{c}\cos 2\omega t'.$$

This quantity is c_s/c times smaller than the quantity (18). Since $\psi_{23} \equiv 0$, the average over a period of the energy flux in the direction x equals

$$\frac{dI}{R^2\, do} \approx \frac{G}{c^3 R^2}(\rho c^2)^2 \Delta^2 \left(\frac{c_s}{c}\right)^6 r_0^2\lambda^2 = \frac{G}{c^3 R^2}(\tilde{\rho} c^2)^2 \left(\frac{c_s}{c}\right)^6 r_0^2 \lambda^2. \quad (21)$$

Here $\tilde{\rho}$ denotes the amplitude of the alternating part of the density $\tilde{\rho} = \Delta\rho$. Equation (21) is universal for the calculation of the radiation generated by elastic stresses. This expression may still contain a large factor $(l_1/\lambda)^2$, but only under the condition that the source be coherent.

It is interesting to note that Eq. (20) illustrates the possibility of using an insignificant electromagnetic field in a regime close to resonance as a mechanism capable to excite effectively and synchronously masses situated at a distance.

3. A SPHERICAL RESONANT CAVITY WITH A CONSTANT MAGNETIC FIELD

We consider a standing electromagnetic wave in a cavity in which a static homogeneous magnetic field is present. (We, of course, do not pay attention here to the technical difficulties of realizing a magnetic field in a cavity with superconducting walls.) In this case the components of the magnetic field strength (16) will have the additional term $\mathbf{H}^c$. In the energy-momentum tensor there will appear, in addition to the enumerated terms proportional to $\cos 2\omega t$, also cross terms varying like $\cos\omega t$, with which we deal now.

Since $\mathbf{H}^c$ does not depend on the coordinates and time, the calculation of ψ_{ik} reduces to the computation of retarded integrals of the magnetic field of the wave, $\int[\mathbf{H}]dV$. If $\mathbf{n}$ is a unit vector pointing to the observation point, one can write the result of the integration in the form

$$\int [H_x]\, dV = \alpha n_1 n_3 \cos\omega t', \qquad \int [H_y]\, dV = \alpha n_2 n_3 \cos\omega t',$$
$$\int [H_z]\, dV = \alpha(1 - n_3^2)\cos\omega t', \quad (22)$$

where we have used the notation

$$\alpha = -2(2\pi)^{3/2}\varepsilon^{1/2} r_0^2 k^{-1} \sin kr_0.$$

From the quantities (22) it is easy to construct the $\psi_{\alpha\beta}$:

$$\psi_{22} - \psi_{33} = \frac{2G\alpha}{\pi c^4 R}[H_2^c(1 - n_3^2) - H_3^c n_2 n_3]\cos\omega t'$$
$$\psi_{23} = \frac{G\alpha}{\pi c^4 R}[H_2^c(1 - n_3^2) + H_3^c n_2 n_3]\cos\omega t',$$
$$\psi_{11} - \psi_{33} = \frac{2G\alpha}{\pi c^4 R}[H_1^c(1 - n_3^2) - H_3^c n_1 n_3]\cos\omega t',$$
$$\psi_{13} = \frac{G\alpha}{\pi c^4 R}[H_1^c(1 - n_3^2) + H_3^c n_1 n_3]\cos\omega t', \quad (23)$$
$$\psi_{11} - \psi_{22} = \frac{2G\alpha}{\pi c^4 R}[H_1^c n_1 n_3 - H_2^c n_2 n_3]\cos\omega t',$$
$$\psi_{12} = \frac{G\alpha}{\pi c^4 R}[H_1^c n_2 n_3 + H_2^c n_1 n_3]\cos\omega t'.$$

There is no radiation in the directions of the poles on the frequency ω as well as on the frequency 2ω, and in other directions the radiation on the frequency ω depends on the orientation of the magnetic field. We determine the average over one period of the energy flux in the direction of the x axis:

$$\frac{dI}{R^2\,do}=2\pi^2\frac{G}{c^3R^2}(H_z^c)^2\varepsilon r_0^4\sin^2 kr_0\approx\frac{G}{c^3R^2}\varepsilon\varepsilon^c r_0^4, \tag{24}$$

where ϵ^c is the energy density of the constant magnetic field.

The presence of ϵ^a in this formula allows one, in principle, to reduce the intensity of the alternating field increasing accordingly the intensity of the static magnetization field. It is clear that the general form of (24) (the product $\epsilon\epsilon^c$ instead of ϵ^2) is characteristic not only of the concrete problem under consideration, but also in the general case of radiation in the presence of an external field.

4. THE GRAVITATIONAL RADIATION OF RECTANGULAR ELECTROMAGNETIC RESONATOR

In a rectangular parallelepiped of dimensions l_1, l_2, l_3 the electromagnetic field is of the form

$$\begin{aligned}E_x&=A_1\cos k_1x\sin k_2y\sin k_3z\cos\omega t,\\ H_x&=B_1\sin k_1x\cos k_2y\cos k_3z\sin\omega t,\\ E_y&=A_2\cos k_2y\sin k_3z\sin k_1x\cos\omega t,\\ H_y&=B_2\sin k_2y\cos k_3z\cos k_1x\sin\omega t,\\ E_z&=A_3\cos k_3z\sin k_1x\sin k_2y\cos\omega t,\\ H_z&=B_3\sin k_3z\cos k_1x\cos k_2y\sin\omega t.\end{aligned} \tag{25}$$

The components of the propagation vector are

$$k_1=m_1\pi/l_1,\quad k_2=m_2\pi/l_2,\quad k_3=m_3\pi/l_3,$$

where m_1, m_2, m_3 are natural numbers and $k_1^2+k_2^2+k_3^2=k^2=\omega^2/c^2$. Of the six constants A_α and B_α only two are independent, since

$$B_1=-k^{-1}(A_3k_2-A_2k_3),\quad B_2=-k^{-1}(A_1k_3-A_3k_1),$$
$$B_3=-k^{-1}(A_2k_1-A_1k_2),$$
$$B_1k_1+B_2k_2+B_3k_3=0,\quad A_1k_1+A_2k_2+A_3k_3=0.$$

The total energy $\mathscr{E}$ contained in the cavity is

$$\mathscr{E}=\frac{1}{64\pi}l_1l_2l_3(A_1^2+A_2^2+A_3^2),$$

if all $k_\alpha\neq 0$, and

$$\mathscr{E}=\frac{1}{32\pi}l_1l_2l_3A_\alpha^2,$$

if $k_\alpha=0$. Obviously, the field inside the cavity vanishes whenever any two components of the propagation vector vanish.

The time-dependent terms of the components of the energy-momentum tensor are

$$\begin{aligned}T_{22}-T_{33}=&-\frac{1}{8\pi}\cos 2\omega t[\sin^2 k_1x(A_2^2\cos^2 k_2y\sin^2 k_3z-A_3^2\cos^2 k_3z\sin^2 k_2y)\\ &-\cos^2 k_1x(B_2^2\sin^2 k_2y\cos^2 k_3z-B_3^2\sin^2 k_3z\cos^2 k_2y)],\\ T_{23}=&-\frac{1}{8\pi}\cos 2\omega t(A_2A_3\sin^2 k_1x-B_2B_3\cos^2 k_1x)\sin k_2y\cos k_2y\cdot\\ &\cdot\sin k_3z\cos k_3z.\end{aligned} \tag{26}$$

The other components are obtained by cyclic permutation of the variables.

Let q_α be a vector of length k directed to the observation point. Then, as a result of the integration we find the gravitational field at distance R:

$$\begin{aligned}\psi_{22}-\psi_{33}=&\frac{G}{16\pi c^4R}\frac{\sin q_1l_1\sin q_2l_2\sin q_3l_3}{q_1q_2q_3(k_1^2-q_1^2)(k_2^2-q_2^2)(k_3^2-q_3^2)}\\ &\times[A_2^2k_1^2k_3^2(k_2^2-2q_2^2)-A_3^2k_1^2k_2^2(k_3^2-2q_3^2)-B_2^2k_2^2(k_1^2-2q_1^2)(k_3^2-2q_3^2)\\ &+B_3^2k_3^2(k_1^2-2q_1^2)(k_2^2-2q_2^2)]\cos(2\omega t'-\mathbf{ql}),\end{aligned}$$

$$\begin{aligned}\psi_{23}=&\frac{G}{16\pi c^4R}\frac{k_2k_3\sin q_1l_1\sin q_2l_2\sin q_3l_3}{q_1(k_1^2-q_1^2)(k_2^2-q_2^2)(k_3^2-q_3^2)}\\ &\times[B_2B_3(k_1^2-2q_1^2)-A_2A_3k_1^2]\cos(2\omega t'-\mathbf{ql}).\end{aligned} \tag{27}$$

Here $\mathbf{q}\cdot\mathbf{l}=q_1l_1+q_2l_2+q_3l_3$. The other components of $\psi_{\alpha\beta}$ can be obtained by cyclic permutation of the variables.

The vanishing of any of the quantities in the denominator does not lead to difficulties, since the indeterminacy is easily lifted and yields the right result. If the dimensions of the system are large compared to $1/k$, one can see from (27) that the radiation pattern of the gravitational radiation consists of a large number of lobes.

We determine the energy flux in the direction of the x axis ($q_1=k$, $q_2=0$, $q_3=0$). In this direction $\psi_{23}\equiv 0$ and $\psi_{22}-\psi_{33}$ has the form

$$\psi_{22}-\psi_{33}=\frac{G}{16\pi c^4R}\frac{1}{k}l_2l_3\sin kl_1\left[2(A_2^2-A_3^2)+(A_1^2+A_2^2+A_3^2)\frac{k_2^2-k_3^2}{k_2^2+k_3^2}\times\left(1+\frac{k_2^2+k_3^2}{k^2}\right)\right]\cos(2\omega t'-kl_1).$$

We consider for simplicity the case $\mathbf{k}_1=0$ corresponding to the field in the cavity being independent of x. Then the square bracket in $\psi_{22}-\psi_{33}$ can be transformed to $64\pi\epsilon k^{-2}(k_2^2-k_3^2)$. The flux density averaged over a period has the form

$$\frac{dI}{R^2\,do}=\frac{1}{2\pi}\frac{G}{c^3R^2}\varepsilon^2\frac{(k_2^2-k_3^2)^2}{k^8}\sin^2 kl_1(l_2k)^2(l_3k)^2.$$

If k_2 and k_3 are not too close to one another and l_1 is selected so that $\sin kl_1=1$, we have, finally

$$\frac{dI}{R^2\,do}\approx\frac{G}{c^3R^2}\varepsilon^2\left(\frac{l_2}{\lambda}\right)^2\left(\frac{l_3}{\lambda}\right)^2\lambda^4. \tag{28}$$

We compare this formula with (4). In our case only part of the cavity, with dimension along the x axis of the order of $\lambda=\pi c/\omega$, radiates effectively. Since the source is not coherent, it obviously does not make sense to increase the size l_1 of the resonator in order to increase the output. It is more help to select it equal to $\lambda/2$. An increase of the flux can be achieved by means of a series of cavities situated one behind the other. In order for the cavities to function coherently the electromagnetic oscillations must be phase shifted by $\pi/2$. Then at the observation point the amplitudes of the gravitational waves add and the resulting amplitude is proportional to the number N of resonators, and the energy flux density is proportional to N^2.

As an illustration we consider a cavity with dimensions $l_1=1$ cm, $l_2\approx l_3=10^3$ cm with a standing wave of wavelength 4 cm ($kl_1=\pi/2$). We substitute these values into the Eq. (28) and set $\epsilon=10^{10}$ erg/cm^3. Then at the upper limit for $R=10^3$ cm we obtain a flux density from one cavity of the order of 10^{-13} erg/s-cm^2. As was already noted, strictly speaking, the cavity should not be rectangular but bent or consisting of a set of resonators of smaller sizes l_2 and l_3, but having the same total area l_2l_3, and arranged at the same distance from the reception point. A system of a thousand such resonant cavities with effective sizes $l_1=1$ cm, $l_2=l_3=10^2$ cm (i.e., a radiator of the same volume as the spherical cavity of radius $r_0=10^3$ cm) with the condition that the phases of the oscillations in all of them have been properly selected and are properly maintained, is capable of creating an energy flux of the order of 10^{-7} erg/s-cm^2 in a focal spot of area ~1 cm^2.

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[1)]The coherence idea has been used in the papers by Gertsenshteĭn [5] and Kopvillem [6]. In the problem, considered by Gertsenshteĭn, of emission of gravitational waves and a traveling electromagnetic wave in the presence of a static magnetic field, the coherence of the source is realized in a remarkable way automatically by the traveling wave itself.

[2)]The difficulties in defining the energy and momentum of the gravitational field by means of pseudotensors are well known (as well as by other "complexes" of energy-momentum). Usually one assumes that an acceptable "complex" must consist of the metric and its derivatives, must exhibit definite transformation properties and satisfy some "conservation laws." It is quite possible that no mathematical construction exists which satisfies all the enumerated requirements and to which we would be inclined unconditionally to attribute the meaning of a density of energy and momentum of a gravitational field. It should be stressed, however, that an "experimenter" can himself introduce a concept like the flux of gravitational energy, since the interaction of the receiving antenna with a gravitational wave can be described in terms of the components of the curvature tensor or in terms of ψ_{ik}, their amplitudes, frequencies etc. The result of this interaction in the wave zone, as expressed through the absorption of energy by some dissipative element will be the answer to our question. Therefore a rigorous reader may consider the quantities in the text which have dimension erg/s-cm^2 etc., as simply an economic notation for the properties of the field in the wave zone.

[3)]We neglect here the gravitational radiation of the currents which are concentrated in a thin surface layer.

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