Statistical properties of the energy spectrum of "gliding" electrons with mixed classical trajectories

G. M. Zaslavskiĭ and N. N. Filonenko

Physics Institute, Siberian Division, USSR Academy of Sciences
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An investigation is made of the statistical properties of the distribution of distances between energy levels in the quasiclassical approximation, for a finite system with mixed classical trajectories in phase space. The model is that of electrons "drifting" in a magnetic field along a periodically corrugated surface which is convex at all points in the region of motion of the electrons. Quantization conditions in the quasiclassical approximation are obtained. Estimates are presented for the probability of a given spacing ΔE between the levels. A Gaussian distribution is obtained for large values of ΔE; for small values of ΔE the probability is mainly a power function of ΔE. The exponent depends on the mixing properties of the trajectories.

INTRODUCTION

Particles that move with trajectory mixing in phase space have the following property: a small perturbation of the initial conditions leads to an exponential growth of the distance between phase trajectories with time. The mixing property leads to a decoupling of the time correlations of the physical quantities and to a possibility of describing the system statistically.[1] We assume now that the Hamiltonian of the particle and the boundary conditions ("walls") determine in the classical case particle motion with mixing. What are the properties possessed by the energy spectrum of the particle in the quantum case?

This question is part of the more general problem of obtaining the connection between quasiclassical quantization rules and the properties of classical particle trajectories. The quasiclassical Bohr-Sommerfeld quantization rules are usually defined in terms of the actions of the particle (see, e.g.,[2]). The action variables are at the same time integrals of the motion and describe conditionally-periodic motion of the particle. An essential property of such particle trajectories is their stability. Recently, however, the question has been raised concerning the form that the quasiclassical quantization rules should assume in the case of unstable classical trajectories[3,4], which can occur in different physical problems (and, in particular, not only in problems dealing with a particle in the field of scattering centers, but also in problems dealing with oscillations of resonators with boundaries of complex shape, where a beam trajectory is considered rather than particle trajectories). Some of the difficulties of quantizing unstable trajectories are discussed in[4]. In the case when the classical trajectories are mixed in phase space, some of the integrals of motion decay and the corresponding action variables can no longer be quantized. The present paper is devoted to the construction of quasiclassical quantization rules for stochastically unstable (mixing) classical trajectories of a particle and to an explanation of the properties of the energy spectrum in the quasiclassical region. The investigation is carried out for a model in which the electrons "hop" along a periodically corrugated surface in an external magnetic field.

We note that simpler variants of the model of "hopping" electrons were considered[5-7] in connection with the problem of electronic surface levels in a metal. The effects obtained in the present paper are not realistically observable in metals at present. The selected model, however, not only contains the most characteristic properties of the problem posed above, but is apparently one of the simplest, and therein lies the utility of its study.

We consider a surface with periodic inhomogeneity, having a definite negative curvature. The classical motion of the electrons, if certain conditions are satisfied, is motion with mixing. This causes the action of the particle, which determines the energy spectrum in the quasiclassical approximation, to be a random function of the coordinate. Thus, when the mixing conditions are satisfied, the electron energy-level distribution becomes stochastic. We calculate in this paper different characteristics of this distribution. The most interesting of them is a quantity connected with the probability of a given distance ΔE between neighboring levels. It turns out that as $\Delta E \to 0$ the sought probability tends to zero like a certain power of ΔE. A distribution of this type was introduced for excited levels of heavy nuclei by Dyson[8] in the form of a hypothesis, in order to satisfy the property of "pushing asunder" of the levels[9]. We shall show here that the character of this "pushing asunder" is determined by the properties of the mixing motion of the classical trajectories. For the models considered, it is possible to obtain directly not only a distribution of the Dyson type, but also to determine the limits between which it is valid. At large values of the distance ΔE between nearest levels, the probability of a given ΔE has a Gaussian form. Thus, the distribution of the distances between levels is closer in shape to a Wigner distribution[8].

1. INVESTIGATION OF CLASSICAL ELECTRON TRAJECTORIES

We consider the motion of electrons in a plane (x, y) and in a magnetic field $\mathcal{H}$ directed along z; the motion is bounded by a surface $\mathcal{P}(x, y) = 0$ (Fig. 1). The following is assumed concerning the curve that determines the shape of the boundary from the equation $\mathcal{P}(x, y) = 0$:

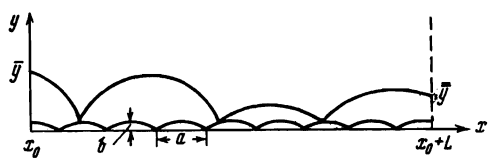

FIG. 1

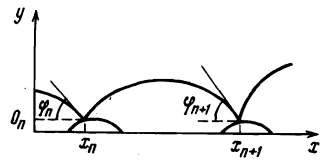

FIG. 2

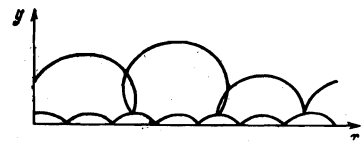

FIG. 3

the boundary $y = y(x)$ is a periodic function of x with period a and with amplitude b, and is always convex into the region of electron motion. The reflections of the electron from the boundary are absolutely elastic.

The Hamiltonian of the electron

$$H=\frac{1}{2m}\left(p_x+\frac{e}{c}\mathcal{H}y\right)^2+\frac{1}{2m}p_y^2 \tag{1.1}$$

with generalized momenta p_x and p_y determines the equation for the wave function[2]

$$\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+i\hbar\Omega y\frac{\partial\psi}{\partial x}+\left[E-\frac{m\Omega^2y^2}{2}\right]\psi=0, \tag{1.2}$$
$$\Omega\equiv e\mathcal{H}/mc.$$

The wave function satisfies the following boundary conditions:

$$\psi|_{\mathcal{S}(x,\,y)=0}=0, \tag{1.3}$$

$$\psi(x_0,y)=\psi(x_0+L,y). \tag{1.4}$$

The first of them corresponds to the law of elastic reflection from the boundary, and the second is the usual condition of periodicity with respect to x and makes it possible to simplify further the problem of determining the eigenvalues. We shall consider later on $L\to\infty$, and regard L and a as not commensurate. This makes the problem of determining the eigenfunctions and eigenvalues strongly dependent on the phase of the periodic boundary to which the position of the point x_0 corresponds (Fig. 1). We shall say that different choices of the position of the point x_0 correspond to different configurations of the boundary.

To solve the problem (1.2)—(1.4), we use the quasiclassical approximation and deal only with the behavior of the spectrum in the region of large values of E. To obtain the quasiclassical wave functions we investigate first the classical motion of the electron.

Let x_n by the coordinate and φ_n the angle between the x axis and the direction of motion of the electrons at the instant of the n-th collision with the boundary (Fig. 2). We assume that $b/a=\epsilon\ll 1$. We write down the equation of motion of the electron with allowance for the fact that ϵ is small, in the form

$$x_{n+1}=x_n+2R\sin\varphi_{n+1},\quad \varphi_{n+1}=\varphi_n+\varepsilon\chi(\xi_n),$$
$$\xi\equiv\left\{\frac{x}{a}\right\},\quad R=\left[\frac{2E}{m\Omega^2}\right]^{1/2}. \tag{1.5}$$

here $\chi(\xi)$ is determined by the shape of the boundary ($\max\chi\sim 1$) and the curly brackets in the definition of ξ denote the fractional part of the argument. We rewrite the transformation (1.5) in the form

$$\xi_{n+1}=\{\xi_n+2Ra^{-1}[1-(p_x^{(n+1)}/m\Omega R)^2]^{1/2}\},$$
$$p_x^{(n+1)}=m\Omega R\cos[\varphi_n+\varepsilon\chi(\xi_n)] \tag{1.6}$$
$$\approx p_x^{(n)}-\varepsilon m\Omega R\chi(\xi_n)[1-(p_x^{(n)}/m\Omega R)^2]^{1/2}-\tfrac{1}{2}\varepsilon^2\chi^2(\xi_n)p_x^{(n)},$$

where $p_x^{(n)}$ is the component of the electron momentum p_x after the n-th step. A transformation of the type (1.5), (1.6) was investigated in[10] (see also[11]), and its properties are determined by the "stretching" parameter K:

$$K=\frac{d\xi_{n+1}}{d\xi_n}\sim 1+2\frac{R}{a}\varepsilon\chi'\cos\varphi, \tag{1.7}$$

where the prime denotes differentiation with respect to the argument. At $|K-1|\ll 1$, i.e., at sufficiently small R, and consequently at low energies E, the motion of the electron is stable and conditionally periodic. When $K\gg 1$, to the contrary, the electron trajectories have the property of mixing in phase space. The mixing process is characterized in this case by rapid establishment of uniform distribution with respect to the phase ξ and by slow diffusion relaxation with respect to the variable φ (or p_x).

The electron energy region for which the trajectories become mixed can be obtained from the condition $K\gg 1$. This yields

$$E\gg m\Omega^2a^2/2(2\varepsilon\chi')^2\equiv E_0. \tag{1.8}$$

It is seen from (1.7) that the condition $K\gg 1$ is not satisfied if φ falls in the interval $(\pi/2-\Delta\varphi,\ \pi/2)$, where

$$\Delta\varphi=a/2R\varepsilon\chi'\sim 1/K\ll 1. \tag{1.9}$$

The condition (1.9) defines small "stability islands" of the trajectories in the electron phase space[11]. Stochastic motion of the electron takes place mainly in the phase-space region outside the stability islands. Since the change of the angle φ in each step, according to (1.5), is equal to $\epsilon\chi$, the condition

$$\varepsilon\chi\ll\Delta\varphi\sim 1/K \tag{1.10}$$

means that the range of angles $(\pi/2-\Delta\varphi,\ \pi/2)$ cannot be traversed by the electron in a single collision. In this case the electron motion takes place with the angles φ lying in the region $(0,\ \pi/2-\Delta\varphi)$, and the upper limit of this region is equivalent to the presence of a reflecting wall in phase space[10]. The inequality (1.10) means a stronger limitation than (1.8) imposed on the energy region:

$$E\gg E_0/\varepsilon^2\chi^2. \tag{1.11}$$

We shall investigate from now on the motion of an electron in the energy region satisfying condition (1.11). In addition to the fact that most electron trajectories in phase space correspond to stochastic motion, the inequality (1.11) singles out a class of trajectories having still another attribute, which can be readily understood from Fig. 3. If φ can be larger than $\pi/2$, then the trajectories can "glide" and self-intersection of the trajectories in configuration space sets in. Thus, the condition (1.11) means selection of the class of trajectories corresponding to the so-called "glancing" electrons (Fig. 2). Accurate to small "penetrations" of the values of φ into the region $(\pi/2-\Delta\varphi,\ \pi/2)$, the property wherein a trajectory is glancing is an invariant of the motion, so that these trajectories can be treated separately from the others.

We now describe the process of slow diffusion relaxation with respect to the angle φ. We note first that

the quantity $\chi(\xi)$ is proportional to the factor sign $(1/2 - \xi)$. As a result of the fast mixing with respect to the variable ξ, a nearly uniform distribution with respect to ξ is established[11], and averaging over this distribution yields

$$\overline{\chi(\xi)} = 0, \quad \overline{\chi^2(\xi)} = \chi_0^2, \tag{1.12}$$

where χ_0 is of the order of unity. The process of slow relaxation with respect to the angle φ or, equivalently, with respect to the momentum $p_x = R\cos\varphi$, will be described by a distribution function $w(p_x, x)$ satisfying the normalization condition

$$\int_0^{\max p_x} w(p_x, x)\, dp_x = 1.$$

We can write for the distribution $w(p_x, x)$ an equation of the Fokker-Planck type

$$\frac{\partial w}{\partial x} = -\frac{\partial}{\partial p_x}\left[\left(\frac{\overline{\Delta p_x}}{d}\right)w\right] + \frac{1}{2}\frac{\partial^2}{\partial p_x^2}\left[\left(\frac{\overline{(\Delta p_x)^2}}{d}\right)w\right], \tag{1.13}$$

where Δp_x is the change of the momentum p_x in one step, d is the length of the step, and the bar denotes averaging over ξ. From (1.6) we obtain, accurate to terms $\sim\epsilon^2$,

$$d = 2R\sin\varphi = 2R\mathcal{R}, \quad \mathcal{R} = \sqrt{1 - (p_x/m\Omega R)^2},$$
$$\Delta p_x = -\varepsilon m\Omega R\chi\mathcal{R} - {}^1/_2\varepsilon^2\chi^2 p_x,$$

whence

$$\left(\frac{\overline{\Delta p_x}}{d}\right) = -\frac{\varepsilon^2\chi_0^2 p_x}{4R\mathcal{R}}, \quad \left(\frac{\overline{(\Delta p_x)^2}}{d}\right) = (m\Omega R\varepsilon\chi)^2\frac{\mathcal{R}}{2R},$$

and Eq. (1.13) takes the form

$$\frac{\partial w}{dx} = \frac{1}{4R}(m\Omega R\varepsilon\chi)^2\frac{\partial}{\partial p_x}\left(\mathcal{R}\frac{\partial w}{\partial p_x}\right). \tag{1.14}$$

The stationary solution of (1.14), satisfying the condition that there be no electron flow at $\varphi = 0$ and $\varphi = \pi/2 - \Delta\varphi$, takes the simple form

$$w(p_x) = \text{const} = \frac{1}{m\Omega R(1-\Delta\varphi)} \approx \frac{1+\Delta\varphi}{m\Omega R},$$
$$w(\varphi) = w(p_x)\frac{dp_x}{d\varphi} \approx (1+\Delta\varphi)\sin\varphi. \tag{1.15}$$

It follows also from (1.14) that the characteristic length l_D over which the equilibrium distribution (1.15) is established is equal to

$$l_D \sim R/\varepsilon^2\chi_0^2 \sim R/\varepsilon^2 = \varepsilon^{-2}[2E/m\Omega^2]^{1/3}. \tag{1.16}$$

The condition $L \to \infty$ will henceforth mean $L \gg l_D$.

2. QUANTIZATION CONDITIONS

To derive the quantization conditions we use a quasiclassical representation of the wave function. Let (x, y) be a certain spatial point through which the trajectory of a "glancing" electron of energy E can pass, and let $x_n \le x \le x_{n+1}$ (see Fig. 2a). We express the wave function in this interval in the form[12,13]

$$\psi_n(x,y) = \Pi_n \exp\left\{\frac{i}{\hbar}S_x^{(n)} - \frac{1}{2}i\pi\theta(x - x(p_y^{(n)} = 0))\right\}\cdot$$
$$\times\left[A_n\exp\left(\frac{i}{\hbar}S_y^{(n)}\right) + B_n\exp\left(-\frac{i}{\hbar}S_y^{(n)}\right)\right], \tag{2.1}$$

$(n \ne 0,\ x_n \le x \le x_{n+1})$, where

$$S_x^{(n)} = p_x^{(n)}(x - x_n), \quad S_y^{(n)} = \int_{O_n}^{y} p_y^{(n)}(y')\,dy',$$
$$\Pi_n = |p_y^{(n)}(y)/p_y^{(n)}(x_n + 0)|^{-1/2}, \tag{2.2}$$
$$\theta(\alpha) = \begin{cases} 0, & \alpha < 0 \\ 1, & \alpha > 0 \end{cases}.$$

The momenta $p_x^{(n)}$ and $p_y^{(n)}$ are taken on the particle trajectory at the corresponding point, and the point O_n corresponds to the value of y on the boundary at the point x_n from which the particle trajectory emerges. The wave function in (2.1) is a superposition, each term of which corresponds to one of two possible rays. The wave function in the form (2.1) already takes into account the tangency of the caustic ray at the maximum value of y on the turn[13] (the momentum p_y vanishes at this point). Thus, the wave function satisfies the boundary condition $\psi \to 0$ as $y \to \infty$ in the interval $x_n \le x \le x_{n+1}$. This is attained by introducing the step function θ in (2.1), where $x(p_y^{(n)} = 0)$ is the coordinate along the x axis of the ray at the caustic tangency point. A solution in the form (2.1), (2.2) actually corresponds to the case of separable variables in the interval (x_n, x_{n+1}).

In the interval (x_0, x_1) where the trajectory generally speaking does not make a complete turn, the wave function $\psi_0(x, y)$ differs somewhat from (2.1) (see Fig. 1):

$$\psi_0(x,y) = \Pi_0\exp\left\{\frac{i}{\hbar}S_x^{(0)} - \frac{1}{2}i\pi\theta(x - x(p_y^{(0)} = 0))\right\}$$
$$\times\left[A_0\exp\left(\frac{i}{\hbar}S_y^{(0)}\right) + B_0\exp\left(-\frac{i}{\hbar}S_y^{(0)}\right)\right]; \tag{2.3}$$

$$S_x^{(0)} = p_x^{(0)}(x - x_0), \quad S_y^{(0)} = \int_{\bar y}^{y} p_y^{(0)}(y')\,dy',$$
$$\Pi_0 = |p_y^{(0)}(y)/p_y^{(0)}(\bar y)|^{-1/2}. \tag{2.4}$$

Here $\bar y \equiv y(x_0)$ is the initial y coordinate of the particle trajectory. From the Schrödinger equation (1.2) we get $\psi^*(x) = \psi(-x)$, whence

$$A_0 = B_0^* = |A_0|e^{i\vartheta_0}. \tag{2.5}$$

Here $A_0 = A_0(\bar y)$ determines the wave function on the line $x = x_0$:

$$\psi(x_0, \bar y) = 2|A_0(\bar y)|\cos\vartheta_0(\bar y) \tag{2.6}$$

for a trajectory emerging from the point $(x_0, \bar y)$.

We write down an expression analogous to (2.1) for ψ_{n+1} in the interval (x_{n+1}, x_{n+2}) and stipulate that the wave function vanish at the point x_{n+1} in accordance with the condition (1.3):

$$\psi_n(x_{n+1} - 0, O_{n+1}) + \psi_{n+1}(x_{n+1} + 0, O_{n+1}) = 0. \tag{2.7}$$

Near the point (x_{n+1}, O_{n+1}) the wave function can always be represented in the form of a superposition of plane waves, so that (2.7) results in the following recurrence relations

$$A_{n+1} = A_n\Pi_n|_{x_{n+1}-0}\exp\left\{\frac{i\pi}{2} + \frac{i}{\hbar}(\Delta S_x^{(n)} + \Delta S_y^{(n)})\right\},$$
$$B_{n+1} = B_n\Pi_n|_{x_{n+1}-0}\exp\left\{\frac{i\pi}{2} + \frac{i}{\hbar}(\Delta S_x^{(n)} - \Delta S_y^{(n)})\right\}$$
$$\Delta S_x^{(n)} = p_x^{(n)}(x_{n+1} - x_n), \tag{2.8}$$
$$\Delta S_y^{(n)} = \oint_{O_n}^{O_{n+1}} p_y^{(n)}(y')\,dy' \quad (n \ne 0),$$
$$\Delta S_y^{(0)} = \oint_{\bar y}^{O_1} p_y^{(0)}(y')\,dy'.$$

The value $\Pi_n|_{x_{n+1}-0}$ is taken here at the final point $(x_{n+1} - 0, O_{n+1})$ of the n-th turn. The circle on the integral in (2.8) denotes that the integration with respect to y′ is performed along the particle trajectory.

Let $y = y(x_0 + L)$ be the y-coordinate of the point of

the trajectory at $x = x_0 + L$, and let the number of the corresponding turn be N. We write down the wave function $\psi_N(x_0 + L, y)$, using formulas (2.3)–(2.6) and the recurrence relations (2.8):

$$\psi_N(x_0+L,\bar{y})=2Q|A_0(\bar{y})|\exp\left\{\frac{i}{\hbar}\sum_{n=0}^{N}\Delta S_x^{(n)}+\frac{iN\pi}{2}\right\}$$

$$\times\cos\left(\sum_{n=0}^{N}\Delta S_y^{(n)}+\vartheta_0(\bar{y})\right), \qquad (2.9)$$

where

$$Q=\Pi_0|_{x_1=0}\dots\Pi_{N-1}|_{x_N=0}\cdot\Pi_N|_{x_0+L},$$

$$\Delta S_x^{(N)}=p_x^{(N)}(x_0+L-x_N),\quad \Delta S_y^{(N)}=\oint p_y^{(N)}(y')\,dy'.$$

We stipulate satisfaction of the boundary condition (1.4), which in the notation of the present section takes the form

$$\psi_0(x_0,\bar{\bar{y}})=\psi_N(x_0+L,\bar{y}). \qquad (2.10)$$

An expression for $\psi_0(x_0, \bar{y})$ is obtained from (2.6) by replacing $\bar{y}$ by $\bar{\bar{y}}$. Then, taking (2.6) and (2.9) into account, we obtain from (2.10)

$$|A_0(\bar{\bar{y}})|\cos\vartheta_0(\bar{\bar{y}})=Q|A_0(\bar{y})|$$

$$\times\cos\left[\sum_{n=0}^{N}\Delta S_y^{(n)}+\vartheta_0(\bar{y})\right]\exp\left\{\frac{i}{\hbar}\sum_{n=0}^{N}\Delta S_x^{(n)}+i\frac{\pi}{2}N\right\}. \qquad (2.11)$$

From (2.11) follow the quantization conditions

$$\frac{1}{\hbar}\sum_{n=0}^{N}\Delta S_x^{(n)}=\pi\left(M_x+\frac{1}{2}\right),$$
$$\frac{1}{\hbar}\sum_{n=0}^{N}\Delta S_y^{(n)}+\vartheta_0(\bar{y})\pm\delta=\pi M_y \qquad (2.12)$$

and

$$\cos\delta=\left|\frac{\cos\vartheta_0(\bar{\bar{y}})}{Q}\frac{A_0(\bar{\bar{y}})}{A_0(\bar{y})}\right|\leq 1, \qquad (2.13)$$

where M_x and M_y are certain positive numbers, and the inequality (2.13) should be regarded as a condition supplementing the preceding two equalities.

We note that the quantity $\vartheta_0(\bar{y}) \pm \delta$ in (2.12) can be neglected since $\vartheta_0(\bar{y}) \pm \delta \leq 4\pi$, and the phase advances due to the change in the action are much larger than unity[1)]. The inequality (2.13) can be simplified by recognizing that the quantity Q is self-averaging. It can be shown that

$$Q=|p_y^{(0)}(\bar{y})/p_y^{(N)}(\bar{\bar{y}})|^{1/2}[1+O(N^{-1/2})]. \qquad (2.14)$$

Before we proceed to investigate the conditions (2.12) and (2.13), let us stop to discuss certain essential aspects of the considered boundary-value problem.

In accordance with the equations of motion (1.5) and (1.6), the parameters x_0, y, p_{x_0}, and E determine uniquely the entire particle trajectory, and consequently, the quantities $S_x^{(n)}$, $S_y^{(n)}$, and Π_n, in terms of which the wave function is expressed. In particular, the final coordinate $\bar{\bar{y}}$ is a definite function of $\bar{y}$. The sequence $[p_x^{(n)}]$, $[p_y^{(n)}]$ is random under the condition (1.8). Therefore the sequence of the phases of $[S_x^{(n)}]$ and $[S_y^{(n)}]$ is also random. This leads to the conclusion that our problem is analogous to the problem of the properties of disordered systems, i.e., systems with random potentials. A feature of the considered model is that all the interactions introduced into the problem (external field and walls) are regular (non-random), and the randomness is the consequence of a definite type of instability in the problem. A simple representation of the resultant situation can be obtained by starting from the Feynman representation of the wave function as a functional of all possible particle paths. In the quasiclassical approximation, the integration along the paths reduces to an integrand determined by the extremal (saddle-point) trajectory and by virtual trajectories in a small vicinity of the extremal one. Since the saddle-point trajectory is a real trajectory of a classical particle, defined by equations (1.5) and (1.6), the mixing property of the latter becomes immediately manifest in the randomness of the phase of the wave function.

We put

$$S_x(x_0,L|E,p_{x_0}(\bar{y}))=\sum_{n=0}^{N-1}p_x^{(n)}(x_{n+1}-x_n)+p_x^{(N)}(x_0+L-x_N),$$
$$S_y(x_0,L|E,p_{x_0}(\bar{y}))=\sum_{n=0}^{N-1}\oint_{o_n}^{o_{n+1}}p_y^{(n)}(y')\,dy'+\oint_{o_N}^{\bar{\bar{y}}}p_y^{(N)}(y')\,dy'. \qquad (2.15)$$

The quantities S_x and S_y determine respectively the x and y components of the action. The number of steps N is a function of the same variables as S_x and S_y. Rewriting (2.12) with allowance for (2.15) and (2.8), we obtain

$$\hbar^{-1}S_x\approx\pi M_x,\quad \hbar^{-1}S_y\approx\pi M_y. \qquad (2.16)$$

In spite of their simple form, the obtained necessary quantization conditions call for a definite discussion.

The system (2.16) defines two two-parameter families of quantities:

$$E=E(M_x,M_y),\quad p_{x_0}=p_{x_0}(\bar{y},M_x,M_y). \qquad (2.17)$$

In the case of a plane boundary, the quantization conditions take the form

$$E=E(M_x,M_y),\quad p_{x_0}=p_x=p_x(M_x),$$

where the momentum p_x is an integral of the motion ($p_x = -m\Omega Y$, where Y is the distance from the center of the circle of the glancing electron to the plane x = y, i.e., to the boundary). In this case, the integral of motion p_x decays because of the stochastic instability. The second relation in (2.17) then expresses the connection between the initial value of the momentum p_{x_0} and the initial coordinate $\bar{y}$ on the trajectory corresponding to the proper wave function with quantum numbers M_x and M_y. Just as in the plane case, the set of eigenfunctions and eigenvalues turns out to be of the two-parameter type, although the two parameters now define one integral of motion and not two. Since there is no degeneracy (the degeneracy is lifted in the magnetic field even in the case of a plane boundary), different pairs of the numbers M_x and M_y correspond to different values of E and $p_{x_0}(y)$.

3. STATISTICAL CHARACTERISTICS OF THE ENERGY LEVEL DISTRIBUTION

The conditions (2.16), which quantize the energy spectrum of the particle, contain in the left-hand side the random quantities S_x and S_y. This causes the distribution of the energy levels to constitute a certain random sequence of the energy values. As already noted in Sec. 1, the position of the point x_0 (and hence also the point $x_0 + L$) with respect to the phase of the boundary corrugation determines the concrete configuration of the system. The spectrum corresponding to a given configuration will be called the representative of

the statistical ensemble of the levels of the problem, or realization. The entire ensemble is made up of representatives (realizations) corresponding to all possible configurations. The fact that the introduced ensemble is statistical follows immediately from the properties of the particle trajectory mixing, since the change in the position of the point x_0 leads to a perturbation of the initial conditions in Eqs. (1.5) and to stochastic instability of the values of S_x and S_y.

The main task of the present section is to estimate a quantity connected with the probability that two levels in one realization, with energies E and E + ΔE, are neighbors. Namely, let the level with energy E be the eigenvalue of a certain realization, (i.e., the eigenvalue of these realizations falls in the interval (E, E + dE)). Let us find the probability $P(E|\Delta E)$ that there are no other energy eigenvalues for the same realizations) in the (E, E + ΔE) interval. We consider two asymptotic forms of $P(E|\Delta E)$: 1) $\Delta E \to 0$, 2) ΔE large enough.

Let the pair of numbers M_x and M_y correspond to a pair of solutions E and p_{x_0} of the system (2.17), and let another pair M_x' and M_y' correspond to the pair of solutions E′ and p'_{x_0}. We assume that $E' - E \geq \Delta E$ that E′ is upper level closest to E. From (2.16) we have

$$\begin{aligned}\left(\frac{\delta S_x}{\delta E}\right)_{p_{x_0}}\Delta E+\left(\frac{\delta S_x}{\delta p_{x_0}}\right)_E\Delta p_{x_0}=\pi\hbar\mu_x,\\ \left(\frac{\delta S_y}{\delta E}\right)_{p_{x_0}}\Delta E+\left(\frac{\delta S_y}{\delta p_{x_0}}\right)_E\Delta p_{x_0}=\pi\hbar\mu_y,\end{aligned} \tag{3.1}$$

where μ_x and μ_y are certain integers that do not vanish simultaneously, $\Delta p_{x_0} = p'_{x_0} - p_{x_0}$, and the subscripts at the parentheses indicate the corresponding partial derivatives. From the definitions of S_x and S_y it follows that in the statistical sense they are equivalent. This means that they satisfy identical distribution laws with moments of the same order of magnitude.

We determine ΔE from (3.1):

$$\Delta E=\pi\hbar\frac{\mu_x(\delta S_y/\delta p_{x_0})_E-\mu_y(\delta S_x/\delta p_{x_0})_E}{(\delta S_x/\delta E)_{p_{x_0}}(\delta S_y/\delta p_{x_0})_E-(\delta S_y/\delta E)_{p_{x_0}}(\delta S_x/\delta p_{x_0})_E}. \tag{3.2}$$

From the definitions of S_x and S_y it also follows that

$$\left(\frac{\delta S_x}{\delta p_{x_0}}\right)_E\sim\left(\frac{\delta S_y}{\delta p_{x_0}}\right)_E,\quad \left(\frac{\delta S_x}{\delta E}\right)_{p_{x_0}}\sim\left(\frac{\delta S_y}{\delta E}\right)_{p_{x_0}}.$$

Hence

$$\Delta E\sim\pi\hbar\mu/(\delta S_x/\delta E)_{p_{x_0}}, \tag{3.3}$$

where $\mu \neq 0$. As $\Delta E \to 0$ this means that a small change of the parameter E by an amount ΔE should lead to a large change of the action S_x (or S_y). Only in this case can the derivative $(\delta S_x/\delta E)_{p_{x_0}}$ become large, satisfy the condition (3.3), and by the same token ensure the appearance of a level near the energy E + ΔE. It is now obvious that for very small perturbations of the parameter E the condition for strong instability of the function $S_x(x_0 L|E, p_{x_0})$ should indeed determine the sought probability.

We consider the first equation of (1.6) for a fixed configuration with fixed p_{x_0}. A change in the parameter E (or R) by an amount ΔE leads to a change of the initial condition

$$\Delta\xi\sim\Delta E/[m\Omega^2a^2E/2]^{1/2}. \tag{3.4}$$

Since the system (1.6) describes motion with mixing, the initial difference Δξ between the trajectories with energy E and E + ΔE increases with each step of the transformation. The number of steps N_0 after which the difference between the phases ξ of the trajectories becomes of the order of unity is determined from the condition $KN_0\Delta\xi \sim 1$, whence

$$N_0\sim\ln(1/\Delta\xi)/\ln K. \tag{3.5}$$

After N_0 steps, the correlations of the phases of the trajectories with energies E and E + ΔE become uncoupled, and subsequently the trajectories become statistically independent. This means that regardless of the value of $S_x(x_0, L|E, p_{x_0})$ there exists a nonzero probability that $S_x(x_0, L|E + \Delta E, p_{x_0})$ has an arbitrary prescribed value (in the interval dS_x), provided that the segment of path traversed after N_0 steps is smaller than L. Hence the probability $P(l(N_0) > L; E, p_{x_0})$ that a path $l(N_0) > L$ will be traversed after N_0 steps is indeed the probability that if E is the eigenvalue of a given configuration, then the distance to the next higher eigenvalue in the same configuration is not smaller than ΔE. Indeed, if $l(N_0) > L$, then the number of steps subtended by L is smaller than N_0, the instability of S_x does not have time to develop within the length L, and S_x does not experience the change necessary for a new state to be produced. To the contrary, the quantity

$$P(l(N_0)<L;E,p_{x_0})=1-P(l(N_0)>L;E,p_{x_0})$$

determines the probability that if E is an eigenvalue of a given configuration, then at least one other eigenvalue falls in the interval (E, E + ΔE).

It is useful next to bear in mind the following two circumstances: 1) if the probability $P(l(N_0) < L; E, p_{x_0})$ is very small, then it determines a quantity close to the probability for the appearance of a single level in the interval (E, E + ΔE), since the probability of the appearance of more than one level can be neglected; 2) if furthermore $P(l(N_0) < L; E, p_{x_0})$ decreases sufficiently rapidly as $\Delta E \to 0$, then it determines a quantity that is close to the probability of the appearance of the next higher level to E in a narrow region close to E + ΔE.

We proceed to calculate $P(l(N_0) < L; E, p_{x_0})$. Let l_n be the length of the particle path along the x axis after n steps. From (1.5) it follows that

$$l_{n+1}=l_n+d_n,\quad d_n=2R\sin\varphi_n.$$

Using (1.15) for lengths $l > l_D$ (see (1.16)), we have (neglecting the quantity $\Delta\varphi \ll 1$)

$$\begin{aligned}\langle\Delta l\rangle=\langle d\rangle=2R\int_0^{\pi/2}\sin\varphi\,w(\varphi)\,d\varphi=\pi R/2,\\ \langle(\Delta l)^2\rangle=\langle d^2\rangle={}^8/_3R^2.\end{aligned} \tag{3.6}$$

From (3.6) follows an expression for the probability density $\Phi(l, n)$ that the path segment traversed along the x axis by the particle after n steps lies in the interval (l, l + dl):

$$\Phi(l,n)=\left(\frac{8}{3}\pi R^2n\right)^{-1/2}\exp\left\{-\frac{(l-\pi Rn/2)^2}{16\pi^2n/3}\right\}. \tag{3.7}$$

Hence

$$P(l(N_0)<L;E,p_{x_0})=\int_0^L dl\,\Phi(l,N_0). \tag{3.8}$$

We are interested in the asymptotic behavior as ΔE $\to$ 0, i.e., $N_0 \to \infty$. If $N_0\langle d\rangle \gg L$, then we obtain from (3.6)–(3.8) the sought probability:

$$P(E|\Delta E)=P(l(N_0)<L;E,p_{x_0})\approx\frac{L}{(\pi\langle d^2\rangle N_0/2)^{1/2}}\exp\left\{-\frac{N_0}{2}\frac{\langle d\rangle^2}{\langle d^2\rangle}\right\}$$

or after substituting (3.5):

$$P(E|\Delta E)\approx L\left(\frac{3m^2\Omega^2}{16\pi E}\ln K\right)^{1/2}\times\left[\frac{\Delta E}{(m\Omega^2a^2E/2)^{1/2}}\right]^{1/2\ln K}\left[\ln\frac{\Delta E}{(m\Omega^2a^2E/2)^{1/2}}\right]^{-1/2},\qquad(3.9)$$

$$K=\frac{2}{a}R\varepsilon\chi'=\frac{2e\chi'}{a\Omega}\left(\frac{2E}{m}\right)^{1/2}.$$

We have obtained for the probability an expression of the Dyson type[8], namely, as $\Delta E\to 0$ the probability of appearance of at least one level in the interval (E, E + ΔE) tends to zero mainly in proportion to some power of ΔE. This power is determined essentially by the mixing parameter K. In general, as seen from the entire derivation of (3.9), the power law in $P(E|\Delta E)$ is due to the property of mixing of the classical trajectories. The formula (3.9) expresses the appearance of "pushing asunder" of the levels when they come closer together[2)].

We now consider another limiting case, when

$$N_0\ll L/\langle d\rangle\equiv\langle N\rangle,\qquad(3.10)$$

i.e., at sufficiently large ΔE. After a small number ($\sim N_0$) of the first steps, the trajectories corresponding to the eigenvalues E and E + ΔE become statistically independent. The probability that within the length L of the fixed configuration the quantity $S_X(x_0, L|E, p_{X_0})$ falls in the interval $(\pi\hbar M_X, \pi\hbar M_X + dS_X)$ is equal to

$$P(E|S_x)=\left[\frac{\pi}{2}L\left\langle\frac{(\Delta S_x)^2}{d}\right\rangle_E\right]^{-1/2}\times\exp\left\{-\left(S_x-\left\langle\frac{\Delta S_x}{d}\right\rangle_E L\right)^2\Big/2L\left\langle\frac{(\Delta S_x)^2}{d}\right\rangle_E\right\}dS_x,\qquad(3.11)$$

where ΔS_X and d are given by

$$\Delta S_x=p_xd=m\Omega Rd\cos\varphi,\quad d=2R\sin\varphi,\qquad(3.12)$$

and the averaging is carried out with the aid of the distribution function $w(\varphi)$. The subscript E at the averaging brackets shows that all the parameters that depend on the energy are taken at the value E. In analogy with (3.11) we can write

$$P(E+\Delta E|S_x')=\left[\frac{\pi}{2}L\left\langle\frac{(\Delta S_x')^2}{d}\right\rangle_{E+\Delta E}\right]^{-1/2}\times\exp\left\{-\left(S_x'-L\left\langle\frac{\Delta S_x'}{d}\right\rangle_{E+\Delta E}\right)^2\left[2L\left\langle\frac{(\Delta S_x')^2}{d}\right\rangle_{E+\Delta E}\right]^{-1}\right\}dS_x'.\qquad(3.13)$$

The probability that the difference $S_X' - S_X$ assumes a given value $S_X' - S_X = \pi\hbar\mu_X$ independently of the value of S_X is

$$P(E|E+\Delta E;\mu_x)=\int_0^\infty dS_x\,P(E|S_x)P(E+\Delta E|S_x+\pi\hbar\mu_x)=\left(\frac{L}{2\pi D_x}\right)^{1/2}\exp\left\{-\frac{L}{2D_x}\left[\left\langle\frac{\Delta S_x}{d}\right\rangle_{E+\Delta E}-\left\langle\frac{\Delta S_x}{d}\right\rangle_E-\frac{\pi\hbar\mu_x}{L}\right]^2\right\},$$

$$D_x=\frac{1}{2}\left[\left\langle\frac{(\Delta S_x)^2}{d}\right\rangle_{E+\Delta E}+\left\langle\frac{(\Delta S_x)^2}{d}\right\rangle_E\right].\qquad(3.14)$$

Finally, the probability that the level will appear at a distance ΔE in a realization with an eigenvalue lying in the interval (E, E + dE) is determined at fixed μ_X and μ_Y in terms of expression (3.14) in the following manner:

$$P(E|E+\Delta E;\mu_x,\mu_y)=P(E|E+\Delta E;\mu_x)P(E|E+\Delta E;\mu_y)\qquad(3.15)$$

where $P(E|E+\Delta E;\mu_Y)$ is determined with the aid of expressions analogous to (3.11)–(3.14). If the quantities $\bar\mu_X$ and $\bar\mu_Y$ characterize certain mean values of the quantities μ_X and μ_Y for two neighboring levels with energies E and E + ΔE, and if the inequality

$$\left|\left\langle\frac{\Delta S_{x,y}}{d}\right\rangle_{E+\Delta E}-\left\langle\frac{\Delta S_{x,y}}{d}\right\rangle_E\right|\gg\frac{\pi\hbar}{L}\bar\mu_{x,y}$$

is satisfied, then at large values of ΔE we obtain from (3.15) the following estimate of the probability $P(E|E+\Delta E)$ of a specified distance between two neighboring levels E and E + ΔE:

$$P(E|E+\Delta E)\sim P(E|E+\Delta E;\mu_x=0,\mu_y=0)\approx\frac{L}{2\pi(D_xD_y)^{1/2}}\exp\left\{-\gamma\frac{L}{R}\left(\frac{\Delta E}{E}\right)^2\right\}\qquad(3.16)$$

(γ is a numerical constant of the order of unity), i.e., a distribution of the Gaussian type.

Thus, in the considered model the distribution of the distances between levels turns out to be closer in its physical properties to the Wigner hypothesis[8] than to the Dyson hypothesis for levels of heavy nuclei: a Gaussian distribution for large ΔE and a power-law distribution at small ΔE, ensuring the "pushing asunder" of the levels.

4. REMARKS

1. In the investigation of the spectra of complicated systems, the analysis is usually carried out for certain series of levels[8,8,15]. In our case we consider a series of levels corresponding to a set of intermixing trajectories of "glancing" electrons. A much smaller fraction (on the order of K^{-1}) are stable conditionally-periodic trajectories. The series of levels corresponding to trajectories lying in the stability islands can apparently be treated by a suitable renormalization of the Hamiltonian.

2. Our problem is of the so-called "billiard" type, since the particle motion is analogous to the motion of a billiard ball. In this case we have a billiard table with walls of negative curvature.

3. We have used a quasiclassical approximation in the derivation of the quantization conditions (2.16). However, the inclusion of small corrections cannot change the statistical properties of the distribution of the quantities S_X, S_Y, and E. The reason is that the considered series of levels is defined on classical mixing trajectories.

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[1)] It is useful to compare (2.12) with the quantization rules obtained in [12]; where conditions similar to (2.12) are imposed on the phase as a result of the uniqueness of the definition of a certain canonical operator.

[2)] For a one-dimensional system with a random potential, the "pushing-asunder" between levels is exponential [14]:

$$P(E|\Delta E)\sim\exp[-\text{const}/(\Delta E)^2],\quad \Delta E\to 0.$$

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66