Damping of spin waves in antiferromagnetic substances with magnetic anisotropy of the "easy plane" type

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The spin-wave interaction amplitudes which describe processes involving three or four magnons are calculated in the long-wave limit. Both exchange and relativistic interactions are taken into account. It is shown that the interaction amplitudes tend to zero when the nonactivated particle wave vectors vanish and the laws of conservation of energy and momentum are obeyed. The dependences of the low- and high-frequency magnon damping coefficients on energy and temperature are calculated in a broad range of energy and temperature.

1. INTRODUCTION

Relaxation processes in magnetically-ordered crystals have been the subject of many experimental and theoretical studies (see the monographs[1-6], where references to the original papers are given), and the main mechanisms of relaxation phenomena in the magnon system of dielectric ferromagnets are by now clearly understood. When it comes to antiferromagnets (AFM), the situation is much more complicated, although many experiments have been devoted to the antiferromagnetic resonance line widths, and intensive studies of nonlinear antiferromagnetic resonance are being carried out[7-9] and yield the experimental dependence of the spin-wave damping coefficient on the temperature and on the wave vector.

At present there are only about ten papers[10-18] devoted to the theory of relaxation processes in AFM, i.e., to the calculation of the magnon damping coefficient on the basis of a microscopic or phenomenological Hamiltonian of the AFM. The situation is made complicated by the fact that the published results fail to agree, but frequently contradict one another (a detailed analysis of the published data is contained in the paper by Harris et al.[18]). This circumstance is not accidental, but is due to the fact that in the case of AFM the determination of the Hamiltonian for the magnon-magnon interaction in the existing spin-wave theory is a rather complicated problem. (It is necessary to sum about 100 separate terms to determine the amplitude of magnon-magnon scattering.) Naturally the fact that neither the formalism of either Holstein and Primakoff nor that of Dyson and Maleev provides a general principle for the determination of the magnon-interaction amplitudes makes it very difficult to verify the results. Thus, Harris et al.[18] believe that the discrepancies in the results on the magnon damping are due to the fact that different authors have actually used different magnon-magnon interaction amplitudes.

This situation is typical of modern theory of a system of magnons, but is entirely different, for example, for a system of phonons. In the investigation of phonon-phonon interactions one uses in explicit form the fact that the lattice Hamiltonian is invariant to the shift of the lattice as a unit, and must therefore be made up (in the long-wave limit) of powers of the strain tensor. This makes it possible to determine in general form the dependence of the interaction amplitude on the phonon wave vectors, namely $\Psi(k_1, \ldots, k_n) \sim (k_1, \ldots, k_n)^{1/2}$.

It has become clear recently that the main properties of the spin-wave spectrum are governed by the symmetry of the Hamiltonian of the spin-system—by the invariance of the Hamiltonian to rotations[19]. This has made it possible to develop a hydrodynamic theory of spin waves in AFM[20]. It is shown in the theory of the phenomenological Lagrangians of elementary particles (see, for example, Volkov's review[21]) that the symmetry properties of the Hamiltonian not only make it possible to assess the particle spectrum (the Goldstone theorem), but also to draw definite conclusions with respect to the dependence of the scattering matrix on the particle wave vectors (Adler's principle). In the present paper, when checking on the magnon-magnon interaction amplitudes, we start from the fact that the Adler principle should be satisfied for activationless magnons (they are Goldstone particles for AFM), meaning that the scattering amplitude should vanish, when energy and momentum conservation is taken into account, if the wave vector of the activationless magnons tends to zero. We note that the amplitudes used by Harris et al.[18] also satisfy this requirement.

In this paper we investigate the damping of magnons in an antiferromagnet with uniaxial magnetic anisotropy of the easy plane type, and take the Dzyaloshinskiĭ interaction and the inter-ion anisotropy into account. We use the Holstein-Primakoff formalism, which is more convenient for this problem. We calculate the spin-wave damping coefficients in a wide range of temperatures and magnon wave vectors, and obtain the mean magnon relaxation times.

2. AMPLITUDES OF SPIN-WAVE INTERACTION WITH ONE ANOTHER

To describe the interaction processes in a system of spin waves, we start from the following expression for the Hamiltonian:

$$\mathscr{H} = \sum_{gf} \{J(R_{gf})\mathbf{S}_g\mathbf{S}_f + \beta(R_{gf})(\mathbf{S}_g\mathbf{n})(\mathbf{S}_f\mathbf{n}) + d(R_{gf})\mathbf{n}[\mathbf{S}_g\mathbf{S}_f]\} - 2\mu\mathbf{H}\Big\{\sum_g \mathbf{S}_g + \sum_f \mathbf{S}_f\Big\}, \quad (1)$$

where $J(R_{gf})$ is the exchange integral between sublattices, $\beta(R_{gf})$ is a quantity describing the magnetic-anisotropy energy, $\mathbf{S}_g$ and $\mathbf{S}_f$ are the sublattice spin vectors, $\mathbf{R}_{gf} = \mathbf{R}_g - \mathbf{R}_f$, where $\mathbf{R}_g$ and $\mathbf{R}_f$ are the radius vectors of the sublattice sites, n is a unit vector along the z axis, $d(R_{gf})$ is a quantity describing the Dzyaloshinskiĭ interaction, and $\mathbf{H} = (H, 0, 0)$ is the external magnetic field.

It is convenient to represent the spin operators in the form

$$\mathbf{S}_g = S_g^{\zeta}\mathbf{e}_{g\zeta} + S_g^{\eta}\mathbf{e}_{g\eta} + S_g^{\xi}\mathbf{e}_{g\xi},\quad \mathbf{S}_f = S_f^{\zeta}\mathbf{e}_{f\zeta} + S_f^{\eta}\mathbf{e}_{f\eta} + S_f^{\xi}\mathbf{e}_{f\xi}, \tag{2}$$

where $\mathbf{e}_{\alpha\zeta} = \mathbf{S}_\alpha/S_\alpha$ (α = g, f) are unit vectors along the quantization axes, and the operators S_α^{ζ}, $S_\alpha^{\pm} = S_\alpha^{\xi} \pm iS_\alpha^{\eta}$ are connected with the Holstein-Primakoff operators by the relations

$$\begin{aligned} S_\alpha^{\zeta} &= S - a_\alpha^+ a_\alpha, \\ S_\alpha^+ &= \sqrt{2S}\left(1 - \frac{a_\alpha^+ a_\alpha}{2S}\right)^{1/2} a_\alpha \approx \sqrt{2S}\left(a_\alpha - \frac{a_\alpha^+ a_\alpha a_\alpha}{4S}\right), \\ S_\alpha^- &= \sqrt{2S}\, a_\alpha^+\left(1 - \frac{a_\alpha^+ a_\alpha}{2S}\right)^{1/2} \approx \sqrt{2S}\left(a_\alpha^+ - \frac{a_\alpha^+ a_\alpha^+ a_\alpha}{4S}\right). \end{aligned} \tag{3}$$

Using (2) and (3) and changing over with the aid of a canonical transformation[1)] from the Holstein-Primakoff operators to the spin-wave creation and annihilation operators $c_{\mathbf{k}}$ and $d_{\mathbf{k}}$ with wave vector $\mathbf{k}$:

$$\begin{aligned} a_{1\mathbf{k}} &= u_{1\mathbf{k}} c_{\mathbf{k}} + v_{1\mathbf{k}}^* c_{-\mathbf{k}}^+ + u_{2\mathbf{k}} d_{\mathbf{k}} + v_{2\mathbf{k}}^* d_{-\mathbf{k}}^+, \\ a_{2\mathbf{k}} &= u_{1\mathbf{k}} c_{\mathbf{k}} + v_{1\mathbf{k}}^* c_{-\mathbf{k}}^+ - u_{2\mathbf{k}} d_{\mathbf{k}} - v_{2\mathbf{k}}^* d_{-\mathbf{k}}^+, \end{aligned} \tag{4}$$

We can represent the spin-system Hamiltonian (1) in the form

$$\mathcal{H} = \mathcal{E} + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4, \tag{4'}$$

where $\mathcal{H}_2$ is given by

$$\mathcal{H}_2 = \sum_{\mathbf{k}} \{\varepsilon_{1\mathbf{k}} c_{\mathbf{k}}^+ c_{\mathbf{k}} + \varepsilon_{2\mathbf{k}} d_{\mathbf{k}}^+ d_{\mathbf{k}}\}$$

and the energies of the spin waves are determined by the formulas

$$\begin{aligned} \varepsilon_{1\mathbf{k}}^2 &= (A_0 + B_{\mathbf{k}})^2 - C_{\mathbf{k}}^2,\quad \varepsilon_{2\mathbf{k}}^2 = (A_0 - B_{\mathbf{k}})^2 - C_{\mathbf{k}}^2; \\ A_0 &= S[2\mu h\cos\theta + d\sin 2\theta - J_0\cos 2\theta], \end{aligned} \tag{5}$$

$$\begin{aligned} B_{\mathbf{k}} &= S[J_{\mathbf{k}}\cos^2\theta + {}^1/_2\beta - {}^1/_2 d\sin 2\theta], \\ C_{\mathbf{k}} &= S[J_{\mathbf{k}}\sin^2\theta + {}^1/_2\beta + {}^1/_2 d\sin 2\theta]. \end{aligned} \tag{5'}$$

In these formulas $J_{\mathbf{k}}$ is the Fourier component of the exchange integral, 2θ is the angle between the equilibrium directions of the sublattice spins, the angle θ is chosen from the condition that the term linear in the spin-wave creation and annihilation operators be missing from expression (4′), $h = H/S$, and β and d are the Fourier components of the corresponding quantities in the Hamiltonian (1) at $\mathbf{k} = 0$.

In the region of small wave vectors, expressions (5) for the spin-wave energies take the form[22]

$$\begin{aligned} \varepsilon_{1\mathbf{k}}^2 &= S^2[2\mu h\cos\theta - I_0(ak)^2\cos 2\theta][2J_0 + \beta + d\,\mathrm{ctg}\,\theta - I_0(ak)^2], \\ \varepsilon_{2\mathbf{k}}^2 &= S^2[I_0(ak)^2 - \beta + d\,\mathrm{ctg}\,\theta][2J_0 + 2d\,\mathrm{ctg}\,\theta - \\ &\quad - 2\mu h\cos\theta + \cos 2\theta I_0(ak)^2], \end{aligned} \tag{6}$$

where a is the lattice constant and I_0 is a coefficient in the expansion of the exchange integral $J_{\mathbf{k}} = J_0 - I_0(ak)^2$.

The Hamiltonians $\mathcal{H}_3$ and $\mathcal{H}_4$, which describe the interaction of the spin waves with one another, are given by

$$\begin{aligned} \mathcal{H}_3 = \sum_{123} \{ &\Psi^{(1)}(1;23) d_1^+ c_2 c_3 + \Psi^{(2)}(1,2,3) d_1^+ c_2^+ c_3 + \Psi^{(3)}(1;23) d_1^+ c_2^+ c_3^+ \\ &+ \Psi^{(4)}(1;23) d_1^+ d_2 d_3 + \Psi^{(5)}(123) d_1^+ d_2^+ d_3^+ + \text{h.c.}\}, \end{aligned}$$

$$\begin{aligned} \mathcal{H}_4 = \sum_{1234} \{ &\Psi^{(1)}(12;34) c_1^+ c_2^+ c_3 c_4 + \Psi^{(2)}(12;34) d_1^+ d_2^+ d_3 d_4 \\ &+ \Psi^{(3)}(1,2;3,4) c_1^+ d_2^+ c_3 d_4 + \Psi^{(4)}(12;34) c_1^+ c_2^+ d_3 d_4 + \Psi^{*(4)}(12;34) d_4^+ d_3^+ c_2 c_1\}. \end{aligned}$$

We have written out in the Hamiltonian $\mathcal{H}_4$ only the terms that contribute to the mass operator.

The amplitudes that enter in these Hamiltonians are given in Appendix I for arbitrary values of the wave vectors. We confine ourselves here to the expressions for the amplitudes in the case of small wave vectors and in the absence of an external field:

$$\begin{aligned} \Psi^{(1)}(1;23) &= i\left(\frac{S}{N}\right)^{1/2} d\left(\frac{H_D}{H_E}\right)^2 \left(\frac{A_0}{2\varepsilon_{21}\varepsilon_{12}\varepsilon_{13}}\right)^{1/2} (\varepsilon_{21} + \varepsilon_{12} + \varepsilon_{13})\Delta(1-2-3), \\ &\Psi^{(2)}(1,2,3) \\ &= i\left(\frac{S}{N}\right)^{1/2} d\left(\frac{H_D}{H_E}\right)^2 \left(\frac{2A}{\varepsilon_{21}\varepsilon_{12}\varepsilon_{13}}\right)^{1/2} (\varepsilon_{12} - \varepsilon_{13} - \varepsilon_{21})\Delta(1+2-3), \\ &\Psi^{(3)}(1;23) \\ &= i\left(\frac{S}{N}\right)^{1/2} d\left(\frac{H_D}{H_E}\right)^2 \left(\frac{A_0}{2\varepsilon_{21}\varepsilon_{12}\varepsilon_{13}}\right)^{1/2} (\varepsilon_{21} - \varepsilon_{12} - \varepsilon_{13})\Delta(1+2+3), \\ &\Psi^{(4)}(1;23) \\ &= i\left(\frac{S}{N}\right)^{1/2} d\left(\frac{H_D}{H_E}\right)^2 \left(\frac{A_0}{2\varepsilon_{21}\varepsilon_{22}\varepsilon_{23}}\right)^{1/2} (\varepsilon_{21} - \varepsilon_{22} - \varepsilon_{23})\Delta(1-2-3), \end{aligned} \tag{7}$$

where

$$2\mu H_D = Sd,\quad \mu H_E = SJ_0.$$

In these formulas we use the following notation:

$$1 \equiv \mathbf{k}_1,\ 2 \equiv \mathbf{k}_2,\ 3 \equiv \mathbf{k}_3,\ 4 \equiv \mathbf{k}_4,\ \varepsilon_{1\alpha} \equiv \varepsilon_1(\mathbf{k}_\alpha),\ \varepsilon_{2\alpha} \equiv \varepsilon_2(\mathbf{k}_\alpha),\ \alpha = 1, 2, 3, 4.$$

$$\begin{aligned} \Psi^{(1)}(12;34) &= \left\{\frac{J_0 + {}^1/_2\beta}{64N}\frac{(\varepsilon_{11} + \varepsilon_{12} - \varepsilon_{13} - \varepsilon_{14})^2}{(\varepsilon_{11}\varepsilon_{12}\varepsilon_{13}\varepsilon_{14})^{1/2}} \right. \\ &\quad \left. + J_0^{-2}(\varepsilon_{11}\varepsilon_{12}\varepsilon_{13}\varepsilon_{14})^{1/2} f(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4)\right\}\Delta(1+2-3-4), \\ \Psi^{(2)}(12;34) &= \frac{\Delta(1+2-3-4)}{4N(\varepsilon_{21}\varepsilon_{22}\varepsilon_{23}\varepsilon_{24})^{1/2}} \\ &\quad \times \left\{\frac{1}{16}\left(J_0 - \frac{1}{2}\beta\right)(\varepsilon_{21} + \varepsilon_{22} + \varepsilon_{23} + \varepsilon_{24})^2 + \frac{1}{2}J_0\Delta^2\right\}, \\ \Psi^{(3)}(1,2;3,4) &= \frac{\Delta(1+2-3-4)}{4N(\varepsilon_{11}\varepsilon_{22}\varepsilon_{13}\varepsilon_{24})^{1/2}} \\ &\quad \times \left\{\frac{1}{4}J_0(\varepsilon_{11} + \varepsilon_{22} - \varepsilon_{13} - \varepsilon_{24})^2 + 2J_0 s^2(k_1 k_3 - \mathbf{k}_1\mathbf{k}_3)\right\}, \\ \Psi^{(4)}(12;34) &= \frac{\Delta(1+2-3-4)}{8N(\varepsilon_{11}\varepsilon_{12}\varepsilon_{23}\varepsilon_{24})^{1/2}} \\ &\quad \times \left\{J_0 s^2(k_1 k_2 - \mathbf{k}_1\mathbf{k}_2) - \frac{1}{8}J_0(\varepsilon_{11} + \varepsilon_{12} - \varepsilon_{23} - \varepsilon_{24})^2\right\}, \\ \Delta^2 &= 2S^2 J_0|\beta|,\quad s = \Theta_c a,\quad \Theta_c^2 = 2S^2 J_0 I_0, \\ \varepsilon_{1\mathbf{k}} &= sk,\quad \varepsilon_{2\mathbf{k}} = [\Delta^2 + (sk)^2]^{1/2}, \end{aligned} \tag{8}$$

where N is the number of lattice sites and $f(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, and $\mathbf{n}_4)$ is a function on the order of unity and depends on the unit vectors

$$\mathbf{n}_\alpha = \mathbf{k}_\alpha / k_\alpha.$$

We note that the amplitudes (7) of the interaction of three spin waves contain the Dzyaloshinskiĭ constant as a coefficient; in other words, these processes do not occur in a collinear antiferromagnet. Processes in which four magnons participate occur both in the collinear and in the canted phases. We call attention to the fact that the amplitudes of those processes of (8) in which two types of magnons (activation and activationless) take part vanish when the wave vectors of the corresponding activationless particles tend to zero and the energy and momentum conservation laws are satisfied ($\Psi^{(3)}(1, 2; 3, 4) \to 0$ as $k_1, k_3 \to 0$; $\Psi^{(4)}(12; 34) \to 0$ as $k_1, k_2 \to 0$)[2)]. In addition, the amplitude $\Psi^{(1)}(12; 34)$ vanishes, up to terms with k^4, for a real scattering process when the energy conservation law is satisfied, i.e., the interaction of the activationless magnons with one another is much weaker than the interaction between the activation and activationless magnons. This circumstance is connected with the definite choice of the phase of the wave function of the ground state. In our case, this choice of the phase of the wave function of the ground state was dictated by the fact that we have considered a crystal with magnetic anisotropy of the easy plane type. If we take only exchange interaction into account then the phase of the ground state remains arbitrary and, generally speaking, all the amplitudes are of the same order $\Psi \sim k^2$. This was the case, neglecting the relativistic interactions, of the amplitudes in[18], where the authors started from a ground state corresponding to magnetic anisotropy of the easy axis type.

3. SPIN-WAVE DAMPING DUE TO THREE-MAGNON PROCESSES

Knowing the Hamiltonian $\mathscr{H}_3$ that describes scattering processes in which three magnons take part, we can determine the spin-wave damping coefficient as the imaginary part of the mass operator[23]. Proceeding in standard fashion, we obtain the following expression for the damping of the activationless branch of the spectrum:

$$\gamma_1^{(3)}(\mathbf{k})=4\pi\sum_{12}|\Psi^{(1)}(2;\,1\mathbf{k})|^2(n(\varepsilon_{11})-n(\varepsilon_{22}))\,\delta(\varepsilon_{1\mathbf{k}}+\varepsilon_{11}-\varepsilon_{22}). \quad (9)$$

In (9) we have recognized that the conservation laws permit only the process of coalescence of two activationless magnons into one activation magnon and the decay of an activation magnon into two activationless ones; $n(\epsilon) = [e^{\epsilon/T} - 1]^{-1}$.

Analogously, for the activation branch of the spectrum we obtain

$$\gamma_2^{(3)}(\mathbf{k})=2\pi\sum_{12}|\Psi^{(1)}(\mathbf{k};\,12)|^2(1+n(\varepsilon_{11})+n(\varepsilon_{12}))\,\delta(\varepsilon_{2\mathbf{k}}-\varepsilon_{11}-\varepsilon_{12}). \quad (10)$$

Performing the integration in (9) and (10), we can obtain the dependence of the damping coefficients $\gamma_{1,2}^{(3)}(\mathbf{k})$ on the wave vector and on the temperature. Without stopping to describe the straight-forward but cumbersome manipulations, we present the final results:

$$\gamma_1^{(3)}(\mathbf{k})=J_0\left(\frac{H_D}{H_E}\right)^4\left(\frac{\mu H_D}{\Theta_c}\right)^2\left(\frac{T}{\Theta_c}\right)\times\begin{cases}\dfrac{1}{16\pi}\left(\dfrac{\Delta}{\varepsilon_{1\mathbf{k}}}\right)^4\left[1+\left(\dfrac{2\varepsilon_{1\mathbf{k}}}{\Delta}\right)^2\right]^2\left(1-\exp\left(-\dfrac{\varepsilon_{1\mathbf{k}}}{T}\right)\right)\exp\left\{-\dfrac{\Delta^2}{4\varepsilon_{1\mathbf{k}}T}\right\}, \\ \qquad \varepsilon_{1\mathbf{k}}\ll\dfrac{\Delta^2}{T},\quad T<\Delta, \\ \dfrac{1}{\pi}\ln\dfrac{4\varepsilon_{1\mathbf{k}}T}{\Delta^2},\quad \varepsilon_{1\mathbf{k}}\gg\dfrac{\Delta^2}{T},\quad T<\Delta.\end{cases} \quad (11)$$

These formulas determine the damping coefficient of the activationless magnons at low temperatures ($T < \Delta$).

At high temperatures ($T > \Delta$) the damping coefficient of the activationless magnons is given by

$$\gamma_1^{(3)}(\mathbf{k})=J_0\left(\frac{H_D}{H_E}\right)^4\left(\frac{\mu H_D}{\Theta_c}\right)^2\times\begin{cases}\dfrac{1}{16\pi}\left(\dfrac{\Delta}{\Theta_c}\right)\left(\dfrac{\Delta}{\varepsilon_{1\mathbf{k}}}\right)^3\exp\left\{-\dfrac{\Delta^2}{4\varepsilon_{1\mathbf{k}}T}\right\}, & \varepsilon_{1\mathbf{k}}\ll\dfrac{\Delta^2}{T},\quad T>\Delta, \\ \dfrac{\pi}{6}\left(\dfrac{T}{\Theta_c}\right)\left(\dfrac{T}{\varepsilon_{1\mathbf{k}}}\right), & \varepsilon_{\mathbf{k}1}\gg\dfrac{\Delta^2}{T},\quad T>\Delta, \\ \dfrac{1}{\pi}\left(\dfrac{T}{\Theta_c}\right)\ln\dfrac{4\varepsilon_{1\mathbf{k}}T}{\Delta^2}, & \varepsilon_{1\mathbf{k}}>T>\Delta,\end{cases} \quad (12)$$

It is seen from these formulas that the magnon damping coefficient increases with increasing magnon energy. We note that as $\epsilon_{1k}\to 0$, the damping due to the triple processes is exponentially small both at low and at high temperatures. The reason is that the coalescence of two activationless magnons into an activation magnon is a threshold process. It follows from the conservation law that the process is possible only when the energies of the activationless magnons satisfy the condition $4\epsilon_{1k}\epsilon_{11} \geq \Delta^2$, and therefore $\epsilon_{11} \geq \Delta^2/4\epsilon_{1k}$, and as $\epsilon_{1k} \to 0$ the number of such magnons is $n(\epsilon_{11}) \sim \exp(-\Delta^2/4\epsilon_{1k}T)$, and this is why the factor $\exp(-\Delta^2/4\epsilon_{1k}T)$ appears in the formulas for the damping coefficient. The damping coefficient of the activation branch of the spin waves is given by

$$\gamma_2^{(3)}(\mathbf{k})=\frac{8J_0}{\pi}\left(\frac{H_D}{H_E}\right)^4\left(\frac{\mu H_D}{\Theta_c}\right)^2\left(\frac{T}{\Theta_c}\right)\frac{\varepsilon_{2\mathbf{k}}}{(\varepsilon_{2\mathbf{k}}^2-\Delta^2)^{1/2}}\times\ln\left\{\operatorname{sh}\left[\frac{\varepsilon_{2\mathbf{k}}+(\varepsilon_{2\mathbf{k}}^2-\Delta^2)^{1/2}}{4T}\right]\Big/\operatorname{sh}\left[\frac{\varepsilon_{2\mathbf{k}}-(\varepsilon_{2\mathbf{k}}^2-\Delta^2)^{1/2}}{4T}\right]\right\}.$$

To illustrate the dependence of $\gamma_2^{(3)}(\mathbf{k})$ on the wave vector in more lucid fashion, we present the asymptotic forms of these expressions for small and large wave vectors.

In the case of low temperatures ($T \ll \Delta$)

$$\gamma_2^{(3)}(\mathbf{k})=\frac{4J_0}{\pi}\left(\frac{H_D}{H_E}\right)^4\left(\frac{\mu H_D}{\Theta_c}\right)^2\times\begin{cases}\left(\dfrac{\Delta}{\Theta_c}\right)\left[1+\dfrac{1}{2}\left(\dfrac{sk}{\Delta}\right)^2\right]\left(1+2\exp\left(-\dfrac{\Delta}{2T}\right)\right), & sk\ll T\ll\Delta, \\ \left(\dfrac{\varepsilon_{2\mathbf{k}}}{\Theta_c}\right)\left[1+\dfrac{1}{4}\left(\dfrac{\Delta}{\varepsilon_{2\mathbf{k}}}\right)^2\right], & T\ll\Delta\ll sk\ll\dfrac{\Delta^2}{T}, \\ \left(\dfrac{\varepsilon_{2\mathbf{k}}}{\Theta_c}\right)\left[1+\dfrac{1}{2}\left(\dfrac{\Delta}{\varepsilon_{2\mathbf{k}}}\right)^2\right]\left(1-\dfrac{2T}{\varepsilon_{2\mathbf{k}}}\ln\dfrac{\Delta^2}{4\varepsilon_{2\mathbf{k}}T}\right), & T\ll\Delta\ll\dfrac{\Delta^2}{T}\ll sk.\end{cases} \quad (13)$$

In the case of high temperatures ($T \gg \Delta$)

$$\gamma_2^{(3)}(\mathbf{k})=\frac{4J_0}{\pi}\left(\frac{H_D}{H_E}\right)^4\left(\frac{\mu H_D}{\Theta_c}\right)^2\times\begin{cases}\left(\dfrac{T}{\Theta_c}\right)\left[1+\left(\dfrac{sk}{\Delta}\right)^2\right], & sk\ll\Delta\ll T, \\ 2\left(\dfrac{T}{\Theta_c}\right)\left[1+\dfrac{1}{2}\left(\dfrac{\Delta}{\varepsilon_{2\mathbf{k}}}\right)^2\right]\ln\dfrac{2\varepsilon_{2\mathbf{k}}}{\Delta}, & \Delta\ll sk\ll T, \\ \left(\dfrac{\varepsilon_{2\mathbf{k}}}{\Theta_c}\right)\left[1+\dfrac{1}{2}\left(\dfrac{\Delta}{\varepsilon_{2\mathbf{k}}}\right)^2\right]\left(1-\dfrac{2T}{\varepsilon_{2\mathbf{k}}}\ln\dfrac{\Delta^2}{4\varepsilon_{2\mathbf{k}}T}\right), & \Delta\ll T\ll sk.\end{cases} \quad (14)$$

The damping coefficient of magnons with activation remains finite as $k \to 0$, and is proportional to the larger of the quantities T or Δ, depending on whether the temperature is high or low.

As already noted, three-magnon processes are due to relativistic interaction, so that it becomes necessary to investigate the role of the processes due to exchange interaction.

4. SPIN-WAVE DAMPING DUE TO FOUR-MAGNON PROCESSES

We proceed now to consider the contribution made to the spin-wave damping by processes in which four magnons take part. The amplitudes of these processes, as seen from formulas (A.1)—(A.4), are due to both exchange and relativistic interaction. Bearing in mind crystals in which the exchange interaction is much larger than the relativistic interactions, we neglect the latter and start out from the amplitudes determined by formulas (8).

In the calculation of the mass operator it is necessary to take into account formally not only diagrams containing two amplitudes of the Hamiltonian $\mathcal{H}_4$, but also diagrams containing each four amplitudes of the Hamiltonian $\mathcal{H}_3$. The reason is that both classes of diagrams contain the small parameter 1/S of the theory of spin waves in antiferromagnets, raised to identical powers; since, however, the amplitudes of the Hamiltonian $\mathcal{H}_3$ are connected with the relativistic interaction and the amplitudes of $\mathcal{H}_4$ are connected with the exchange interaction, the principal contribution to the damping is made by diagrams containing the amplitudes $\mathcal{H}_4$ in second-order perturbation theory.

The spin-wave damping due to four-magnon processes is determined by the following expressions

$$\gamma_1^{(4)}(\mathbf{k})=\frac{\pi}{n_{1k}}\sum_{234}\{8|\Psi^{(1)}(\mathbf{k}2;34)|^2(1+n_{12})n_{13}n_{14}\delta(\varepsilon_{1k}+\varepsilon_{12}-\varepsilon_{13}-\varepsilon_{14})$$
$$+8|\Psi^{(4)}(\mathbf{k}2;34)|^2(1+n_{12})n_{23}n_{24}\delta(\varepsilon_{1k}+\varepsilon_{12}-\varepsilon_{23}-\varepsilon_{24})+$$
$$+|\Psi^{(3)}(\mathbf{k},2;3,4)|^2(1+n_{22})n_{13}n_{24}\delta(\varepsilon_{1k}+\varepsilon_{22}-\varepsilon_{13}-\varepsilon_{24})\}; \quad (15)$$

$$\gamma_2^{(4)}(\mathbf{k})=\frac{\pi}{n_{2k}}\sum_{234}\{8|\Psi^{(2)}(\mathbf{k}2;34)|^2(1+n_{22})n_{23}n_{24}\delta(\varepsilon_{2k}+\varepsilon_{22}-\varepsilon_{23}-\varepsilon_{24})$$
$$+8|\Psi^{(4)}(34;\mathbf{k}2)|^2(1+n_{22})n_{13}n_{14}\delta(\varepsilon_{2k}+\varepsilon_{22}-\varepsilon_{13}-\varepsilon_{14})+$$
$$+|\Psi^{(3)}(2,\mathbf{k};4,3)|^2(1+n_{12})n_{23}n_{14}\delta(\varepsilon_{2k}+\varepsilon_{12}-\varepsilon_{23}-\varepsilon_{14})\}. \quad (16)$$

We consider first the damping of activationless spin waves. Since the amplitude $\Psi^{(1)}(12;34)$ vanishes accurate to terms k^4 when account is taken of the energy conservation law, the principal processes that determine $\gamma_{(1)}^4(\mathbf{k})$ are the scattering of the activationless magnons by activation magnons and the process of conversion of two activationless magnons into two activation magnons. Accordingly, we represent $\gamma_{(1)}^4(\mathbf{k})$ in the form

$$\gamma_1^{(4)}(\mathbf{k})=\Gamma_{11}(\mathbf{k})+\Gamma_{12}(\mathbf{k}). \quad (17)$$

The quantities $\Gamma_{11}(\mathbf{k})$ and $\Gamma_{12}(\mathbf{k})$ are damping coefficients and are due respectively to the processes of conversion of two activationless magnons into two activation magnons and the scattering of activationless magnons by activation magnons. The results of the calculation of $\Gamma_{11}(\mathbf{k})$ and $\Gamma_{12}(\mathbf{k})$ are gathered in Appendix II.

In the case of low temperatures ($T \ll \Delta$) we see that as $k \to 0$ the principal role is assumed by the scattering of activationless magnons by activation magnons. The damping coefficient $\gamma_1^{(4)}(\mathbf{k})$ depends on the spin-wave energy in power-law fashion, and on the temperature in exponential fashion (see Appendix II). Since $\Gamma_{11}(\mathbf{k})$ and $\Gamma_{12}(\mathbf{k})$ are exponentially small at low energies, it is necessary to estimate the role of the processes of scattering of activationless spin waves by one another. Using expression (8) for $\Psi^{(1)}(12;34)$, we find that the contribution $\Gamma_{13}(\mathbf{k})$ of this process to the damping coefficient of the activationless spin waves is determined by the formulas

$$\Gamma_{13}(\mathbf{k})\sim\begin{cases}J_0\left(\frac{J_0}{\Theta_c}\right)\left(\frac{\varepsilon_{1k}}{\Theta_c}\right)^2\left(\frac{T}{\Theta_c}\right)^7, & \varepsilon_{1k}\ll T,\\ J_0\left(\frac{J_0}{\Theta_c}\right)\left(\frac{\varepsilon_{1k}}{\Theta_c}\right)^5\left(\frac{T}{\Theta_c}\right)^4, & T\ll\varepsilon_{1k}\ll\Theta_c.\end{cases}$$

Comparing $\Gamma_{13}(\mathbf{k})$ with the quantities $\Gamma_{11}(\mathbf{k})$ and $\Gamma_{12}(\mathbf{k})$ we see that if

$$\varepsilon_{1k}\ll T(T/\Theta_c)^2e^{\Delta/2T},$$

then the process of scattering of activationless magnons by one another plays the decisive role, $\gamma_1^{(4)}(\mathbf{k})\approx\Gamma_{13}(\mathbf{k})$. When the magnon energy ϵ_{1k} increases, both $\Gamma_{11}(\mathbf{k})$ and $\Gamma_{12}(\mathbf{k})$ increase. At larger values of the energy, $\epsilon_{1k} > \Delta^2/T$, the principal role is played by the process of conversion of two activationless magnons into two magnons with activation, so that the summary damping coefficient is $\gamma_1^{(4)}(\mathbf{k})\approx\Gamma_{11}(\mathbf{k})$ and increases linearly with energy.

We proceed now to the case of high temperatures ($T \gg \Delta$). At sufficiently low values of the energy, $\epsilon_{1k} \ll (T/\Theta_c)^2\Delta^2/T$, the principal role is played by the process of scattering of activationless magnons by one another, $\gamma_1^{(4)}(\mathbf{k})\approx\Gamma_{13}(\mathbf{k})$; in the energy interval

$$\frac{\Delta^2}{T}\left(\frac{T}{\Theta_c}\right)^2\ll\varepsilon_{1k}\ll\frac{\Delta^2}{T}$$

the principal role is played by the process of scattering of activationless magnons by activation magnons $\gamma_1^{(4)}(\mathbf{k})\approx\Gamma_{12}(\mathbf{k})$. At energies $\Delta^2/T \ll \epsilon_{1k}$, on the other hand, the contributions $\Gamma_{11}(\mathbf{k})$ and $\Gamma_{12}(\mathbf{k})$ become comparable and play the decisive role with respect to $\Gamma_{13}(\mathbf{k})$. Thus, $\gamma_1^{(4)}(\mathbf{k})\approx\Gamma_{11}(\mathbf{k})+\Gamma_{12}(\mathbf{k})$.

Let us analyze the results of the calculation of the damping coefficient of magnons with activation, given in Appendix II. The damping of these magnons is due to three processes: the scattering of activation magnons by each other, the conversion of activation magnons into activationless magnons, and the scattering of activation magnons by activationless magnons. It is therefore convenient to express the damping coefficient in the form

$$\gamma_2^{(4)}(\mathbf{k})=\Gamma_{21}(\mathbf{k})+\Gamma_{22}(\mathbf{k})+\Gamma_{23}(\mathbf{k}), \quad (18)$$

where $\Gamma_{21}(\mathbf{k})$, $\Gamma_{22}(\mathbf{k})$, and $\Gamma_{23}(\mathbf{k})$ correspond to the three terms of formula (16).

We note first that as $k \to 0$ the quantity $\gamma_2^{(4)}(\mathbf{k})$ remains finite at both low and high temperatures. At low temperatures ($T \ll \Delta$) the principal contribution is made by the scattering of activation magnons by activationless magnons, $\gamma_2^{(4)}(\mathbf{k})\approx\Gamma_{23}(\mathbf{k})$, and at high temperatures ($T \gg \Delta$) it is made by both processes with participation of activationless magnons: $\gamma_2^{(4)}(\mathbf{k})\approx\Gamma_{22}(\mathbf{k})+\Gamma_{23}(\mathbf{k})$. As seen from Appendix II, when the wave vector of the magnon increases the damping coefficient $\Gamma_{21}(\mathbf{k})$ decreases, because the scattering amplitude of this process decreases with increasing k, see formulas (8). As to $\Gamma_{22}(\mathbf{k})$ and $\Gamma_{23}(\mathbf{k})$, they increase with increasing wave vector.

It is interesting to note that in the wave-vector regions $\Delta \ll sk \ll T$ and $\Delta \ll T \ll sk$ the values of $\gamma_1^{(4)}(\mathbf{k})$ coincide with the values of $\gamma_2^{(4)}(\mathbf{k})$. This is not surprising, since the principal role is played by the same processes, and the activation of the spin wave can be allowed formally to approach zero.

Appendix II gives expressions for the summary values of the damping coefficients $\gamma_1^{(4)}(\mathbf{k})$ and $\gamma_2^{(4)}(\mathbf{k})$.

We present expressions for the mean values of the damping coefficients, defined by the formula

$$\gamma_\alpha=\sum_k\gamma_\alpha(\mathbf{k})n_{\alpha k}\Big/\sum_k n_{\alpha k},\quad \alpha=1,2.$$

If $T \ll \Delta$, then

$$\gamma_1^{(4)}\sim J_0\left(\frac{J_0}{\Theta_c}\right)\left(\frac{T}{\Theta_c}\right)^5\left(\frac{T}{\Delta}\right)^{1/2}e^{-\Delta/T},$$
$$\gamma_1^{(3)}\sim J_0\left(\frac{H_D}{H_E}\right)^4\left(\frac{\mu H_D}{\Theta_c}\right)^2\left(\frac{\Delta}{\Theta_c}\right)\left(\frac{\Delta}{T}\right)^3\exp\left(-\frac{\Delta^2}{4T^2}\right),$$
$$\gamma_2^{(4)}\sim J_0\left(\frac{J_0}{\Theta_c}\right)\left(\frac{T}{\Theta_c}\right)^5\left(\frac{T}{\Delta}\right)^2,\quad \gamma_2^{(3)}\sim J_0\left(\frac{H_D}{H_E}\right)^4\left(\frac{\mu H_D}{\Theta_c}\right)^2\left(\frac{\Delta}{\Theta_c}\right).$$

If $T \gg \Delta$, then

$$\gamma_1^{(4)} \sim J_0\left(\frac{J_0}{\Theta_c}\right)\left(\frac{T}{\Theta_c}\right)^5, \quad \gamma_1^{(3)} \sim J_0\left(\frac{H_D}{H_E}\right)^4\left(\frac{\mu H_D}{\Theta_c}\right)^2\left(\frac{T}{\Theta_c}\right),$$

$$\gamma_2^{(4)} \sim J_0\left(\frac{J_0}{\Theta_c}\right)\left(\frac{T}{\Theta_c}\right)^5, \quad \gamma_2^{(3)} \sim J_0\left(\frac{H_D}{H_E}\right)^4\left(\frac{\mu H_D}{\Theta_c}\right)^2\left(\frac{T}{\Theta_c}\right).$$

CONCLUSION

A comparison of the damping coefficients of activationless magnons in three-magnon and four-magnon processes (under the condition $\mu H_D \sim S\beta$) shows that the principal role is played by four-magnon processes, $\gamma_1^{(4)}(k) > \gamma_1^{(3)}(k)$ at both low temperatures ($T \ll \Delta$) and high temperatures ($T \gg \Delta$). A similar situation obtains when the damping coefficient of the activation magnons are compared, $\gamma_2^{(4)}(k) > \gamma_2^{(3)}(k)$.

Thus, four-magnon processes make the main contribution to the damping coefficients due to the interaction of spin waves.

Processes in which three magnons take part, with allowance for activation in both branches of the spectrum, were considered by Ozhogin[16]. It is easily seen that Ozhogin's formulas give an exponential decrease of the damping coefficient of magnons of type I with wave vector $k = 0$ (these magnons are analogous to our activationless magnons), if their activation, i.e., the external magnetic field, tends to zero. In this sense we can state that our results agree with Ozhogin's. A more detailed comparison, however, is impossible, since the formulas of[16] are correct only as to order of magnitude.

We note, finally, that although the magnon scattering amplitudes given in[11] do not satisfy Adler's principle, and therefore cannot give the correct dependence of the damping coefficients on the wave vectors of the spin waves, nonetheless, correct results were obtained in[11] for the temperature dependence of the mean values of the damping coefficients, since the amplitudes are quadratic forms in the wave vectors. As seen from the formulas of Appendix II, at energies $\Delta \ll sk \ll T$ and $\Delta \ll T \ll sk$ our results coincide fully with the results of A. B. Harris, D. Kumar, B. I. Halperin, and P. C. Hohenberg (if we put in our results $J_0 = 2zJ$, $\Theta_c = JSz$, and $I_0 = zJ/4$). The reason is that the activation dose not play an important role at these energies and can be set formally equal to zero.

In conclusion, the authors thank A. S. Borovik-Romanov, I. A. Akhiezer, and L. A. Prozorova for useful discussions.

APPENDIX I

Amplitudes of spin-wave interaction Hamiltonians at arbitrary values of the wave vectors

We present first the expressions for the amplitudes of the canonical transformation (4):

$$u_{\alpha k} \equiv u_\alpha(k) = \left[\frac{A_0 + (-1)^{\alpha-1}B_k + \varepsilon_{\alpha k}}{2\varepsilon_{\alpha k}}\right]^{1/2}$$

$$v_{\alpha k} \equiv v_\alpha(k) = (-1)^\alpha\left[\frac{A_0 + (-1)^{\alpha-1}B_k - \varepsilon_{\alpha k}}{2\varepsilon_{\alpha k}}\right]^{1/2}$$

where $\alpha = 1$ or 2 and A_0 and B_k are determined by formulas (5′). The amplitudes of the Hamiltonian $\mathscr{H}_3$ are given by

$$\Psi^{(1)}(1;23) = \frac{i}{2}\left(\frac{S}{N}\right)^{1/2}\Delta(1-2-3)\{P_1(u_{12}v_{13}+u_{13}v_{12})(v_{21}-u_{21})$$
$$+P_2(u_{21}u_{13}+v_{21}v_{13})(v_{12}-u_{12})+P_3(u_{21}u_{12}+v_{21}v_{12})(v_{13}-u_{13})\},$$
$$\Psi^{(2)}(1,2,3) = i\left(\frac{S}{N}\right)^{1/2}\Delta(1+2-3)\{P_1(u_{12}u_{13}+v_{12}v_{13})(v_{21}-u_{21})$$
$$-P_2(u_{21}u_{13}+v_{21}v_{13})(v_{12}-u_{12})+P_3(u_{21}v_{12}+u_{12}v_{21})(v_{13}-u_{13})\},$$
$$\Psi^{(3)}(1;23) = \frac{i}{2}\left(\frac{S}{N}\right)^{1/2}\Delta(1+2+3)\{P_3(u_{21}v_{12}+u_{12}v_{21})(u_{13}-v_{13})$$
$$+P_2(u_{21}v_{13}+u_{13}v_{21})(u_{12}-v_{12})-P_1(u_{12}v_{13}+u_{13}v_{12})(u_{21}-v_{21})\}, \quad \text{(A.1)}$$
$$\Psi^{(4)}(1;23) = \frac{i}{2}\left(\frac{S}{N}\right)^{1/2}\Delta(1-2-3)\{P_3(u_{21}u_{22}+v_{21}v_{22})(u_{23}-v_{23})$$
$$+P_2(u_{21}u_{23}+v_{21}v_{23})(u_{22}-v_{22})-P_1(u_{22}v_{23}+u_{23}v_{22})(u_{21}-v_{21})\},$$
$$\Psi^{(5)}(123) = \frac{i}{6}\left(\frac{S}{N}\right)^{1/2}\Delta(1+2+3)\{P_3(u_{21}v_{22}+u_{22}v_{21})(v_{23}-u_{23})$$
$$+P_2(u_{23}v_{21}+u_{21}v_{23})(v_{22}-u_{22})+P_1(u_{22}v_{23}+u_{23}v_{22})(v_{21}-u_{21})\}.$$

In these formulas, $P_k = J_k \sin 2\theta + d\cos 2\theta$. The amplitudes of the Hamiltonian $\mathscr{H}_4$ are ($\alpha = 1, 2$)

$$\Psi^{(\alpha)}(12;34) = N^{-1}\Delta(1+2-3-4)$$
$$\times\{\psi_\alpha(12;34)(u_{\alpha1}u_{\alpha2}u_{\alpha3}u_{\alpha4}+v_{\alpha1}v_{\alpha2}v_{\alpha3}v_{\alpha4})$$
$$+\psi_\alpha(1-3;-24)(u_{\alpha1}v_{\alpha2}v_{\alpha3}u_{\alpha4}+v_{\alpha1}u_{\alpha2}u_{\alpha3}v_{\alpha4})$$
$$+\psi_\alpha(1-4;3-2)(u_{\alpha1}v_{\alpha2}u_{\alpha3}v_{\alpha4}+v_{\alpha1}u_{\alpha2}v_{\alpha3}u_{\alpha4})$$
$$+(-1)^\alpha[\varphi(1,-234)(u_{\alpha1}v_{\alpha2}u_{\alpha3}u_{\alpha4}+v_{\alpha1}u_{\alpha2}v_{\alpha3}v_{\alpha4})$$
$$+\varphi(2,-134)(v_{\alpha1}u_{\alpha2}u_{\alpha3}u_{\alpha4}+u_{\alpha1}v_{\alpha2}v_{\alpha3}v_{\alpha4})$$
$$+\varphi(3,-421)(u_{\alpha1}u_{\alpha2}u_{\alpha3}v_{\alpha4}+v_{\alpha1}v_{\alpha2}v_{\alpha3}u_{\alpha4})$$
$$+\varphi(4,-321)(u_{\alpha1}u_{\alpha2}v_{\alpha3}u_{\alpha4}+v_{\alpha1}v_{\alpha2}u_{\alpha3}v_{\alpha4})]\}, \quad \text{(A.2)}$$

$$\Psi^{(3)}(1,2;3,4) = N^{-1}\Delta(1+2-3-4)$$
$$\times\{-4C(1-3;-24)(u_{11}v_{22}v_{13}u_{24}+v_{11}u_{22}u_{13}v_{24})$$
$$+D(1,2;3,4)(u_{11}u_{22}u_{13}u_{24}+v_{11}v_{22}v_{13}v_{24})$$
$$+D(1,-4;3,-2)(v_{11}u_{22}v_{13}u_{24}+u_{11}v_{22}u_{13}v_{24})$$
$$+2M(1,234)(u_{11}v_{22}u_{13}u_{24}+v_{11}u_{22}v_{13}v_{24})$$
$$+2M(3,412)(u_{11}u_{22}u_{13}v_{24}+v_{11}v_{22}v_{13}u_{24})$$
$$-2M(2,143)(v_{11}u_{22}u_{13}u_{24}+u_{11}v_{22}v_{13}v_{24})$$
$$-2M(4,321)(u_{11}u_{22}v_{13}u_{24}+v_{11}v_{22}u_{13}v_{24})\}, \quad \text{(A.3)}$$

$$\Psi^{(4)}(12;34) = N^{-1}\Delta(1+2-3-4)$$
$$\times\{-C(12;34)(u_{11}u_{12}u_{23}u_{24}+v_{11}v_{12}v_{23}v_{24})$$
$$+{}^1\!/_4D(1,-3;-2,4)(u_{11}v_{12}v_{23}u_{24}+v_{11}u_{12}u_{23}v_{24})$$
$$+{}^1\!/_4D(1,-4;-2,3)(u_{11}v_{12}u_{23}v_{24}+v_{11}u_{12}v_{23}u_{24})$$
$$+{}^1\!/_2M(1,324)(u_{11}v_{12}u_{23}u_{24}+v_{11}u_{12}v_{23}v_{24})$$
$$+{}^1\!/_2M(2,314)(v_{11}u_{12}u_{23}u_{24}+u_{11}v_{12}v_{23}v_{24})$$
$$-{}^1\!/_2M(3,142)(u_{11}u_{12}u_{23}v_{24}+v_{11}v_{12}v_{23}u_{24})$$
$$-{}^1\!/_2M(4,132)(u_{11}u_{12}v_{23}u_{24}+v_{11}v_{12}u_{23}v_{24})\}. \quad \text{(A.4)}$$

In these formulas

$$\psi_\alpha(12;34) = {}^1\!/_{16}\{(J_{1-3}+J_{2-4}+J_{1-4}+J_{2-3})\cos 2\theta$$
$$-4d\sin 2\theta + (-1)^\alpha[(J_1+J_2+J_3+J_4)\cos^2\theta + 2\beta - 2d\sin 2\theta]\},$$
$$\varphi(1,234) = {}^1\!/_{16}\{(J_2+J_3+J_4)\sin^2\theta$$
$$+{}^3\!/_2\beta + {}^3\!/_2 d\sin 2\theta\}, \quad \alpha = 1,2;$$
$$C(12;34) = {}^1\!/_{16}\{(J_{1-3}+J_{2-3}+J_{1-4}+J_{2-4})\cos 2\theta$$
$$-4d\sin 2\theta + (J_1+J_2-J_3-J_4)\cos^2\theta\},$$
$$D(1,2;3,4) = {}^1\!/_{16}\{(J_{1-3}+J_{2-4}-J_{1-4}-J_{2-3})\cos 2\theta$$
$$-(J_1-J_2+J_3-J_4)\cos^2\theta\},$$
$$M(1.234) = {}^1\!/_8\{(J_2-J_3+J_4)\sin^2\theta + {}^1\!/_2\beta + {}^1\!/_2 d\sin 2\theta\}.$$

APPENDIX II

Magnon damping coefficients

We present the calculated damping coefficients $\Gamma_{11}(k)$ and $\Gamma_{12}(k)$ of activationless magnons. At low temperatures ($T \ll \Delta$) we have

$$\Gamma_{11}(k) = J_0\left(\frac{J_0}{\Theta_c}\right)$$

$$
\times\begin{cases}\dfrac{1}{32\pi^2\sqrt{\pi}}\left(\dfrac{\Delta}{\Theta_c}\right)^3\left(\dfrac{T}{\Theta_c}\right)^2\left(\dfrac{T}{\varepsilon_{1k}}\right)^{1/2}\left(1-\exp\left(-\dfrac{\varepsilon_{1k}}{T}\right)\right)\exp\left(-\dfrac{\Delta^2}{\varepsilon_{1k}T}\right), & \varepsilon_{1k}\ll\dfrac{\Delta^2}{T},\\ \dfrac{\pi}{720}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)\left(\dfrac{T}{\Theta_c}\right)^4, & \dfrac{\Delta^2}{T}\ll\varepsilon_{1k}\ll\Theta_c.\end{cases}
$$

The quantity $\Gamma_{12}(\mathbf{k})$ is given by

$$
\Gamma_{12}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)
$$

$$
\times\begin{cases}\dfrac{1}{12\pi^2\sqrt{2\pi}}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)^4\left(\dfrac{T}{\Theta_c}\right)\left(\dfrac{T}{\Delta}\right)^{1/2}e^{-\Delta/T}, & \varepsilon_{1k}\ll T,\quad T\ll\varepsilon_{1k}\ll\Delta\\ \dfrac{1}{96\pi^2\sqrt{2\pi}}\left(\dfrac{\Delta}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^2\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)\left(\dfrac{\Delta}{T}\right)^{1/2}e^{-\Delta/T} & \Delta\ll\varepsilon\ \ \ll\Theta\end{cases}
$$

If $\Delta \ll T \ll \Theta_c$, then

$$
\Gamma_{11}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)
$$

$$
\times\begin{cases}\dfrac{3}{64\pi^3}\left(\dfrac{\Delta}{\Theta_c}\right)^4\left(\dfrac{T}{\Theta_c}\right)\exp\left(-\dfrac{\Delta^2}{\varepsilon_{1k}T}\right), & \varepsilon_{1k}\ll\dfrac{\Delta^2}{T},\\ \dfrac{1}{72\pi}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{\Delta}, & \dfrac{\Delta^2}{T}\ll\varepsilon_{1k}\ll\Delta,\\ \dfrac{1}{72\pi}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{\varepsilon_{1k}}, & \Delta\ll\varepsilon_{1k}\ll T,\\ \dfrac{\pi}{720}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)\left(\dfrac{T}{\Theta_c}\right)^4, & T\ll\varepsilon_{1k}\ll\Theta_c,\end{cases}
$$

and the damping coefficient $\Gamma_{12}(\mathbf{k})$ is given by

$$
\Gamma_{12}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)
$$

$$
\times\begin{cases}\dfrac{4\pi}{75}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)^4\left(\dfrac{T}{\Theta_c}\right)\left(\dfrac{T}{\Delta}\right)^4, & \varepsilon_{1k}\ll\dfrac{\Delta^2}{T},\\ \dfrac{1}{72\pi}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{\Delta}, & \dfrac{\Delta^2}{T}\ll\varepsilon_{1k}\ll\Delta,\\ \dfrac{1}{72\pi}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{\varepsilon_{1k}}, & \Delta\ll\varepsilon_{1k}\ll T,\\ \dfrac{2\pi}{2160}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)\left(\dfrac{T}{\Theta_c}\right)^4, & T\ll\varepsilon_{1k}\ll\Theta_c.\end{cases}
$$

We proceed now to consider the damping coefficients $\Gamma_{21}(\mathbf{k})$, $\Gamma_{22}(\mathbf{k})$, and $\Gamma_{23}(\mathbf{k})$ of magnons with activation. At low temperatures $(T \ll \Delta)$ these quantities are given by the formulas

$$
\Gamma_{21}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)
$$

$$
\times\begin{cases}\dfrac{1}{64\pi^3}\left(\dfrac{\Delta}{\Theta_c}\right)^3\left(\dfrac{T}{\Theta_c}\right)^2e^{-\Delta/T}, & sk\ll T,\\ \dfrac{1}{64\pi^3\sqrt{2\pi}}\left(\dfrac{\Delta}{\Theta_c}\right)^3\left(\dfrac{T}{\Theta_c}\right)^2\dfrac{\sqrt{\Delta T}}{sk}e^{-\Delta/T} & T\ll sk\ll\Delta,\\ \dfrac{1}{64\pi^3\sqrt{2\pi}}\left(\dfrac{\Delta}{\Theta_c}\right)^4\left(\dfrac{T}{\Theta_c}\right)\dfrac{\sqrt{\Delta T}}{sk}e^{-\Delta/T}, & \Delta\ll sk\ll\sqrt{J_0\Delta},\end{cases}
$$

$$
\Gamma_{22}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)\cdot
$$

$$
\times\begin{cases}\dfrac{1}{16\pi^2\sqrt{2\pi}}\left(\dfrac{\Delta}{\Theta_c}\right)^3\left(\dfrac{T}{\Theta_c}\right)^2\left(\dfrac{\Delta}{T}\right)^{1/2}e^{-\Delta/T}, & sk\ll T,\quad T\ll sk\ll\Delta,\\ \dfrac{1}{64\pi^2\sqrt{2\pi}}\left(\dfrac{\Delta}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^2\left(\dfrac{sk}{\Theta_c}\right)\left(\dfrac{\Delta}{T}\right)^{1/2}e^{-\Delta/T}, & \Delta\ll sk\ll\Theta_c,\end{cases}
$$

$$
\Gamma_{23}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)
$$

$$
\times\begin{cases}\dfrac{4\pi}{63}\left(\dfrac{T}{\Theta_c}\right)^5\left(\dfrac{T}{\Delta}\right)^2, & sk\ll T,\quad T\ll sk\ll\Delta,\\ \dfrac{320}{\pi^3}\left(\dfrac{T}{\Theta_c}\right)^5\left(\dfrac{T}{\Delta}\right)^2\left(\dfrac{sk}{\Delta}\right)^4, & \Delta\ll sk\ll\dfrac{\Delta^2}{T},\\ \dfrac{2\pi}{2160}\left(\dfrac{sk}{\Theta_c}\right)\left(\dfrac{T}{\Theta_c}\right)^4, & \dfrac{\Delta^2}{T}\ll sk\ll\Theta_c.\end{cases}
$$

If $\Delta \ll T \ll \Theta_c$, then

$$
\Gamma_{21}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)
$$

$$
\times\begin{cases}\dfrac{1}{64\pi^3}\left(\dfrac{\Delta}{\Theta_c}\right)^4\left(\dfrac{T}{\Theta_c}\right)\ln^2\dfrac{T}{\Delta}, & sk\ll\Delta,\\ \dfrac{1}{64\pi^3}\left(\dfrac{\Delta}{\Theta_c}\right)^3\left(\dfrac{T}{\Theta_c}\right)^2\ln\dfrac{T}{\Delta}, & \Delta\ll sk\ll T,\\ \dfrac{1}{384\pi}\left(\dfrac{\Delta}{\Theta_c}\right)^3\left(\dfrac{T}{\Theta_c}\right)^2\left(\dfrac{\Delta}{sk}\right), & T\ll sk\ll\sqrt{J_0\Delta},\end{cases}
$$

$$
\Gamma_{22}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)
$$

$$
\times\begin{cases}\dfrac{1}{96\pi}\left(\dfrac{\Delta}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{\Delta}, & sk\ll\Delta,\\ \dfrac{1}{72\pi}\left(\dfrac{sk}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{sk}, & \Delta\ll sk\ll T,\\ \dfrac{\pi}{720}\left(\dfrac{sk}{\Theta_c}\right)\left(\dfrac{T}{\Theta_c}\right)^4, & T\ll sk\ll\Theta_c,\end{cases}
$$

$$
\Gamma_{23}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)
$$

$$
\times\begin{cases}\dfrac{1}{96\pi}\left(\dfrac{\Delta}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{\Delta}, & sk\ll\Delta,\\ \dfrac{1}{72\pi}\left(\dfrac{sk}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{sk}, & \Delta\ll sk\ll T,\\ \dfrac{2\pi}{2160}\left(\dfrac{sk}{\Theta_c}\right)\left(\dfrac{T}{\Theta_c}\right)^4, & T\ll sk\ll\Theta_c.\end{cases}
$$

Finally, we present the results for the summary damping coefficients $\gamma_1^{(4)}$ and $\gamma_2^{(4)}(\mathbf{k})$ due to processes with participation of four magnons.

In the case of low temperatures $(T \ll \Delta)$ we have

$$
\gamma_1^{(4)}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)
$$

$$
\times\begin{cases}\dfrac{1}{12\pi^2\sqrt{2\pi}}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)^4\left(\dfrac{T}{\Theta_c}\right)\left(\dfrac{T}{\Delta}\right)^{1/2}e^{-\Delta/T}, & \varepsilon_{1k}\ll T,\quad T\ll\varepsilon_{1k}\ll\Delta,\\ \dfrac{1}{96\pi^2\sqrt{2\pi}}\left(\dfrac{\Delta}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^2\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)\left(\dfrac{\Delta}{T}\right)^{1/2}e^{-\Delta/T}, & \Delta\ll\varepsilon_{1k}\ll\dfrac{\Delta^2}{T},\\ \dfrac{\pi}{720}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)\left(\dfrac{T}{\Theta_c}\right)^4, & \dfrac{\Delta^2}{T}\ll\varepsilon_{1k}\ll\Theta_c,\end{cases}
$$

$$
\gamma_2^{(4)}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)\cdot
$$

$$
\times\begin{cases}\dfrac{4\pi}{63}\left(\dfrac{T}{\Theta_c}\right)^5\left(\dfrac{T}{\Delta}\right)^2, & sk\ll T,\quad T\ll sk\ll\Delta,\\ \dfrac{320}{\pi^3}\left(\dfrac{T}{\Theta_c}\right)^5\left(\dfrac{T}{\Delta}\right)^2\left(\dfrac{sk}{\Delta}\right)^4, & \Delta\ll sk\ll\dfrac{\Delta^2}{T},\\ \dfrac{2\pi}{2160}\left(\dfrac{sk}{\Theta_c}\right)\left(\dfrac{T}{\Theta_c}\right)^4, & \dfrac{\Delta^2}{T}\ll sk\ll\Theta_c.\end{cases}
$$

At high temperatures $\Delta \ll T \ll \Theta_c$, the dependence of the damping coefficients on the wave vector is determined by the formulas

$$
\gamma_1^{(4)}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)
$$

$$
\times\begin{cases}\dfrac{4\pi}{75}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)^4\left(\dfrac{T}{\Theta_c}\right)\left(\dfrac{T}{\Delta}\right)^4, & \varepsilon_{1k}\ll\dfrac{\Delta^2}{T},\\ \dfrac{1}{36\pi}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{\Delta}, & \dfrac{\Delta^2}{T}\ll\varepsilon_{1k}\ll\Delta,\\ \dfrac{1}{36\pi}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{\varepsilon_{1k}}, & \Delta\ll\varepsilon_{1k}\ll T,\\ \dfrac{\pi}{432}\left(\dfrac{\varepsilon_{1k}}{\Theta_c}\right)\left(\dfrac{T}{\Theta_c}\right)^4, & T\ll\varepsilon_{1k}\ll\Theta_c,\end{cases}
$$

$$
\gamma_2^{(4)}(\mathbf{k})=J_0\left(\frac{J_0}{\Theta_c}\right)
$$

$$
\times\begin{cases}\dfrac{1}{48\pi}\left(\dfrac{\Delta}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{\Delta}, & sk\ll\Delta,\\ \dfrac{1}{36\pi}\left(\dfrac{sk}{\Theta_c}\right)^2\left(\dfrac{T}{\Theta_c}\right)^3\ln\dfrac{T}{sk}, & \Delta\ll sk\ll T,\\ \dfrac{\pi}{432}\left(\dfrac{sk}{\Theta_c}\right)\left(\dfrac{T}{\Theta_c}\right)^4, & T\ll sk\ll\Theta_c.\end{cases}
$$

[1)]Expressions for the amplitudes of the canonical transformation of $u_{\beta k}$ and $v_{\beta k}$ ($\beta = 1, 2$) are given in Appendix I.

[2)]By starting from the exact formulas (A.1), (A.3) and (A.4) for the amplitudes of the Hamiltonian $\mathcal{H}_4$, one can show that these amplitudes vanish if the momenta of the activationless particles tend to zero and when relativistic interactions (the magnetic anisotropy energy and the Dzyaloshinskii interaction) are taken into account.

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