Deviation of an expanding universe that is isotropic in the mean from the Friedmann universe

A. V. Byalko

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
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Expansion of a universe, which on the average is isotropic, is considered at the stage of applicability of the equation of state $p=\epsilon/3$. A high-frequency expansion of the Einstein equations is carried out under the assumption that the characteristic wavelengths of the metric deviations from an isotropic one are much smaller than the radius of curvature. Averaging of the derived equations over large scales yields the deviation of the universe expansion from the isotropic Friedmann solution, owing to the reaction of high-frequency metric perturbations. The dependence of the metric perturbation amplitudes in the case of high-frequency expansion differs by a logarithmically slowly varying factor from that obtained from an analysis of linear perturbations.

Different model assumptions are used in the description of the universe during different stages of its expansion. Complete homogeneity and isotropy lead to Friedmann models. A study of homogeneous anisotropic models during earlier stages of expansion reveals the possibility of an oscillatory approach to the singularity (Belinskii, E. M. Lifshitz, Khalatnikov[1]).

As is well known [2], the general solution of Einstein's equations should contain eight coordinate functions; four describe the distribution of the density and velocity of matter at a fixed instant of time, and the four others remain the initial conditions for the gravitational field that is not connected with the matrix—for gravitational waves. It is therefore obvious beforehand that any model with artificial limitations is unstable. This is particularly clearly pronounced when the singularity is approached. An analysis of linear perturbations of a Friedmann world (E. M. Lifshitz[3]) shows that when the cosmological time η is decreased, the amplitudes of the high-frequency perturbations of all three types (scalar $\propto \eta^{-2}$, vector $\propto \eta^{-1}$, and tensor $\propto \eta^{-1}$) increase. In final analysis, it is precisely this increase which causes the transition to the oscillatory approach to the singularity (general solution [4]).

So far no one has succeeded in connecting the parameters of the oscillatory regime of the amplitude of the perturbations of the isotropic world. A study of the isotropization of homogeneous cosmological models, connected with inclusion of matter, has no bearing on the general solutions, since there is not enough leeway in the initial conditions for the homogeneous models.

Einstein's equations are nonlinear. Most nonlinear systems are subject to stochastization [5]. This stochastization consists in the fact that the system, after the lapse of a time equal in order of magnitude to the characteristic time of its development, forgets its initial conditions. If they are included in a fixed volume of phase space at the initial instant of time, then after the stochastization time this initial phase volume spreads out, with a certain probability density, over the entire phase space. This circumstance uncovers a possibility of statistically describing such systems. A clear-cut physical example of stochastization is turbulence that arises mathematically as a result of the instability of degenerate (laminar) solutions of the equations of hydrodynamics.

From this point of view, the question of the initial conditions in cosmology is to a certain degree meaningless: the initial conditions are random functions of the coordinates. On the other hand, interest attaches only to characteristics averaged over space, such as correlation functions and spectral densities. In the present paper it is assumed that isotropy holds in the mean, i.e., the averaging will be carried out with the aid of the base metric of homogeneous and isotropic space.

The problem consists of finding such spectra of cosmological perturbations of the base metric, which develop in the nonlinear region as a result of stochastization, and are therefore self-consistent. Their subsequent evolution during the linear stage changes their amplitudes only in a known manner. Since an isotropic base space is chosen, self-consistency requires that the density of the spectra obtained as a result of the solution of the problem not increase too rapidly with increasing wavelength, to values commensurate with the average scale factor. Namely, we assume that the perturbation spectra, which depend on the dimensionless wave number n, are such that the essential region in the integration over phase space d^3n are wave numbers of order n_0, which we shall assume to be much larger than unity[1)].

A high-frequency expansion of Einstein's equations was first carried out by Brill and Hartle[6] and by Isaacson[7] for problems of a different kind. Its gist lies in the fact that the Einstein-equation terms containing second derivatives of the high-frequency increments to the smooth base metric are relatively large. In the high-frequency expansion in cosmology near a singularity, however, it is necessary to distinguish essentially between the temporal and spatial derivatives. Namely, it turns out that the parameter with respect to which it is possible to expand the Einstein equations is $(n_0\eta)^{-1} \ll 1$.

We represent the space time metric in the form

$$ds^2 = r^2(\eta)\,[d\eta^2 - \gamma_{\alpha\beta}(\mathbf{x},\eta)\,dx^\alpha dx^\beta],$$

$$\gamma_{\alpha\beta} = a_{\alpha\beta}(\mathbf{x}) + \frac{1}{r}\varphi_{\alpha\beta}(\mathbf{x},\eta) + \frac{1}{2r^2}\varphi_\alpha^\gamma\varphi_{\gamma\beta} + \ldots,$$

where $a_{\alpha\beta}$ is the isotropic-space metric with the aid of which the indices of the functions $\varphi_{\alpha\beta}$ are raised and lowered. The powers of r were separated in front of $\varphi_{\alpha\beta}$ to facilitate the estimates, because the amplitudes of the gravitational waves and of the rotational perturbations are proportional to r^{-1}. We use in the estimates the Friedmann expansion law $r \approx R_0\eta$ for an equation of state $p=\epsilon/3$.

Terms of zeroth order (in powers of φ) in Einstein's equations for the mixed components are equal to $r^{-2}\eta^{-2}$. The largest term of first order contains two differentiations and therefore have the quantity $n_0^2 r^{-3}\varphi$. The largest terms, which are quadratic in powers of φ, are of order $n_0^2 r^{-4}\varphi^2$. The instant η_0 of the onset of the essential nonlinearity is determined by the condition that all the terms of the expansion become of the same order. Comparison of the orders of magnitude of these terms at $\eta \sim \eta_0$ yields

$$\eta_0 \sim n_0^{-1}, \quad \varphi \sim R_0\eta_0 \sim R_0 n_0^{-1}$$

Thus, the principal terms $\sim n_0 R_0^{-2}\eta^{-3}$ in the Einstein equations are those linear in φ and containing two high-frequency differentiations. The terms of next order $R_0^{-2}\eta^{-4}$ result: 1) from the terms of zeroth order in φ, 2) from terms linear in φ and containing one high-frequency ($\sim n_0$) and one smooth ($\sim \eta^{-1}$) differentiation, and 3) from terms quadratic in φ and containing two differentiations $\sim n_0$. As will be shown below, the presented estimates pertain to increments of the tensor and vector type, while the scalar ones are of one order lower in $(n_0\eta)^{-1}$. Finally, terms of magnitude $R_0^{-2}n_0^{-1}\eta^{-5}$ and of higher order in $(n_0\eta)^{-1}$ are of no interest to us.

It must be emphasized beforehand that the result of this expansion does not coincide with the linear analysis of E. M. Lifshitz[3], since the amplitudes of the perturbations are considered in it to be so small that any quadratic term is much smaller than the linear terms; in our case, on the other hand, the order of magnitude of the amplitudes is determined by the instant η_0 when the essential nonlinearity sets in. On the other hand, as seen from the estimates, the terms of zeroth order in φ, which determine the time dependence of the scale factor $r(\eta)$, are of the same order of magnitude as the terms quadratic in φ. This makes it possible to establish for the expansion of the world a law that differs more from the Friedmann law the closer the time is to the instant η_0.

The region of applicability of the asymptotic oscillatory solution of Belinskiĭ, E. M. Lifshitz, and Khalatnikov (see Fig. 1), can be determined in this formulation of the problem from similar qualitative considerations. As seen from the derivation of the equations describing the general solution [4], most spatial derivatives in Einstein's equations are neglected in comparison with the time derivatives. If the characteristic wave number does not change too strongly on passing through the region $\eta \sim \eta_0$, then the onset of the asymptotic oscillatory regime requires the satisfaction of the relation $\eta \ll 1/n \sim \eta_0$.

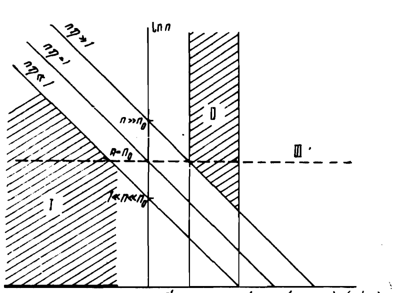

FIG. 1. Regions of applicability: I—of the general solution of Belinskiĭ, E. M. Lifshitz, and Khalatnikov $n\eta \ll 1$; thick line—homogeneous models; II—of high-frequency expansion of the present paper: $n\eta \gg 1$, $\eta_0 \sim n_0^{-1} \ll \eta \ll 1$; III—of the Friedmann solution with perturbations.

It is interesting to note that the simultaneous energy density of gravitational waves of cosmological origin, estimated with allowance for $\varphi \sim n_0^{-1}R_0$, does not depend on the parameter n_0 and is equal to

$$\varepsilon_g \sim \frac{c^4}{16\pi G}\frac{n_0^2\varphi^2}{r^4} \sim \frac{c^4 R_0^2}{16\pi G r^4} \sim 10^{-14}\ \mathrm{erg/cm^3}.$$

Here $r \sim 10^{28}$ cm is the present-day value of the scale factor, and $R_0 \sim 10^{25}$ cm. This value of the gravitational-wave energy density is close in order of magnitude to the energy density of the relict black radiation ($T = 2.7°$ K): $\epsilon_r = 4\sigma T^4/c = 4.05\times 10^{-13}$ erg/cm^3.

1. CORRELATION RELATIONS

We consider the average, over the asymptotic space, of a product of arbitrary symmetrical tensors—random functions of the coordinates—taken at different points of space:

$$\langle f_\alpha^\beta(\mathbf{x}) f_\gamma^\delta(\mathbf{x}+\mathbf{R})\rangle = F_{\alpha\gamma}^{\beta\delta}(\mathbf{R}). \tag{1.1}$$

The symmetry of the left-hand side of (1.1) leads to the conclusion that the tensor $F_{\alpha\gamma}^{\beta\delta}$ is expressed in terms of five scalar functions of $R = |\mathbf{R}|$, corresponding to the number of independent scalar projections: $F_1 = F_{\alpha\gamma}^{\alpha\gamma}$

$$F_2 = F_{\alpha\gamma}^{\beta\gamma}\rho^\alpha\rho_\beta, \quad F_3 = F_{\alpha\gamma}^{\beta\delta}\rho^\alpha\rho_\beta\rho^\gamma\rho_\delta, \quad F_4 = F_{\alpha\beta}^{\beta\delta}\rho^\alpha\rho_\delta,$$
$$F_5 = F_{\alpha\beta}^{\beta\alpha}, \quad \rho_\alpha = R_\alpha/R.$$

We expand the tensor $f_\alpha^\beta(\mathbf{x})$ in a Fourier integral

$$f_\alpha^\beta(\mathbf{x}) = (2\pi)^{-3}\int f_\alpha^\beta(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}\,d^3n. \tag{1.2}$$

Substituting this expansion in (1.1), we easily verify that in order for the correlator $F_\alpha^{\delta\lambda}$ to depend only on $\mathbf{R}$, it is necessary to have

$$\langle f_\alpha^\beta(\mathbf{n}) \overset{*}{f}{}_\gamma^\delta(\mathbf{n}_1)\rangle = F_{\alpha\gamma}^{\beta\delta}(n)\delta(\mathbf{n}-\mathbf{n}_1), \tag{1.3}$$

$$F_{\alpha\gamma}^{\beta\delta}(\mathbf{R}) = (2\pi)^{-3}\int F_{\alpha\gamma}^{\beta\delta}(\mathbf{n})e^{i\mathbf{n}\mathbf{R}}\,d^3n. \tag{1.4}$$

The function $F_{\alpha\gamma}^{\beta\delta}(\mathbf{n})$—the spectral density—can be expressed, in analogy with the correlator, in terms of five scalar spectra, functions of $n = |\mathbf{n}|$. We write this expansion in the form

$$\begin{aligned} n^2F_{\alpha\gamma}^{\beta\delta}(\mathbf{n}) = {} & A(n)\delta_\alpha^\beta\delta_\gamma^\delta + E(n)(\delta_\alpha^\beta\nu_\gamma\nu^\delta + \delta_\gamma^\delta\nu_\alpha\nu^\beta) + B(n)\nu_\alpha\nu^\beta\nu_\gamma\nu^\delta \\ & + C(n)(\delta_\alpha^\delta\nu^\beta\nu_\gamma + \delta_\gamma^\beta\nu_\alpha\nu^\delta + a_{\alpha\gamma}\nu^\beta\nu^\delta + a^{\beta\delta}\nu_\alpha\nu_\gamma - 4\nu_\alpha\nu^\beta\nu_\gamma\nu^\delta) \\ & + D(n)(-\delta_\alpha^\beta\delta_\gamma^\delta + \delta_\alpha^\delta\delta_\gamma^\beta + a_{\alpha\gamma}a^{\beta\delta} + \delta_\alpha^\beta\nu_\gamma\nu^\delta + \delta_\gamma^\delta\nu_\alpha\nu^\beta - \delta_\alpha^\delta\nu_\gamma\nu^\beta - \delta_\gamma^\beta\nu_\alpha\nu^\delta \\ & - a_{\alpha\gamma}\nu^\beta\nu^\delta - a^{\beta\delta}\nu_\alpha\nu_\gamma + \nu_\alpha\nu^\beta\nu_\gamma\nu^\delta), \quad \nu_\alpha = n_\alpha/n. \end{aligned} \tag{1.5}$$

This choice of the expansion is connected with the representation $f_\alpha^\beta(\mathbf{n})$ in the form of a sum of terms of three types: scalar, vector, and tensor:

$$\begin{aligned} f_\alpha^\beta(\mathbf{n}) = a(\mathbf{n})\delta_\alpha^\beta + b(\mathbf{n})\nu_\alpha\nu^\beta + c_\alpha(\mathbf{n})\nu^\beta + c^\beta(\mathbf{n})\nu_\alpha + d_\alpha^\beta(\mathbf{n}), \\ c_\alpha\nu^\alpha = d_\alpha^\alpha = d_\alpha^\beta\nu_\beta = 0. \end{aligned} \tag{1.6}$$

This representation corresponds to E. M. Lifshitz's[3] expansion of the metric perturbation into scalar and rotational perturbations and gravitational waves. The quadratic mean values of the random functions $a(\mathbf{n})$, $b(\mathbf{n})$, $c_\alpha(\mathbf{n})$, and $d_\alpha^\beta(\mathbf{n})$ are expressed in terms of the scalar spectra

$$\langle a(\mathbf{n})\overset{*}{a}(\mathbf{n}_1)\rangle = A(n)n^{-2}\delta(\mathbf{n}-\mathbf{n}_1), \quad \langle a(\mathbf{n})\overset{*}{b}(\mathbf{n}_1)\rangle = E(n)n^{-2}\delta(\mathbf{n}-\mathbf{n}_1),$$
$$\langle b(\mathbf{n})\overset{*}{b}(\mathbf{n}_1)\rangle = B(n)n^{-2}\delta(\mathbf{n}-\mathbf{n}_1), \quad \langle c_\alpha(\mathbf{n})\overset{*}{c}{}^\beta(\mathbf{n}_1)\rangle = C(n)n^{-2}v_\alpha^\beta\delta(\mathbf{n}-\mathbf{n}_1),$$
$$\langle d_\alpha^\beta(\mathbf{n})\overset{*}{d}{}_\gamma^\delta(\mathbf{n}_1)\rangle = D(n)n^{-2}u_{\alpha\gamma}^{\beta\delta}\delta(\mathbf{n}-\mathbf{n}_1), \quad \langle a(\mathbf{n})\overset{*}{c}_\alpha(\mathbf{n}_1)\rangle$$

$$= \langle b(\mathbf{n})\dot{c}_\alpha(\mathbf{n}_1)\rangle = \langle a(\mathbf{n})\dot{d}_\alpha^\beta(\mathbf{n}_1)\rangle = \langle b(\mathbf{n})\dot{d}_\alpha^\beta(\mathbf{n}_1)\rangle = \langle c_\alpha(\mathbf{n})\dot{d}_\gamma^\delta(\mathbf{n}_1)\rangle = 0,$$
$$v_\alpha^\beta = \delta_\alpha^\beta - \nu_\alpha\nu^\beta, \qquad u_{\alpha\gamma}^{\beta\delta} = -v_\alpha^\beta v_\gamma^\delta + v_\alpha^\delta v_\gamma^\beta + v_{\alpha\gamma}v^{\beta\delta}. \tag{1.7}$$

Thus, the isotropy in the mean leads to the absence of crossing average tensors of different types. Omitting the simple derivation we write down the expressions connecting the scalar projections $F_1, \ldots, F_5$ and the spectra $A(n)$, $B(n)$, $C(n)$, $D(n)$, $E(n)$:

$$F_1(R) = \frac{1}{2\pi^2}\int_0^\infty (9A + 6E + B)\frac{\sin nR}{nR}\,dn,$$

$$F_2(R) = \frac{1}{2\pi^2}\int_0^\infty \Big[(3A+E)\frac{\sin nR}{nR}$$
$$+ (3E+B)\Big(\frac{\sin nR}{nR} - \frac{2\cos nR}{(nR)^2} + \frac{2\sin nR}{(nR)^3}\Big)\Big]\,dn,$$

$$F_3(R) = \frac{1}{2\pi^2}\int_0^\infty \Big[(A+D)\frac{\sin nR}{nR} + (2E + 4C - 2D)\Big(\frac{\sin nR}{nR} - \frac{2\cos nR}{(nR)^2}$$
$$+ \frac{2\sin nR}{(nR)^3}\Big) + (B - 4C + D)\Big(\frac{\sin nR}{nR} + \frac{4\cos nR}{(nR)^2} \tag{1.8}$$
$$- \frac{12\sin nR}{(nR)^3} - \frac{24\cos nR}{(nR)^4} + \frac{24\sin nR}{(nR)^5}\Big)\Big]\,dn,$$

$$F_4(R) = \frac{1}{2\pi^2}\int_0^\infty \Big[(3A - C)\frac{\sin nR}{nR}$$
$$+ (2E + B + C - 2D)\Big(\frac{\sin nR}{nR} - \frac{2\cos nR}{(nR)^2} + \frac{2\sin nR}{(nR)^3}\Big)\Big]\,dn,$$

$$F_5(R) = \frac{1}{2\pi^2}\int_0^\infty (3A + 2E + B + 4C + 4D)\frac{\sin nR}{nR}\,dn.$$

The eight coordinate functions of the general solution of Einstein's equations could yield in principle 28 correlation functions. The assumption of the isotropy of space in the mean limits their number to five.

2. EXPANSION OF EINSTEIN'S EQUATIONS IN POWERS OF $(n_0\eta)^{-1}$

We shall use the equation of state $p = \epsilon/3$, and effect the operation of raising the indices with the aid of the positive-definite metric $\gamma^{\alpha\beta} = -r^2 g^{\alpha\beta}$. We introduce the tensor $\lambda_{\alpha\beta} = \gamma'_{\alpha\beta} \equiv \partial\gamma_{\alpha\beta}/\partial\eta$. Then Einstein's equations take the form

$$R_0^0 = \frac{3}{r^4}(r'^2 - rr'') - \frac{1}{2r^2}\lambda' - \frac{r'}{2r^3}\lambda - \frac{1}{4r^2}\lambda_\alpha^\beta\lambda_\beta^\alpha = \frac{\varkappa\varepsilon}{3}\Big(4\frac{u_0^2}{r^2} - 1\Big), \tag{2.1}$$

$$R_\alpha^0 = \frac{1}{2r^2}(\lambda_{\alpha;\beta}^\beta - \lambda_{;\alpha}) = \frac{4\varkappa\varepsilon}{3r^2}u_0u_\alpha, \tag{2.2}$$

$$R_\alpha^\beta = -\frac{1}{r^2}P_\alpha^\beta - \frac{\delta_\alpha^\beta}{r^4}(rr'' + r'^2) - \frac{1}{2r^2}\lambda_\alpha^{\beta\prime} - \frac{r'}{2r^3}(2\lambda_\alpha^\beta + \lambda\delta_\alpha) - \frac{\lambda\lambda_\alpha^\beta}{4r^2}$$
$$= -\frac{\varkappa\varepsilon}{3}\Big(4\frac{u_\alpha u^\beta}{r^2} + \delta_\alpha^\beta\Big). \tag{2.3}$$

We express the energy density ϵ of the matter and the velocity components u_0 and u_α in terms of R_i^k, and consequently in terms of the metric tensor. We put

$$Z_\alpha^\beta = -R_\alpha^0 R_0^\beta/(R_0^0)^2; \qquad Z = Z_\alpha^\alpha > 0.$$

Then Eqs. (2.1) and (2.2), together with the condition

$$u_iu^i = \frac{1}{r^2}(u_0^2 - u_\alpha u^\alpha) = 1$$

yield

$$\varkappa\varepsilon = R_0^0[(4-3Z)^{1/2} - 1] \approx R_0^0(1 - 3Z/4), \tag{2.4}$$

$$u_0^2 = r^2[2 - Z + (4-3Z)^{1/2}]/4(1-Z) \approx r^2(1 + 9Z/16), \tag{2.5}$$

$$u_\alpha u^\alpha = r^2[-2 + 3Z + (4-3Z)^{1/2}]/4(1-Z) \approx 9r^2Z/16. \tag{2.6}$$

We have presented the first terms of the expansion in powers of Z. Since $\epsilon > 0$ and $u_0^2 > 0$, we have $0 < Z < 1$.

Equation (2.3) serves to determine the metric tensor, and when account is taken of (2.4)–(2.6), it takes the form

$$R_\alpha^\beta + {}^1/_3\delta_\alpha^\beta R_0^0 = R_0^0({}^1/_3 Z\delta_\alpha^\beta - Z_\alpha^\beta)\frac{2 - (4-3Z)^{1/2}}{Z}. \tag{2.7}$$

We note that the fraction in the right-hand side of (2.7) ranges merely from 3/4 at $Z = 0$ to unity at $Z = 1$, and the contraction of (2.7) yields, as it should, $R_i^i = R_\alpha^\alpha + R_0^0 = 0$.

We now represent the tensor $\gamma_{\alpha\beta}$ in the form of a sum of an isotropic basic metric $a_{\alpha\beta}$ and an arbitrary metric that depends on the spatial coordinates and on η:

$$\gamma_{\alpha\beta} = a_{\alpha\beta}(\mathbf{x}) + h_{\alpha\beta}(\mathbf{x}, \eta),$$

and raise the indices with the aid of $a^{\alpha\beta}$. The contravariant metric tensor $\gamma^{\alpha\beta}$ is written in the form

$$\gamma^{\alpha\beta} = a^{\alpha\beta} + H^{\alpha\beta}(\mathbf{x}, \eta).$$

Then the condition $\gamma_{\alpha\beta}\gamma^{\beta\delta} = \delta_\alpha^\delta$ yields an equation connecting h_α^β and H_α^β:

$$h_\alpha^\beta + H_\alpha^\beta + h_\alpha^\gamma H_\gamma^\beta = 0.$$

It is most convenient to expand Einstein's equations in powers of the function $r^{-1}\varphi_\alpha^\beta$ (which coincides in the linear approximation with $h_\alpha^\beta \approx -H_\alpha^\beta$), so that $\langle\varphi_\alpha^\beta\rangle = 0$. To determine this function it is necessary to impose an additional condition on the mean values $\langle h_\alpha^\beta\rangle$ and $\langle H_\alpha^\beta\rangle$. We choose this condition to be that the co- and contravariant metric tensors $\gamma_{\alpha\beta}$ and $\gamma^{\alpha\beta}$ be on a par:

$$\langle h_\alpha^\beta\rangle = \langle H_\alpha^\beta\rangle. \tag{2.8}$$

Using these relations, we express everything in terms of the tensor φ_α^β:

$$h_\alpha^\beta = \frac{1}{r}\varphi_\alpha^\beta + \frac{1}{2r^2}\varphi_\alpha^\gamma\varphi_\gamma^\beta - \frac{1}{8r^4}\varphi_\alpha^\gamma\varphi_\gamma^\delta\varphi_\delta^\varepsilon\varphi_\varepsilon^\beta + \ldots,$$
$$H_\alpha^\beta = -\frac{1}{r}\varphi_\alpha^\beta + \frac{1}{2r^2}\varphi_\alpha^\gamma\varphi_\gamma^\beta - \frac{1}{8r^4}\varphi_\alpha^\gamma\varphi_\gamma^\delta\varphi_\delta^\varepsilon\varphi_\varepsilon^\beta + \ldots, \tag{2.9}$$
$$\lambda_\alpha^\beta = \Big(\frac{1}{r}\varphi_\alpha^\beta\Big)' + \frac{1}{2r^2}(\varphi_\alpha^\gamma\varphi_\gamma^{\beta\prime} - \varphi_\alpha^{\gamma\prime}\varphi_\gamma^\beta) - \frac{1}{2r^2}\varphi_\alpha^\gamma\Big(\varphi_\gamma^\delta\frac{1}{r}\Big)'\varphi_\delta^\beta + \ldots,$$
$$\ln|\gamma| = \int\lambda_\alpha^\alpha\,d\eta = \mathrm{const} + \frac{1}{r}\varphi_\alpha^\alpha - \frac{1}{6r^3}\varphi_\alpha^\gamma\varphi_\gamma^\delta\varphi_\delta^\alpha + \ldots.$$

We substitute the obtained expansions in the equation for the metric tensor (2.7), and retain in it terms up to second order in φ_α^β inclusive, containing not less than two differentiations in second order. This produces in the right-hand side of (2.7) a denominator

$${}^1/_3 r^2R_{00} = r'^2 - rr'' - {}^1/_6 r\varphi_\alpha^{\alpha\prime\prime} - {}^1/_{24}\varphi_\alpha^{\gamma\prime}\varphi_\gamma^{\alpha\prime} = W. \tag{2.10}$$

The order of magnitude of the terms retained here, disregarding their coefficients, is the same. Thus, W has a regular component and a random part. Since the numerator of the right-hand side of (2.7) is quadratic in φ_α^β, and since R_{00} is rigorously larger than zero and its random part has a small coefficient, we shall approximately regard $r^2R_{00}/3$ as a regular function of $W(\eta)$, equal to the mean value $r^2\langle R_{00}\rangle/3$.

We change over to the Fourier representation; in this case the covariant derivatives are transformed as ordinary quotients, since

$$\varphi_{\alpha;\gamma}^\beta(\mathbf{x}) = \int e^{i\mathbf{n}\mathbf{x}}(in_\gamma\varphi_\alpha^\beta(\mathbf{n}) + \Gamma_{\gamma\delta}^\beta\varphi_\alpha^\delta - \Gamma_{\alpha\gamma}^\delta\varphi_\delta^\beta)\frac{d^3n}{(2\pi)^3}$$
$$= i(1 + O(n_0^{-1}))\int n_\gamma\varphi_\alpha^\beta e^{i\mathbf{n}\mathbf{x}}\frac{d^3n}{(2\pi)^3}.$$

$\Gamma^{\beta}_{\gamma\delta} \sim 1$ are Christoffel symbols calculated with the metric $a_{\alpha\beta}$. The terms quadratic in φ^{β}_{α} in the Fourier representations take the form

$$[\varphi^{\beta}_{\alpha}\varphi^{\delta}_{\gamma}]_{n} = \int (2\pi)^{-3} d^3l\, d^3m\, \delta(\mathbf{n}-\mathbf{l}-\mathbf{m})\, \varphi^{\beta}_{\alpha}(\mathbf{l})\, \varphi^{\delta}_{\gamma}(\mathbf{m}).$$

As a result, the equation for the increments to the metric takes the form

$$\begin{aligned}
&\varphi^{\beta\prime\prime}_{\alpha} + \frac{\delta_{\alpha}}{3}\left(\varphi'' + \frac{2r'}{r}\varphi'\right) + n^2\varphi^{\beta}_{\alpha} - n_{\alpha}n^{\delta}\varphi^{\beta}_{\delta} - n^{\beta}n_{\delta}\varphi^{\delta}_{\alpha} + n_{\alpha}n^{\beta}\varphi + 4r''\delta^{\beta}_{\alpha}(2\pi)^3\delta(\mathbf{n}) \\
&= \frac{1}{2r}\int d\Gamma\Big\{\varphi^{\delta\prime\prime}_{\alpha}\tilde{\varphi}^{\beta}_{\delta} - \varphi^{\delta}_{\alpha}\tilde{\varphi}^{\beta\prime\prime}_{\delta} - \varphi'\tilde{\varphi}^{\beta\prime}_{\alpha} - \frac{\delta^{\beta}_{\alpha}}{3}\varphi^{\delta\prime}_{\gamma}\tilde{\varphi}^{\gamma\prime}_{\delta} - n^2\varphi^{\gamma}_{\alpha}\tilde{\varphi}^{\beta}_{\gamma} \\
&\quad + n^{\beta}n_{\delta}\varphi^{\gamma}_{\alpha}\tilde{\varphi}^{\delta}_{\gamma} + n_{\alpha}n^{\delta}\varphi^{\beta}_{\gamma}\tilde{\varphi}^{\gamma}_{\delta} + 2n_{\delta}\varphi^{\delta}_{\gamma}(\tilde{\varphi}^{\beta}_{\alpha}m^{\gamma} - \tilde{\varphi}^{\gamma}_{\alpha}m^{\beta} - \tilde{\varphi}^{\gamma\beta}m_{\alpha}) \\
&\quad - (\varphi_{\alpha\delta}l^{\gamma} - \varphi^{\gamma}_{\alpha}l_{\delta} - \varphi^{\gamma}_{\delta}l_{\alpha})(\tilde{\varphi}^{\beta}_{\gamma}m^{\delta} - \tilde{\varphi}^{\delta}_{\gamma}m^{\beta} - \tilde{\varphi}^{\delta\beta}m_{\gamma})\Big\} \\
&+ \frac{r^2}{8W}\int d\Gamma\Big\{\frac{\delta^{\beta}_{\alpha}}{3}\Big[\Big(\frac{\varphi^{\gamma}_{\varepsilon}}{r}\Big)' l_{\gamma} - \Big(\frac{\varphi}{r}\Big)' l_{\varepsilon}\Big]\Big[\Big(\frac{\tilde{\varphi}_{\delta}}{r}\Big)' m^{\delta} - \Big(\frac{\tilde{\varphi}}{r}\Big)' m^{\varepsilon}\Big] \\
&\quad - \Big[\Big(\frac{\varphi^{\gamma}_{\alpha}}{r}\Big)' l_{\gamma} - \Big(\frac{\varphi}{r}\Big)' l_{\alpha}\Big]\Big[\Big(\frac{\tilde{\varphi}^{\beta}_{\delta}}{r}\Big)' m^{\delta} - \Big(\frac{\tilde{\varphi}}{r}\Big)' m^{\beta}\Big]\Big\}.
\end{aligned} \tag{2.11}$$

Here $d\Gamma = (2\pi)^{-3}d^3l\,d^3m\,\delta(\mathbf{n}-\mathbf{l}-\mathbf{m})$; φ^{β}_{α} under the integral sign has the argument $\mathbf{l}$, and the tensors with the tilde $\tilde{\varphi}^{\beta}_{\alpha}$ depend on $\mathbf{m}$.

The term proportional to $\delta(\mathbf{n})$, which is the zeroth approximation in φ^{β}_{α}, leads to the Friedmann solution $r(\eta) = R_0\eta$. The solution of Eq. (2.11) in the linear approximation (neglecting the right-hand side) was obtained by E. M. Lifshitz. If we represent $\varphi^{\beta}_{\alpha}(\mathbf{n})$ in the form (1.6)—expansion by types—then the equations for $c_{\alpha}(\mathbf{n})$ and $d^{\beta}_{\alpha}(\mathbf{n})$ are obtained directly, and for the equations describing the functions $a(\mathbf{n})$ and $b(\mathbf{n})$ it is necessary to take the scalar projections of the left-hand side of (2.11), which is multiplied by δ^{α}_{β} and $\nu^{\alpha}\nu_{\beta}$.

E. M. Lifshitz's solution at $p = \epsilon/3$ and $n\eta \gg 1$ is[2)]

$$\varphi^{\beta}_{\alpha}(\mathbf{n},\eta) = (\delta^{\beta}_{\alpha} + \nu_{\alpha}\nu^{\beta})Q(\mathbf{n})\eta^{-1}\exp\{in\eta/\sqrt{3}\} + S_{\alpha}(\mathbf{n})\nu^{\beta} + S^{\beta}(\mathbf{n})\nu_{\alpha} + G^{\beta}_{\alpha}(\mathbf{n})e^{in\eta},$$
$$S_{\alpha}\nu^{\alpha} = G^{\alpha}_{\alpha} = G^{\beta}_{\alpha}\nu_{\beta} = 0. \tag{2.12}$$

To abbreviate the notation, we have not written out the complex-conjugate terms. The perturbations (2.12) are the general solution of Einstein's equation, since they contain eight arbitrary but small functions. The condition for applicability of (2.12) is that the functions $Q(\mathbf{n})$, $S_{\alpha}(\mathbf{n})$, and $G^{\beta}_{\alpha}(\mathbf{n})$ be small enough to neglect the right-hand side of (2.11), i.e.,

$$\begin{aligned}
&R_0 Q(n_0) \gg n_0^2\int Q^2(l)\,d^3l \sim n_0^5 Q^2(n_0), \qquad |S_{\alpha}(n_0)| \gg r^{-1}\int S_{\gamma}S^{\gamma}\,d^3l \\
&\sim r^{-1}|S_{\alpha}(n_0)|^2 n_0^3, \qquad |G^{\beta}_{\alpha}(n_0)| \gg r^{-1}|G^{\beta}_{\alpha}(n_0)|^2 n_0^3,
\end{aligned} \tag{2.13}$$

n_0 is the characteristic wave number of the functions $Q(\mathbf{n})$, $S_{\alpha}(\mathbf{n})$, and $G^{\beta}_{\alpha}(\mathbf{n})$.

The difference between the criterion for the applicability of $Q(\mathbf{n})$ and the inequalities for S_{α} and G^{β}_{α} is connected with the tensor structure of the second integral in (2.11), an integral that stems from the right-hand side of (2.7). It contains a small number of differentiations (four), but vanishes identically for the function G^{β}_{α} and gives the same order of magnitude for the terms quadratic in S_{α} as the first integral, since the vector perturbations do not depend on the time in the first-order approximation.

We, to the contrary, will not assume that these inequalities are satisfied and assume, in accordance with the Introduction, that the right and left hand sides of (2.13) are of the same order of magnitude. Then the terms of principal $(n_0^{-2}R_0)$ order in (2.11) yield for the vector function $c_{\alpha}(\mathbf{n},\eta)$ and tensor function $d^{\beta}_{\alpha}(\mathbf{n},\alpha)$ equations coinciding with the corresponding equations of the linear approximation. Their solutions are

$$c_{\alpha}(\mathbf{n},\eta) = c_{\alpha}(\mathbf{n}), \qquad d^{\beta}_{\alpha}(\mathbf{n},\eta) \sim e^{in\eta}.$$

We therefore seek a solution of the next-higher approximation of (2.11) in the form

$$\begin{aligned}
&\varphi^{\beta}_{\alpha}(\mathbf{n},\eta) = \gamma(\eta)\{d^{\beta}_{\alpha}(\mathbf{n})e^{in\eta} + c_{\alpha}(\mathbf{n})\nu^{\beta} + c^{\beta}(\mathbf{n})\nu_{\alpha} \\
&+ r^{-1}e^{inq(\eta)}[a(\mathbf{n})\delta^{\beta}_{\alpha} + b(\mathbf{n})\nu_{\alpha}\nu^{\beta}]\}, \qquad c_{\alpha}\nu^{\alpha} = d^{\alpha}_{\alpha} = d^{\beta}_{\alpha}\nu_{\beta} = 0.
\end{aligned} \tag{2.14}$$

Here $a(\mathbf{n})$, $b(\mathbf{n})$, $c_{\alpha}(\mathbf{n})$, and $d^{\beta}_{\alpha}(\mathbf{n})$ are random functions, and the functions $\gamma(\eta)$ and $q(\eta)$, which are of the order of unity, separate the slow $(\gamma' \sim \eta^{-1}\gamma)$ dependence on η. The orders of magnitude of the wave-vector functions are: $a(n_0) \sim b(n_0) \sim n_0^{-5}R_0^2$, $|c_{\alpha}(n_0)| \sim |d^{\beta}_{\alpha}(n_0)| \sim n_0^{-4}R_0$.

Substituting the expansion in terms of the types (2.14) in (2.11), we obtain

$$\begin{aligned}
&4rr''\delta^{\beta}_{\alpha}(2\pi)^3\delta(\mathbf{n}) + \gamma a n^2[(1-2q'^2)\delta^{\beta}_{\alpha} + \nu_{\alpha}\nu^{\beta}]e^{inq} + 2inr\gamma' d^{\beta}_{\alpha}e^{in\eta} \\
&- \gamma b n^2 q'^2({}^1\!/_3\delta^{\beta}_{\alpha} + \nu_{\alpha}\nu^{\beta})e^{inq} = \frac{\gamma^2}{2}\int d\Gamma\,\{d^{\delta}_{\gamma}\tilde{d}^{\gamma}_{\delta}({}^1\!/_3\delta^{\beta}_{\alpha}lm - l_{\alpha}m^{\beta}) \\
&+ d^{\gamma}_{\alpha}\tilde{d}^{\delta}_{\gamma}n_{\delta}(m^{\beta} - l^{\beta}) + d^{\gamma}_{\delta}\tilde{d}^{\beta}_{\gamma}n^{\delta}(l_{\alpha} - m_{\alpha}) + 2(d^{\beta}_{\alpha}\tilde{d}^{\delta}_{\gamma} - d^{\delta}_{\alpha}\tilde{d}^{\beta}_{\gamma})n_{\delta}n^{\gamma}\}e^{i(l+m)\eta} \\
&+ \frac{i\gamma^2}{2}\int\frac{d\Gamma}{l}\{-n^2c_{\alpha}\tilde{d}^{\beta}_{\gamma}n^{\gamma} + c^{\gamma}\tilde{d}^{\beta}_{\gamma}(n^2l_{\alpha} - 2m^2l_{\alpha} - n_{\delta}l^{\delta}n_{\alpha}) + c_{\alpha}\tilde{d}^{\beta}_{\gamma}n^{\gamma}n_{\delta}n^{\beta} \\
&+ c^{\beta}\tilde{d}^{\delta}_{\gamma}n^{\gamma}n_{\delta}n_{\alpha} - c^{\delta}\tilde{d}^{\gamma}_{\alpha}n_{\delta}n_{\gamma}n^{\beta} - c^{\delta}\tilde{d}^{\beta}_{\gamma}n_{\delta}n^{\gamma}n_{\alpha} + c^{\beta}\tilde{d}^{\gamma}_{\alpha}n_{\gamma}(2m^2 - n^2) \\
&+ c_{\gamma}\tilde{d}^{\gamma}_{\alpha}(n^2l^{\beta} - n_{\delta}l^{\delta}n^{\beta}) - c^{\gamma}\tilde{d}^{\delta}_{\gamma}n_{\delta}(n_{\alpha}l^{\beta} + n^{\beta}l_{\alpha}) + 2c^{\gamma}\tilde{d}^{\beta}_{\alpha}n_{\gamma}(n^2 - m^2)\}e^{imn} \\
&- \frac{\gamma^2}{2}\int\frac{d\Gamma}{lm}\{c_{\alpha}\tilde{c}^{\gamma}n_{\gamma}(n^2l^{\beta} - l^2n^{\beta}) + c_{\gamma}\tilde{c}^{\beta}n^{\gamma}(n^2m_{\alpha} - m^2n_{\alpha}) \\
&- c_{\alpha}\tilde{c}^{\beta}n^2l_{\delta}m^{\delta} - c_{\gamma}\tilde{c}^{\delta}n_{\delta}n^{\gamma}n_{\alpha}n^{\beta} - 2c_{\gamma}\tilde{c}^{\gamma}(m^2l_{\alpha}l^{\beta} + m_{\delta}l^{\delta}m_{\alpha}m^{\beta})\} \\
&+ \frac{\gamma^2}{8W}\int d\Gamma\, lm\Big\{4q'^2a\tilde{a}\Big(l_{\alpha}m^{\beta} - \frac{1}{3}\delta^{\beta}_{\alpha}l_{\gamma}m^{\gamma}\Big)e^{i(l+m)q} - 2iq'r'(a\tilde{c}^{\beta}l_{\alpha} + a\tilde{c}_{\alpha}l^{\beta} \\
&- 2a\tilde{c}^{\gamma}n_{\gamma}\delta^{\beta}_{\alpha})e^{ilq} - r'^2(c_{\alpha}\tilde{c}^{\beta} - {}^1\!/_3\delta^{\beta}_{\alpha}c_{\gamma}\tilde{c}^{\gamma})\Big\}.
\end{aligned} \tag{2.15}$$

The functions without the tilde under the integral sign depend on $\mathbf{l}$, and those with the tilde depend on $\mathbf{m}$. In the differentiation of the functions under the integral sign in (2.11) with respect to η, we took into account only the rapidly-oscillating exponential dependence. The terms of principal order in $(n_0\eta)^{-1}$ cancel out, and all the terms of the equation (2.15) are of the order $n_0^{-3}R_0^2$.

Let us examine the order of magnitude of the discarded terms. In the left-hand, linear part we have omitted the term $r\gamma'' c_{\alpha}\nu^{\beta} \sim R_0^2 n_0^{-4}\eta^{-1}$ and the term proportional to $(br^{-1})'$, which is of the same order of magnitude. The terms proportional to a and b from the first integral of (2.11) are also of the next order of magnitude in $(n_0\eta)^{-1}$.

3. AVERAGING

Equation (2.15) describes the behavior of the random functions in the high-frequency approximation. Physical interest attaches, however, not to the solution of this equation in the exact sense, but to a determination of the space-averaged characteristics of the random functions—spectra and correlators—and to a determination of the time dependences of the scale factor $r(\eta)$ and of the amplitude $\gamma(\eta)$.

Following Sec. 1, it is necessary to average Eq. (2.15). Direct averaging yields an equation for the scale factor. In the right-hand, nonlinear part of (2.15), this gives rise to a factor $\delta(\mathbf{n})$, which cancels out the δfunction of the zeroth-approximation term in the perturbations. It turns out that in this averaging, a nonzero contribution of the required order from the quadratic part of the equation can be made only by the first term under the integral sign in (2.15):

$$\frac{\gamma^2}{2}\int\frac{d^3l\,d^3m}{(2\pi)^3}\delta(\mathbf{n}-\mathbf{l}-\mathbf{m})D(l)\delta(\mathbf{l}+\mathbf{m})u^{\delta\gamma}_{\gamma\delta}\left(\frac{1}{3}\delta^\beta_\alpha-\frac{l_\alpha m^\beta}{lm}\right)$$

$$=\frac{16\pi\gamma^2}{3(2\pi)^3}\delta(\mathbf{n})\delta^\beta_\alpha\int_0^\infty D(l)l^2\,dl.$$

As a result, the equation for the scale factor (the radius of curvature) takes the form

$$rr''=\frac{4\pi}{3(2\pi)^6}\gamma^2\int_0^\infty D(l)l^2\,dl. \qquad (3.1)$$

Returning to the definition of $W(\eta)$ as the mean value $r^2R_{00}/3$, we substitute in (2.10) the expansion over the types (2.14) and average. It turns out here that only the integral that enters in (3.1) is retained in the required order in $(n_0\eta)^{-1}$. As a result we have

$$W(\eta)=r'^2-{}^3/_2rr''. \qquad (3.2)$$

To obtain equations for the spectra, we multiply Eq. (2.15) by similar complex-conjugate $(\overset{\delta}{\gamma})$-equation with argument n_1 (the linear part by the linear and the quadratic by the quadratic). After averaging the obtained expressions, we use the definitions of the spectra (1.7). The five scalar projections of this $(^{\beta\gamma}_{\alpha\gamma})$ equation will indeed be the scalar equations of the spectra. In practice it is more convenient to obtain them directly. To this end we multiply (2.15) by: 1) δ^α_β, 2) $\nu^\alpha\nu_\beta$, 3) ν_β. The results are designated by Φ_1, Φ_2, and Φ_α, and Eq. (2.15) itself by Φ^β_α. By multiplying and averaging we then obtain five independent equations:

$$\langle\Phi_1\dot{\Phi}_1\rangle,\quad\langle\Phi_1\dot{\Phi}_2\rangle,\quad\langle\Phi_2\dot{\Phi}_2\rangle,\quad\langle\Phi_\alpha\dot{\Phi}^\alpha\rangle,\quad\langle\Phi^\beta_\alpha\dot{\Phi}^\alpha_\beta\rangle.$$

In the right-hand sides of these equations there appear integrals of the mean values of the four random functions. The random functions φ^β_α have Gaussian distributions at least in the principal approximation in $(n_0\eta)^{-1}$. We shall use the law for the separation of the mean product of four functions into paired averages, which holds true for Gaussian random functions. For example, for the tensor type this takes the form

$$\langle d^\beta_\alpha(\mathbf{l}_1)d^\delta_\gamma(\mathbf{m}_1)\dot{d}^{*\varkappa}_\varepsilon(\mathbf{m}_2)\dot{d}^{*\nu}_\mu(\mathbf{l}_2)\rangle$$

$$=D(l_1)D(m_1)\left[u^{\beta\varkappa}_{\alpha\varepsilon}(\mathbf{l}_1)u^{\delta\nu}_{\gamma\mu}(\mathbf{m}_1)\delta(\mathbf{l}_1-\mathbf{m}_2)\delta(\mathbf{m}_1-\mathbf{l}_2)\right.$$

$$\left.+u^{\beta\nu}_{\alpha\mu}(\mathbf{l}_1)u^{\delta\varkappa}_{\gamma\varepsilon}(\mathbf{m}_1)\delta(\mathbf{l}_1-\mathbf{l}_2)\delta(\mathbf{m}_1-\mathbf{m}_2)\right].$$

We decouple analogously also the groups of four average functions a and c_α and the mixed averages $\langle c_\alpha d^{\beta*}_\nu c\,\overset{*}{\delta} d^\kappa_\epsilon\rangle$ and $\langle ac_\alpha\overset{*}{a}c^\beta\rangle$, which result from the integrals of the right-hand side of (2.15). All the cross-over mean values of the different integrals (2.15) vanish because of the identical vanishing of the majority of the mixed mean values in (1.7).

With such an averaging of (2.15), the rapidly-oscillating exponentials vanish when multiplied by the complex-conjugates, and we are left as a result with a system of five nonlinear integral equations for the spectra. The calculation of the kernels of the integrals is quite cumbersome, but entails no fundamental difficulties.

Since the right-hand side of (2.15) does not contain the function b(n), the nonlinear part of the obtained system does not contain the spectra B(n) and E(n). After eliminating them from the left-hand sides of the equations, we are left with a system of three equations for the spectra A(n), C(n), and D(n). In this region, where we can approximately assume

$$W(r')^{-2}=1-3rr''/2r'^2\approx\text{const},$$
$$q'/r'\approx\text{const},$$

it is possible to separate the temporal and spatial variables in this system. For functions that depend on the time there arises (together with (3.1)) the following system of equations:

$$rr''=R_0^2kk_1^2\gamma^2,\quad r\gamma'=R_0k_1\gamma^2,$$
$$1-3q'^2=k_2\gamma^2. \qquad (3.3)$$

We have introduced symbols for the dimensionless constants $k>0$, k_1 and k_2, which are numbers of the order of unity and are expressed in terms of integrals of the spectral functions. Their exact values cannot be determined in the present theory.

The parametric solution of the first two equations of the system (3.3) is given by

$$r=R_0n_0^{-1}\,y^{-k}e^y,\qquad\eta=n_0^{-1}\int_1\xi^{-k}e^\xi\,d\xi,\qquad\gamma=-\frac{1}{k_1y}. \qquad (3.4)$$

Here y is a parameter and $R_0n_0^{-1}$ and n_0^{-1} are the integration constants.

The numerical value of the lower limit in the integral for η can be chosen arbitrarily and corresponds to the choice of the origin of the time. The usually assumed time origin at the singularity is not suitable here, since the solution written out has no singularities at all. The reason, however, is that the region of applicability of (3.4) is $\eta\gg n_0^{-1}$ and $r\gg R_0n_0^{-1}$, which reduces to $y>k$ owing to the exponential character of the dependence on y.

To satisfy the conditions

$$W/r'^2\approx\text{const},\quad q'/r'\approx\text{const}$$

we must have $y\gg1$. At $y\gg1$, the following asymptotic formula holds true

$$\eta\approx\frac{1}{n_0y^k}e^y\left(1+\frac{k}{y}+\frac{k(k+1)}{y^2}+\frac{k(k+1)(k+2)}{y^3}+\ldots\right). \qquad (3.5)$$

We see therefore that the expansion of the world obeys asymptotically the Friedmann law $r=R_0\eta$ (see Fig. 2). It is of interest to note that the synchronous time ($dt=rd\eta/c$) depends on the parameter y in the same manner as η, but with doubling of the constant k:

$$t=\frac{R}{n_0^2c}\int_1^y\xi^{-2k}e^{2\xi}\,d\xi. \qquad (3.6)$$

Since y increases monotonically with increasing y, it follows from the third equation of (3.3) that at sufficiently large η the function $q(\eta)$ approaches the value $\eta/\sqrt{3}$. (The integration constant is an inessential phase factor of the exponential in (2.14)). This $q(\eta)$ dependence is known from the analysis of linear perturbations and reflects the fact that the scalar high-frequency perturbations constitute an acoustic wave propagating with velocity $u=(dp/d\epsilon)^{1/2}=1/\sqrt{3}$.

The system of linear integral equations for the spectra, which remains after separating the variables, makes it possible in principle to determine A(n), C(n), and D(n). Their asymptotic behavior at $n\ll n_0$ (but $n\gg\eta^{-1}$) is clear already from (2.15):

$$A(n)\sim n^2\langle a^2(n)\rangle\sim B(n)\sim E(n)\sim n^{-2},$$
$$C(n)\sim n^2\langle c_\alpha(n)c^\alpha(n)\rangle\sim D(n)\sim\text{const}. \qquad (3.7)$$

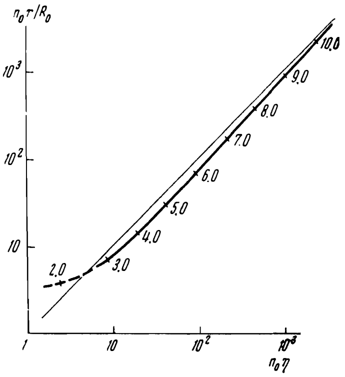

FIG. 2. Dependence of the curvature radius $r(\eta)$ in the case $k = 1$— thick line ($n_0 r/R_0 = e^y/y$, $n_0\eta = \mathrm{Ei}(y) - \mathrm{Ei}(1)$; the numbers along the curve are the values of the parameter y. The dashed curve is the solution of (3.4) outside the region of its applicability. The thin line is the Friedmann law of expansion at $p = \epsilon/3$: $r = R_0\eta$.

Returning to the correlation functions (1.8), we note that all the $F_1, \ldots, F_5$ are of the same order of magnitude, since the integrals of the scalar spectra should be cut off from below at $n \sim \eta^{-1}$.

The time dependences of the curvature radius and of the perturbation amplitudes, obtained from the high-frequency expansion, differ from the dependences of the linear case by a logarithmically slowly varying function. Another important result of the work is that the Friedmann expansion law is patently not applicable already at time t_0 corresponding to η_0. Its numerical value can be quite large. For example, at $n_0 \sim 10^6$ we have $t_0 \sim R_0 n_0^{-2} c^{-1} \sim 10^2$ sec.

A rigorous analysis of the behavior of the world at earlier stages of the expansion should contain an analysis in the region $\eta \sim \eta_0$. A perfectly realistic possibility is that at $\eta \sim \eta_0$ an oscillatory approach to the singularity sets in as a general solution that is isotropic in the mean. On the other hand, the considered model does not exclude the possibility of a positive minimum of the function $r(\eta)$ and an asymptotic approach to the Friedmann solution as $\eta \to -\infty$, in other words, a cosmological solution without a singularity.

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[1)]As follows from E. M. Lifshitz's paper [3], n are positive integers. However, since $n_0 \gg 1$, summation over n is replaced everywhere by integration.

[2)]It should be noted that the pre-exponential η^{-1} dependence of the first term of (2.12) does not satisfy one of the scalar equations.

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