Pair production by a periodic electric field

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The formation of boson and fermion pairs by a homogeneous periodic field during a finite time T is considered. The Klein–Gordon equation and the squared Dirac equation in this field reduce to the Hill equation; however, only in the boson case are all of the equation parameters real. This is one of the factors that affects radically the solutions at physical values of the quantum numbers. In the boson case unstable solutions exist which depend on time exponentially and correspond to exponential growth of the mean number of pairs in a given quantum state. (The exponential dependence of the amplitude on T goes over into the usual linear dependence only for the quantum numbers corresponding to the boundary of the instability region). There are no unstable solutions for the physical values of the quantum numbers in the spinor case and correspondingly the probability for pair production in a given state oscillates in time, remaining, as one would expect, less than unity.

1. INTRODUCTION

The interest in studying the production of pairs in external electromagnetic fields has increased in recent years[1-14]. Such processes and processes of pair annihilation with energy transfer to the external field are the only processes of zeroth order in the radiation field. If the pair-producing field vanishes as $|t| \to \infty$, the probabilities for these processes are expressed in terms of the coefficients c_1 and c_2 (cf. Eq. (3)) of the solution of the Klein-Gordon or Dirac equation[4]. In the sequel we shall call c_1 and c_2 the scattering amplitudes.

A periodic field is specific in that the duration T during which it is different from zero can be varied experimentally within wide limits, and it makes sense to study the dependence of the transition probability on T. The most interesting case is the case of the field of colliding electromagnetic waves. Unfortunately, the corresponding solutions of the Klein-Gordon and Dirac equations have not yet been studied (cf. however,[15,16]). In this situation the first natural step is to consider the simplified case: pair production by a homogeneous electric field which has a sinusoidal time-dependence.

For the first time this still rather complicated problem has been analyzed by Brezin and Itzykson[7] by means of the Schwinger method[17] and by Popov[8] (cf. also[9,10]) by means of the so-called imaginary time method. Both papers claim only to determine the order of magnitude of the effect, and in this respect they agree with each other.

The time-dependence of the probability derived in these papers has the usual form characteristic for small probabilities. For larger probabilities it is natural to expect a more complicated T dependence since the sum of the probabilities of all the events must add up to one. We shall see below that the dependence of the scattering amplitudes c_1 and c_2 on T is indeed peculiar and the simple approximations made in[7-10] are not sufficient for determining it.

2. THE SCALAR CASE

The Klein-Gordon equation in the presence of a vector potential $\mathbf{A}(t)$ reduces to the equation

$$\left[\frac{d^2}{dt^2} + m^2 + \mathbf{p}^2 - 2e\mathbf{A}\mathbf{p} + e^2\mathbf{A}^2\right] f_p(t) = 0, \qquad (1)$$

if $\psi(x)$ is assumed to be of the form $\psi(x) = f_p(t)\exp(i\mathbf{p}\cdot\mathbf{x})$, where p_1, p_2, p_3 are the quantum numbers of the solution. We shall assume that $\mathbf{A}(t) = (0, 0, A_3(t))$ and that the field is different from zero only in the time interval (τ, T)

$$A_3(t) = \begin{cases} a\cos\omega\tau, & t<\tau \\ a\cos\omega t, & \tau<t<T. \\ a\cos\omega T, & t>T \end{cases} \qquad (2)$$

The pair production probabilities are determined[18,2,4] by the scattering amplitudes c_{1p}, c_{2p} of the solution $f_p(t)$, which satisfy the conditions[1)]:

$$f_p(t)|_{t<\tau} = e^{-ip_0(\tau)t}, \quad f_p(t)|_{t>T} = c_{1p}e^{-ip_0(T)t} + c_{2p}e^{ip_0(T)t}. \qquad (3)$$

According to Eq. (1) we have here for the vector potential (2)

$$p_0(t) = [m^2 + p_\perp^2 + (p_3 - ea\cos\omega t)^2]^{1/2}, \quad p_\perp^2 = p_1^2 + p_2^2. \qquad (4)$$

The switching on and off of the field $E_3(t) = E\sin\omega t$, for $\tau < t < T$ will be less abrupt if τ and T are selected to be multiples of π/ω. For values of t inside the interval (τ, T) we have two linearly independent solutions of the equation (1): $f_p^{(1)}(t)$ and $f_p^{(2)}(t)$.

The considerable simplification of the problem related to the model potential chosen in the form (2) becomes obvious: since the quantum numbers of the solutions are the same in the field region and outside it, the conditions of continuity of the amplitudes c_{1p} and c_{2p} are easily expressed in terms of the values of the solutions $f_p^{(1)}(t)$ and $f_p^{(2)}(t)$ (and of their derivatives) at the points τ and T. Thus, the solution which has the form $\exp\{-ip_0(\tau)t\}$ for $t < \tau$ goes over into the solution $\beta_1 f_p^{(1)}(t) + \beta_2 f_p^{(2)}(t)$ when the field is switched on, with the constants β_1 and β_2 determined from the conditions that the function $f_p(t)$ and its derivative be continuous at the point τ. Thus, we obtain

$$\beta_1 = D^{-1}\exp\{-ip_0(\tau)\tau\}[f_p^{(2)\prime}(\tau) + ip_0(\tau)f_p^{(2)}(\tau)],$$
$$\beta_2 = -D^{-1}\exp\{-ip_0(\tau)\tau\}[f_p^{(1)\prime}(\tau) + ip_0(\tau)f_p^{(1)}(\tau)], \qquad (5)$$
$$D = f_p^{(1)}(t)f_p^{(2)\prime}(t) - f_p^{(2)}(t)f_p^{(1)\prime}(t) \equiv f_p^{(1)}(t)\overleftrightarrow{\frac{d}{dt}}f_p^{(2)}(t).$$

Since Eq. (1) does not contain the first derivative, the Wronskian D does not depend on t.

Similarly, matching up the solution and its derivative at the instant T when the field is switched off, determines c_{1p} and c_{2p}:

$$2D\exp\{-ip_0(T)T + ip_0(\tau)\tau\}\, c_{1p}/ip_0(\tau)$$

$$= \Phi_p^{(2)}(\tau,+)\Phi_p^{(1)}(T,-) - \Phi_p^{(1)}(\tau,+)\Phi_p^{(2)}(T,-), \tag{6}$$

$$2D\exp\{ip_0(T)T + ip_0(\tau)\tau\}c_{2p}/ip_0(\tau)$$

$$= \Phi_p^{(2)}(\tau,+)\Phi_p^{(1)}(T,+) - \Phi_p^{(1)}(\tau,+)\Phi_p^{(2)}(T,+); \tag{7}$$

$$\Phi_p^{(1,2)}(t,\pm) = \left[1 \pm \frac{1}{ip_0(t)}\frac{d}{dt}\right]f_p^{(1,2)}(t). \tag{8}$$

We note that on account of charge conservation, or in view of the time-independence of the Wronskian

$$f_p^*(t)\overleftrightarrow{\frac{d}{dt}}f_p(t)$$

it follows from (3) that

$$|c_{1p}|^2 - |c_{2p}|^2 = p_0(\tau)/p_0(T); \tag{9}$$

the quantity $|c_{2p}|^2 p_0(T)/p_0(\tau)$ determines the average number of pairs produced in a given state p (cf.[3,4]). We also note that the instantaneous switching-on at the instant τ is an idealization. In fact, the switching-on must be characterized by some function with a width $\Delta\tau$ in the neighborhood of τ. In the sequel however, we shall be interested in the dependence of the average number of particles on the length of the interval $T - \tau$ during which the field is switched on, and this dependence should not be very sensitive to the concrete form of the switching on and off function as long as $T - \tau \gg \Delta\tau$.

Thus, our problem has been reduced to considering the solutions of the equation (1) with a periodic function A(t), i.e., to the solution of a Mathieu-Hill equation[19-21]. Unfortunately the properties of these solutions are not sufficiently well known at the present time, and this circumstance prevents us from obtaining a complete solution of the problem.

We write the solutions of (1) in the form

$$f_p(t) \equiv f_{q_0}(t) = e^{-iq_0 t}\sum_{n=-\infty}^{\infty} A_n(q_0)e^{in\omega t}, \tag{10}$$

where q_0 is the quasi-energy. A substitution of (10) into (1) yields the recursion relation

$$\left[-\left(\frac{q_0}{\omega} - n\right)^2 + \frac{p_0^2 + {}^1/_2 e^2a^2}{\omega^2}\right]A_n(q_0) - \frac{eap_3}{\omega^2}[A_{n-1}(q_0) + A_{n+1}(q_0)] + \frac{e^2a^2}{4\omega^2}[A_{n-2}(q_0) + A_{n+2}(q_0)] = 0. \tag{11}$$

From it one can determine $A_n(q_0)$ and q_0; it is often more convenient to consider q_0 as given, and to determine the corresponding p_0 from (11).

In the stability region q_0 is by definition real and is not a multiple of $\omega/2$. In the instability region $\operatorname{Im} q_0 \neq 0$, and at the boundary of the stability region $2q_0 = s_0\omega$, $s_0 = 0, 1, 2, \ldots$ In the latter case one cannot obtain a second solution from (10) by means of the substitution $t \to -t$; the second solution can be defined as the limit

$$f_{q_0}^{(2)}(t) = \lim\left\{[f_{q_0}(t) \mp f_{q_0}(-t)]\Big/\sin\frac{2q_0}{\omega}\pi\right\}, \quad 2q_0 \to s_0\omega. \tag{12}$$

The upper (lower) sign corresponds to the even (odd) limit function $f^{(2)}_{s_0\omega/2}(t)$.

Thus, in the instability region $2q_0/\omega = s_0 + i\mu$. The coefficients of the equation (11) do not change under the substitution $q_0 \to q_0^* = {}^1/_2\omega(s_0 - i\mu)$, $n \to s_0 - n$.

Therefore one should expect that

$$A_n(q_0) = A_{s_0-n}(q_0^*) \text{ or } A_n(q_0) = -A_{s_0-n}(q_0^*). \tag{13}$$

Making use of these relations in (10) it is easy to check that $f_{q_0}(t) = f^*_{q_0}(t)$ or $f_{q_0}(t) = -f^*_{q_0}(t)$, i.e., in the instability region the solutions (10) are essentially real; one of the two grows exponentially as $t \to \infty$, and the other decreases exponentially. In agreement with this, the number of pairs in the state p grows, according to (7), (8), exponentially with t: $|c_{2p}|^2 \propto \exp(|\mu|\omega t)$.

We now consider the stability region. In principle one can look for the solution of Eq. (11) in the form of an expansion in powers of the charge. The quantity $q_0 = q_0(p_0)$ (or $p_0 = p_0(q_0)$) is uniquely determined if one requires that $A_n(q_0) \to \delta_{0n}$ for $e \to 0$, where δ_{0n} denotes the Kronecker symbol. The series obtained this way converge slower and slower (or start diverging) as one approaches the boundary of instability (cf. Eqs. (4)–(6), Sec. 2.16, in MacLachlan's book[21]). At the same time it is already clear from the lowest approximation that it is the nearness of this boundary which is responsible for the pair production. Near the boundary there appears a delta-function term in the expression for c_{2p}, describing the conservation of energy $s_0\omega = 2q_0$; the integer s_0 labels the instability zone.

The appearance of the term with $\delta(s_0\omega - 2q_0)$ in the exact solution (7) for c_{2p} can be understood in the following way. We take as $f_p^{(1)}(t)$ the function $f_{q_0}(t)$ in (10) and obtain $f_p^{(2)}(t)$ from (10) by the substitution $q_0 \to -q_0$. Let $2q_0 = (s_0 + \epsilon)\omega$ where ϵ will tend to zero. Since, according to (11), $A_n(q_0) = A_{-n}(-q_0)$, it follows from (10) for finite ϵ that

$$f_p^{(2)}(t) = f_{-q_0}(t) = e^{i\epsilon\omega t}f_{q_0}(t) + \sum_{n=-\infty}^{\infty}[A_n(q_0) - A_{s_0-n}(q_0)]e^{i(q_0-n\omega)t}, \tag{14}$$

where the $A_n(q_0)$ are chosen in such a way that $A_n(q_0) - A_{s_0-n}(q_0)$ as $\epsilon \to 0$ (cf. (13)).

Using (14) in $\Phi_p^{(1)}(\tau, +)$ and $\Phi_p^{(2)}(T, +)$ in (7) we obtain

$$c_{2p} = -\frac{i}{2}p_0(\tau)D^{-1}\exp\left\{\frac{i}{2}\epsilon\omega(T-\tau) - ip_0(T)T - ip_0(\tau)\tau\right\} \times\left[f_{-q_0}(\tau) + \frac{1}{ip_0(\tau)}f'_{-q_0}(\tau)\right]\Phi_p^{(1)}(T,+)\{e^{i\epsilon\omega(T-\tau)/2} - e^{-i\epsilon\omega(T-\tau)/2}\} + \underline{\text{remaining terms}}. \tag{15}$$

In the limit $\epsilon \to 0$ the <u>remaining terms</u> in (15) are a periodic function, and the square bracket can be replaced by $\Phi^{(1)}(\tau, +)$. Since D is proportional to ϵ, the explicitly written terms in (15) contain indeed (for sufficiently large $(T - \tau)$) the term $\delta(\epsilon) = \delta(2q_0/\omega - s_0)$, or, for $\epsilon = 0$, the quantity $T - \tau$.

It should be remarked, however, that all the time we have implicity assumed that the sign of ϵ is fixed. In fact, for fixed ea/ω and p_3/ω there are two different values of p_0 corresponding to the value $2q_0 = s_0\omega$ in (11). If one starts out from the stability region, as p_0 increases the quantity $2q_0$ approaches $s_0\omega$ from below ($\epsilon < 0$). At the lower limit of the instability region $\epsilon = 0$. As p_0 increases further we are in the instability region: $\epsilon = i\mu$ is purely imaginary. Finally, for still larger p_0 we pass through the upper limit of the instability region, after which ϵ becomes positive. (For the case $p_3 = 0$, cf. the stability chart in[21].) In the lowest order of perturbation theory the instability region can be neglected. Then for $\epsilon \to 0$ we have $p_0(s_0/2 + \epsilon) \approx p_0(s_0/2 - \epsilon)$ and the factor in front of $\delta(\epsilon)$ becomes continuous at the point $\epsilon = 0$.

We shall consider further that $\epsilon = 0$, i.e., that $2q_0 = s_0$ and we rewrite (15) in a slightly different form.

The solution $f_p^{(1)}(t) = f_{q_0}^{(1)}(t)$ in (10) will then be periodic. A second solution, corresponding to the given quantum numbers, has the form[19, 21]

$$f_p^{(2)}(t) = f_{q_0}^{(2)}(t) = Ctf_{q_0}^{(1)}(t) + g(t), \quad (16)$$

where $g(t)$ is also a periodic function (but has parity opposite to that of $f_{q_0}^{(1)}(t)$). Making use of (16) in (7) we obtain

$$2c_{2p}D\exp\{ip_0(T)T + ip_0(\tau)\tau\} = -ip_0(\tau)(T-\tau)C\Phi_p^{(1)}(\tau,+)\Phi_p^{(1)}(T,+) + \underline{\text{remaining terms}}. \quad (17)$$

With a suitable normalization of $f_{q_0}^{(1)}(t)$ and $f_{q_0}^{(2)}(t)$ the determinant D can be made equal to one.

The periodic function "remaining terms" in (17), as a function of τ, T, oscillates with frequencies which are multiples of ω. its existence is completely determined by the switching on (and off) of the field, and it depends, naturally, on the concrete form of the switching function. For this reason we shall not consider it further.

Thus, we see that boson pairs are effectively produced with quantum numbers p corresponding to the unstable region, including its boundaries. This circumstance manifests itself clearly in the dependence of the solutions (and the probability) on time. In addition, in the stable region the solutions (10) are normalizable ($j_0 \neq 0$), and upon entering the unstable region they become real (up to a constant phase) and the charge density of such states equals zero. The production of pairs with the quantum numbers of the stable region is also possible, but it is described by a probability oscillating with $T - \tau$. In the next section we shall see that only the latter possibility is realized for spinor particles.

We now consider in more detail the case when ea/ω is so small that one can neglect all but the lowest order of perturbation theory. Consider first that $\mathbf{A}\cdot\mathbf{p} = 0$. We determine c_{2p} in the linear regime (cf. Eq. (17), where we omit the remaining terms). The substitution $\omega t = x + \pi/2$ reduces Eq. (1) to a Mathieu equation

$$\left(\frac{d^2}{dx^2} + \eta_0 - 2\eta\cos 2x\right)f(x) = 0;$$
$$\omega^2\eta_0 = p_0^2 + {}^1/_2 e^2a^2, \quad \eta = (ea/2\omega)^2. \quad (18)$$

We take as its solutions[21]

$$f_p^{(1)}(x) = \mathrm{ce}_n(x,\eta), \quad \mathrm{ce}_n(x,0) = \cos nx;$$
$$f_p^{(2)}(x) = C_n x\,\mathrm{ce}_n(x,\eta) + g_n(x,\eta), \quad g_n(x,0) = \sin nx. \quad (19)$$

To lowest order in η we obtain for C_n in $f_p^{(2)}(x)^{2)}$:

$$C_n = \frac{\eta^n}{n[2^{n-1}(n-1)!]^2}. \quad (20)$$

The final result is

$$|c_{2n}|^2 = \frac{1}{4n^2}\frac{\eta^{2n}}{[2^{n-1}(n-1)!]^4}(T-\tau)^2\omega^2. \quad (21)$$

In the zeroth approximation p_0 in (4) does not depend on t so that (21) yields the average number of pairs in the state p.

Setting $n \gg 1$ and $p_0 \approx m \approx n\omega$, we find

$$|c_{2p}|^2 = \frac{1}{\pi^2}\omega^2(T-\tau)^2\left(\frac{e}{4}\frac{ea}{m}\right)^{2N}, \quad N \approx \frac{2m}{\omega}. \quad (21')$$

Let now p_0 be such that the solutions of Eq. (18) are unstable. According to the Whittaker method[20, 21] we obtain to lowest order

$$f_p^{(1)}(x) \approx e^{\mu_n x}\sin(nx - \sigma),$$
$$f_p^{(2)}(x) \approx e^{-\mu_n x}\sin(nx + \sigma), \quad \mu_n = -{}^1/_2 C_n \sin 2\sigma. \quad (22)$$

Here C_n is the same as in (20) and the parameter σ determines the value of p_0 for which the solution (22) was obtained. From here we obtain according to (7), (8)

$$|c_{2p}|^2 = \frac{\mathrm{sh}^2\,\mu_n\omega(T-\tau)}{\sin^2 2\sigma},$$
$$|c_{1p}|^2 = \frac{\mathrm{sh}[\mu_n\omega(T-\tau)+2i\sigma]\,\mathrm{sh}[\mu_n\omega(T-\tau)-2i\sigma]}{\sin^2 2\sigma}, \quad (23)$$

and in the approximation considered the relation (9) is valid.

The limit $\sigma \to 0$ or $\sigma \to \pi/2$ corresponds to passing to the boundary of the region of instability. In this case (21) follows again from the first equation (23). We note that although the width of the instability region is extremely small for $ea/\omega \ll 1$, it widens as ea/ω increases, due to the exponential dependence of the average number of pairs in it on T, its existence seems to be of interest.

Until now it has been assumed that $p_3 = 0$. Then the production of boson pairs is realized by an even number of photons absorbed from the field: $c_{2p} = c_{2p}(\eta)$, $\eta = (ea/2\omega)^2$. We now take into account p_3, considering it however to be so small that one may retain only its first nonvanishing power. For $p_3 \neq 0$ a contribution to c_{2p} is possible also from an odd number of photons. The corresponding part of the wave function of the bosons can be expanded in momentum space in terms of $P_{2n+1}(\cos \measuredangle(\mathbf{pa}))$. As $\mathbf{p}\cdot\mathbf{a} \to 0$ the Legendre polynomial $P_{2n+1}(\cos \measuredangle(\mathbf{pa}))$ becomes proportional to p_3, which determines the threshold behavior of c_{2p} for an odd number of photons.

For $p_3 \neq 0$, Eq. (1) goes over into the Hill equation

$$\left[\frac{d^2}{dz^2} + \theta_0 + 2\theta\cos 2z + \frac{1}{2}\vartheta^2\cos 4z\right]f(z) = 0,$$
$$\omega^2\theta_0 = 4\left(p_0^2 + \frac{e^2a^2}{2}\right), \quad \theta = -\frac{4eap_3}{\omega^2}, \quad \vartheta = \frac{2ea}{\omega}, \quad z = \frac{\omega t}{2}. \quad (24)$$

In the instability region its solutions in lowest order have the form (22), where now

$$\mu_{2n+1} = (-1)^{n+1}\left[\frac{e}{4}\frac{ea}{n\omega}\right]^{2n+1}\sin 2\sigma\left[\frac{2p_3}{e\omega} + \ldots\right],$$
$$\mu_{2n} = (-1)^{n+1}\frac{2}{\pi}\left[\frac{e}{4}\frac{ea}{n\omega}\right]^{2n}\sin 2\sigma\left[1 - \frac{\pi^2}{2}\frac{p_3^2}{\omega^2} + \ldots\right], \quad n \gg 1. \quad (25)$$

As before, $|c_{2p}|^2$ has the form (23), but $(T-\tau)$ should be taken to mean $(T-\tau)/2$.

The dependence of (25) and (23) on p_3 does not agree with the spectrum obtained by Popov[8]. This seems to be related to the fact that within the framework of the imaginary time method the momentum distribution is pre-exponential, which in this method is not calculated, or is considered to be equal to one. The reason why there is no exponential dependence of c_{2p} on T in the imaginary time method is that in this method one takes into account only one pair of singularities in the complex time plane, i.e., pair production by a single period of the potential. A possible coherent action of many periods is not taken into account. As regards the paper by Brezin and Itzykson[7], one can show that the use of one iteration of the equations (32) of that paper does not suffice to obtain a quantitative result.

3. THE SPINOR CASE

We look for a solution of the squared Dirac equation

$$(\hat{\pi}^2 + m^2)Z = (\pi^2 + g + m^2)Z = 0,$$

$$\pi_\mu = -i\partial/\partial x_\mu - eA_\mu, \quad \hat{\pi} = \gamma_\mu\pi_\mu, \quad g = {}^1/_2 ieF_{\mu\nu}\gamma_\mu\gamma_\nu, \qquad (26)$$
$$F_{\mu\nu} = \partial A_\nu/\partial x_\mu - \partial A_\mu/\partial x_\nu,$$

in the form

$$Z = e^{ipx} f_p(t)\Gamma_i,$$
$$\Gamma_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \Gamma_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \qquad (27)$$

Since for the field under consideration[3)]

$$g\Gamma_{1,2} = ieE(t)\Gamma_{1,2}, \quad g\Gamma_{3,4} = -ieE(t)\Gamma_{3,4}, \quad E(t) = -\frac{\partial}{\partial t}A_3(t), \qquad (28)$$

we obtain for $f_p(t)$

$$\left\{\frac{d^2}{dt^2} + m^2 + (\mathbf{p} - e\mathbf{A})^2 \pm ieE(t)\right\} f_p(t, \pm) = 0. \qquad (29)$$

The plus sign is selected when $f_p(t)$ occurs with Γ_1 and Γ_2, and the minus sign, when it occurs with Γ_3, and Γ_4.

A solution of the Dirac equation is obtained from that of the squared equation in the usual manner:

$$\psi_i(x) = (m - i\hat{\pi}) e^{ipx} f_p(t)\Gamma_i. \qquad (30)$$

The spin states $i = 3, 4$ are expanded in terms of the spin states $i = 1, 2$. Let $f_{q_0}^{(1)}(t, +)$, $f_{q_0}^{(2)}(t, +)$ be two independent solutions of (29). Then the condition of continuity for the function $\psi_1(x)$ at the point where the field is switched on

$$\begin{bmatrix} p_1 - ip_2 \\ m + p_-(\tau) \\ p_1 - ip_2 \\ -m + p_-(\tau) \end{bmatrix} e^{-ip_0(\tau)\tau} = \begin{bmatrix} p_1 - ip_2 \\ m - \pi_3(\tau) + id/d\tau \\ p_1 - ip_2 \\ -m - \pi_3(\tau) + id/d\tau \end{bmatrix}$$
$$\times \{\beta_1 f_{q_0}^{(1)}(\tau, +) + \beta_2 f_{q_0}^{(2)}(\tau, +)\} \qquad (31)$$

implies

$$\beta_1 = D^{-1}\exp\{-ip_0(\tau)\tau\}[F_{q_0}^{(2)}(\tau, +) - f_{q_0}^{(2)}(\tau, +)],$$
$$\beta_2 = D^{-1}\exp\{-ip_0(\tau)\tau\}[f_{q_0}^{(1)}(\tau, +) - F_{q_0}^{(1)}(\tau, +)],$$
$$F_{q_0}^{(1,2)}(t, +) = \frac{1}{p_-(t)}\left[-\pi_3(t) + i\frac{d}{dt}\right] f_{q_0}^{(1,2)}(t, +), \qquad (32)$$
$$D = \frac{i}{p_-(\tau)} f_{q_0}^{(1)}(t, +) \overleftrightarrow{\frac{d}{dt}} f_{q_0}^{(2)}(t, +),$$
$$\pi_3(t) = p_3 - eA_3(t), \quad p_\pm(t) = p_0(t) \pm \pi_3(t).$$

Here D does not depend on t (cf. Eq. (5)).

Similarly, from the continuity of $\psi_1(x)$ at the switching-off point we obtain

$$\frac{2p_0(T)}{p_-(T)}\exp\{-ip_0(T)T\}c_{1n} = \beta_1\left[\frac{p_+(T)}{p_-(T)} f_{q_0}^{(1)}(T, +) + F_{q_0}^{(1)}(T, +)\right]$$
$$+ \beta_2\left[\frac{p_+(T)}{p_-(T)} f_{q_0}^{(2)}(T, +) + F_{q_0}^{(2)}(T, +)\right], \qquad (33)$$
$$\frac{2p_0(T)}{p_-(T)}\exp\{ip_0(T)T\}c_{2n} = \beta_1[f_{q_0}^{(1)}(T, +) - F_{q_0}^{(1)}(T, +)]$$
$$+ \beta_2[f_{q_0}^{(2)}(T, +) - F_{q_0}^{(2)}(T, +)].$$

Here $p_0(T)$ is the same as in (4), c_{1n}, c_{2n} play the same roles as c_{1p}, c_{2p} in Eq. (3).

From the charge conservation condition we obtain:

$$|c_{1n}|^2 \frac{p_0(T)p_-(T)}{p_0(\tau)p_-(\tau)} + |c_{2n}|^2 \frac{p_0(T)p_+(T)}{p_0(\tau)p_-(\tau)} = 1. \qquad (34)$$

The equations (32)–(34) do not in fact depend on the spin index. They can be made more transparent if one takes into account the relation between the functions $f_{q_0}(t, +)$ and $f_{q_0}(t, -)$. Since

$$\left(\pi_3 \pm i\frac{d}{dt}\right)\left(\pi_3 \mp i\frac{d}{dt}\right) = \frac{d^2}{dt^2} + \pi_3^2 \pm ieE(t),$$

it is easy to verify that $(\pi_3 - id/dt)f(t, +)$ is the solution $f(t, -)$. Therefore one can define $f^{(1,2)}(t, -)$ as follows:

$$\left(\pi_3 - i\frac{d}{dt}\right) f_{q_0}^{(1,2)}(t, +) = \frac{1}{2}\omega\nu_{1,2} f_{q_0}^{(1,2)}(t, -). \qquad (35)$$

Here ν_1 and ν_2 depend on the quantum numbers and the field amplitudes.

Below we will deal with the solutions of (29) having the form (10), so that in addition to (35) one may set

$$f_{q_0}^{(1,2)*}(t, +) = f_{q_0}^{(2,1)}(t, -). \qquad (36)$$

It follows from (35), (36), (29) that

$$\nu_1^*\nu_2 = \nu_1\nu_2^* = -4(p_\perp^2 + m^2)/\omega^2.$$

The relations (35) and (36) are useful for the consideration of the normalization and orthogonality of the solutions of the Dirac equation.

We further define the periodic functions $\chi(z)$:

$$\chi_1(z)e^{-i\mu''z} = \frac{2}{\omega}p_-(t) f_{q_0}^{(1)}(t, +) + \nu_1 f_{q_0}^{(1)}(t, -),$$
$$\chi_2(z)e^{i\mu''z} = \frac{2}{\omega}p_-(t) f_{q_0}^{(2)}(t, +) + \nu_2 f_{q_0}^{(2)}(t, -),$$
$$\chi_{1'}(z)e^{-i\mu''z} = \frac{2}{\omega}p_+(t) f_{q_0}^{(1)}(t, +) - \nu_1 f_{q_0}^{(1)}(t, -), \qquad (37)$$
$$\chi_{2'}(z)e^{i\mu''z} = \frac{2}{\omega}p_+(t) f_{q_0}^{(2)}(t, +) - \nu_2 f_{q_0}^{(2)}(t, -);$$

$z = \omega t/2$. The expression for c_{2n} in (33) now takes the form

$$\exp\{ip_0(T)T + ip_0(\tau)\tau\}c_{2n} = \frac{\omega^2}{8p_0(T)p_-(\tau)}D^{-1}$$
$$\times [\chi_1(z_0)\chi_2(Z)\exp\{i\mu''(Z - z_0)\} - \chi_2(z_0)\chi_1(Z)\exp\{-i\mu''(Z - z_0)\}]. \qquad (38)$$

The coefficients c_{1n} are obtained from (38) by means of the substitution

$$c_{2n} \to \exp\{-2ip_0(T)T\}c_{1n},$$
$$\chi_1(Z) \to \chi_{1'}(Z), \quad \chi_2(Z) \to \chi_{2'}(Z).$$

Considering that $Z - z_0 \equiv {}^1/_2\omega(T - \tau) = 2\pi l$, where l is an integer, we see that $c_{2n} \propto \sin 2\pi l\mu''$, i.e., the probability for the production of a spinor pair with the quantum numbers $n = (p, i)$ oscillates as a function of $T - \tau$. This situation is analogous to the periodic transitions of a two-level system (cf. the problem in Sec. 40 in Landau-Lifshitz[23]). Equation (29) for the potential under consideration is a Hill equation and differs from (24) by the presence of the spin term $\pm i2\vartheta \sin 2z$ in the square bracket:

$$\left[\frac{d^2}{dz^2} + \theta_0 + 2\theta\cos 2z + \frac{1}{2}\vartheta^2\cos 4z \pm 2i\vartheta\sin 2z\right] f_{q_0}(z, \pm) = 0. \qquad (39)$$

The solution $f_{q_0}(z, +)$ for sufficiently small ea/ω is conveniently determined by means of the Whittaker method[21]. Then

$$f_{q_0}(z, +) = e^{\mu z}\Phi(z, \sigma), \quad \theta_0 = s_0^2 + \theta f_1^{(\theta)}(\sigma) + i\vartheta f_1^{(\vartheta)}(\sigma) + \ldots;$$
$$\Phi(z, \sigma) = \sin(s_0 z - \sigma) + \theta h_1^{(\theta)}(z, \sigma) + i\vartheta h_1^{(\vartheta)}(z, \sigma) + \ldots \qquad (40)$$

We consider the lowest-order approximation in ea/ω. Taking into account the higher approximations does not modify the qualitative picture.

A. Pair production by a single photon ($s_0 = 1$)

In this case

$$\theta_0 = 1 + \theta\cos 2\sigma + i\vartheta\sin 2\sigma + \vartheta^2[-{}^1/_4 + {}^1/_8\cos 4\sigma]$$

$$+\vartheta^2[{}^1/_4+{}^1/_8\cos 4\sigma]+{}^1/_4 i\vartheta\theta\sin 4\sigma+O(\vartheta^3), \tag{41}$$
$$\mu={}^1/_2\theta\sin 2\sigma-{}^1/_2 i\vartheta\cos 2\sigma+O(\vartheta^3).$$

Since θ_0 is real (cf. (24)), the parameter $\sigma = \sigma' + i\sigma''$ must be such that $\sin 2\sigma' = 0$. Then $\mu = -i\mu''$ is purely imaginary. The absolute value $|\mu''|$ takes a minimal value at σ_0'' when considered as a function of σ''.

In the lowest approximation under consideration we have

$$\mu=-{}^1/_2 i\rho\cos 2\sigma'\operatorname{ch}2(\sigma''-\sigma_0'')=-i\mu'',$$
$$\theta_0=1-\rho\cos 2\sigma'\operatorname{sh}2(\sigma''-\sigma_0'')+{}^1/_4\rho^2+{}^1/_8\rho^2\operatorname{ch}4(\sigma''-\sigma_0''), \tag{42}$$
$$\frac{2ea}{\omega}=\vartheta=\rho\operatorname{ch}2\sigma_0'',\quad \frac{-4eap_3}{\omega^2}=\theta=\rho\operatorname{sh}2\sigma_0''.$$

Selecting $\sigma' = 0$, we find the solution $f^{(1)}_{q_0}(z, +)$:

$$f^{(1)}_{q_0}(z,+)=e^{-i\mu''z}\{\sin(z-i\sigma'')-{}^1/_8 i\rho\cos(3z-i\sigma''-2i\sigma_0'')+\ldots\},$$
$$\mu''={}^1/_2\rho\operatorname{ch}2(\sigma''-\sigma_0''). \tag{43}$$

The second solution, $f^{(2)}_{q_0}(z, +)$, may be obtained from (43) by means of the substitution

$$i\sigma''\to-\pi/2+i(2\sigma_0''-\sigma'').$$

Here μ'' changes its sign and θ_0 remains unchanged.

Simple calculations give the probability of pair production in the state n:

$$|c_{2n}|^2\frac{p_+}{p_-}=\left(\frac{ea}{2p_0}\right)^2\left[1-\left(\frac{p_3}{p_0}\right)^2\right]\left[\frac{\sin\mu''(Z-z_0)}{\mu''}\right]^2,$$
$$Z=\omega T/2,\quad z_0=\omega\tau/2. \tag{44}$$

In the lowest approximation $p_\pm(t)$ in (33) does not depend on t. It is convenient to compare this expression with the absolute value of the matrix element of the usual perturbation theory:

$$\mathfrak{M}_1=\int_\tau^T dt\int d^3x\,\bar\psi^{(+)}_{p1}(x)\hat A(x)\psi^{(-)}_{-p'1}(x)=\delta_{p,p'}\frac{ea}{2p_0}\left[1-\left(\frac{p_3}{p_0}\right)^2\right]^{1/2}$$
$$\times\exp\left\{i\left(\frac{2p_0}{\omega}-1\right)(Z+z_0)\right\}\sin\left[\left(\frac{2p_0}{\omega}-1\right)(Z-z_0)\right]\Big/\left(\frac{2p_0}{\omega}-1\right). \tag{44'}$$

We see that in the usual perturbation theory nothing prevents one from selecting $p_0 = \omega/2$ and then $|\mathfrak{M}_1| \propto (T - \tau)$. In fact the smallest value of μ'' is $\rho/2$ and the quantity (44) oscillates with $(T - \tau)$.

B. Pair production by two photons ($s_0 = 2$)

Here

$$\mu=\frac{i\vartheta\theta}{8}\left\{\cos 2\sigma+\frac{i\theta}{2\vartheta}\sin 2\sigma\right\},$$
$$\theta_0=4+\frac{\theta^2-\vartheta^2}{6}+2\frac{d\mu}{d\sigma}. \tag{45}$$

Setting

$$\sigma=i\sigma'',\quad \operatorname{th}2\sigma''=\theta/2\vartheta\approx-p_3/p_0,$$

we obtain

$$\mu=\frac{i\vartheta\theta}{8\operatorname{ch}2\sigma_0''}\operatorname{ch}2(\sigma''-\sigma_0'')=-i\mu'',$$
$$|c_{2n}|^2\frac{p_+}{p_-}=\left[\frac{\sin\mu''(Z-z_0)}{\operatorname{ch}2(\sigma''-\sigma_0'')}\right]^2. \tag{46}$$

Here new compared to the case $s_0 = 1$ is the vanishing of μ'' and c_{2n} for $\theta = 0$, i.e., for $p_3 = 0$. This circumstance is a reflection of the following fact. The state of a pair produced by the absorption of an even number of photons must be even under charge conjugation. On the other hand, for $p_3 = 0$ only even spherical harmonics are possible in the final state for $p_3 = 0$. Further, from the independence of the pair production amplitude on the spin state it can be seen that the pair is produced in a state with total spin equal to unity, i.e., the final state is odd under charge conjugation. Thus the contributions from an even number of photons must vanish for $p_3 = 0$.

C. Pair production by many photons

If the pair is produced by s_0 photons, then in the lowest order of a concrete calculation one needs only the quantities μ'' and θ_0. Considering them known, we will have, according to (35), (33), (38)

$$f^{(1)}_{q_0}(z,+)\approx e^{-i\mu''z}\sin(s_0z-i\sigma''),$$
$$f^{(2)}_{q_0}(z,+)\approx e^{i\mu''z}\cos(s_0z-2i\sigma_0''+i\sigma''),$$
$$\frac{2p_3}{\omega}=-\nu_1\sin 2i\sigma_0'',\quad -is_0=\nu_1\cos 2i\sigma_0'',\quad \operatorname{th}2\sigma_0''\approx-\frac{p_3}{p_0},$$
$$c_{1n}=\frac{\cos[\mu''(Z-z_0)-2i(\sigma''-\sigma_0'')]}{\operatorname{ch}2(\sigma''-\sigma_0'')},\quad s_0\omega\approx 2p_0, \tag{47}$$
$$|c_{2n}|^2\frac{p_+}{p_-}=\left[\frac{\sin\mu''(Z-z_0)}{\operatorname{ch}2(\sigma''-\sigma_0'')}\right]^2$$

The relation (34) is valid to the given accuracy. We consider further the simple special case $p_3 = 0$. For an even s_0 we expect that $\mu'' = 0$, i.e., the solution $f^{(1)}_{q_0}(z, +)$ is periodic. Indeed, the method of Sec. 2.13 in[21] shows easily that to any order in ϑ one can construct a periodic solution which expands in a sine series. The second solution can be found using the relations (35), (36), and it is also periodic.

For odd s_0 the Whittaker method yields

$$\mu_{s_0}=i(-1)^{k+1}\frac{1}{2}\frac{\cos 2\sigma}{4^{2k}[(2k)!!]^2}\vartheta^{s_0},\quad s_0=2k+1,$$
$$\mu_{s_0}\approx i\frac{2}{e\pi}(-1)^{k+1}\left[\frac{e}{4}\frac{ea}{m}\right]^{2k+1}\cos 2\sigma,\quad k\gg 1. \tag{48}$$

For $|\mu''(Z - z_0)| \ll$ the result agrees apart from a pre-exponential factor with the result of Popov[8].

Until now we have considered the average number of pairs in a given state. If this quantity would turn out to be proportional to $\delta^2(\epsilon)$ (cf. Eq. (15)) and the coefficient of proportionality would not depend on τ and T, then, multiplying it with the density of final states and integrating with respect to d^3p, we reduce one $\delta(\epsilon)$ by integration, and replace the second one by $(T-\tau)/2\pi$. The result would seem quite normal if the probability is small.

We have seen above that for small probabilities ($|\mu| \ll 1$) one gets approximately $c_{2n} \propto \delta(\epsilon)$. However, only in lowest order of perturbation theory with respect to ea/ω the proportionality coefficient does not depend on τ, T. Thus, in (15), (17) for the lowest order of perturbation theory one has to take the function $|\Phi^{(1)}_p(t, +)|$ in zeroth approximation, and then it does not depend on t. (In the transition from the formula (15) to (17) D^{-1} separates a factor C, which determines the dependence of c_{2p} on the field in the lowest approximation.) For $ea/\omega \sim 1$ the function $|\Phi^{(1)}_p(t, +)|$ already depends essentially on t. The dependence of the proportionality factor on τ and T is a dependence on the phase of the switching-on of the field. One gets the impression that the process under consideration exhibits, as a rule, a high sensitivity to the method of switching the field on and off.

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1)After multiplication by the factor $[p_0(T)/p_0(\tau)]^{1/2}$ the amplitudes c_{1p}, c_{2p} in (3) coincide with the c_{1p}, c_{2p} defined by one of the authors [4], where $f_p(t)$ was normalized to unity.

2)In the book [21] the first terms of the expansion C_n for n = 1, 2, 3 are given, with a reference to Ince. Our result is based on the method of Sec. 7.30 in [21].

3)We use the same representation of the γ matrices as in the book of Akhiezer and Berestetskiĭ [22].

4)Since $\psi_j(x)$ was normalized to unity, we had there in place of (34) $|c_{1n}|^2 + |c_{2n}|^2 = 1$. The second term yields the pair production probability in the state n.

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