On the multifocus structure of light beams in nonlinear media

A. L. Dyshko, V. N. Lugovoĭ, and A. M. Prokhorov

P. N. Lebedev Physics Institute, USSR Academy of Sciences
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The propagation of time-stationary light beams in media exhibiting Kerr nonlinearity is discussed with the "attenuation" of the nonlinearity in strong fields taken into account. It is found that light beams of supercritical intensity propagating in such media develop a multifocus structure arises in the case of a medium exhibiting both attenuation of the Kerr nonlinearity and nonlinear absorption. The multifocus structure obtained for a medium exhibiting attenuation of the Kerr nonlinearity but no absorption is compared with that obtained for a medium exhibiting nonlinear absorption but no attenuation of the Kerr nonlinearity.

Some time ago[1] we predicted that a time stationary light beam in a medium having Kerr nonlinearity would exhibit a multifocus structure. Subsequently[2], we advanced the hypothesis that diverse effects such as multiphoton absorption, various types of induced scattering, breakdown, etc., which can actually arise at the focal points of a multifocus structure, cannot change the picture in a qualitative way, but can only affect some of its quantitative characteristics (the sizes and relative positions of the focal regions and the energy density that can actually be achieved in them). Later[3], we made numerical calculations of the propagation of a light beam in which two- and three-photon absorption and Raman scattering were taken into account; these calculations showed that a multifocus structure can arise even when these effects are present. At that time we also investigated the structure of the focal regions themselves under these conditions.[3] From a formal point of view, to take the above factors into account one need only introduce an intensity dependent imaginary part of the refractive index, in addition to the real part that describes the Kerr nonlinearity; and in this way one also takes into account such deviations of the nonlinearity of the medium from the Kerr type as are associated with the presence of an imaginary part of the refractive index.

The literature also reveals considerable interest in the case in which the deviation of the nonlinearity of the medium from the Kerr type in strong fields is associated with a change in the real part of the refractive index itself. For example, Akhmanov, Sukhorukov, and Khokhlov[4] phenomenologically introduced an "attenuation" of the Kerr nonlinearity in strong fields. Brewer, Lifshitz, Garmire, Chiao, and Townes[5] have made a detailed study of the attenuation of the Kerr nonlinearity that, in the case of the orientation mechanism of the Kerr effect, is associated with the alignment of an appreciable fraction of the molecules in the direction of the electric field. Further, Yablonovitch and Bloembergen[6] have recently discussed the possible weakening (compensation) of the Kerr nonlinearity as a result of the increase in the free electron concentration in the focal regions just prior to breakdown. Although certain experimental results have not yet been successfully explained in terms of these ideas (for example, the lack of any dependence of the diameters of the focal regions on the duration and intensity of the incident light pulse under ordinary conditions), it may still be necessary to take the electron avalanche into account in certain cases, since conditions are known from experiment in which breakdown takes place in the focal regions.[7]

Here we consider the propagation of light beams in media in which the deviation of the nonlinearity from the Kerr type (in intense light fields) is associated with a change in the real part of the refractive index. For definiteness, we write the refractive index n in the form

$$n=n_0+\frac{{}^1/_2 n_0 n_2|E|^2}{1+|E|^2/|E_x|^2}, \tag{1}$$

where E is the complex amplitude of the electric field strength[1)]. With $1/|E_X|^2 = 0$, expression (1) describes the Kerr nonlinearity of the medium, while with $1/|E_X|^2 \neq 0$ it takes into account attenuation (and in the limit $|E|^2 \gg |E_X|^2$, complete "saturation") of the Kerr nonlinearity. In what follows we shall consider the propagation of both parallel and focused beams incident on the medium, and in addition, we shall take into account the effect of two- and three-photon absorption on the propagation of focused beams in a medium with Kerr nonlinearity.

Let us consider the propagation of an axially symmetric light beam in a medium whose refractive index is given by (1). Then in cylindrical coordinates, the parabolic equation for the complex amplitude E has the form

$$\frac{\partial^2 E}{\partial r^2}+\frac{1}{r}\frac{\partial E}{\partial r}+2ik\frac{\partial E}{\partial z}+k^2\frac{n_2|E|^2E}{1+|E|^2/|E_x|^2}=0, \tag{2}$$

in which $k = 2\pi/\lambda$, where $\lambda = 2\pi c/\omega n_0$ is the wavelength of the light in the medium, ω being the frequency of the light wave. We write the boundary condition at z = 0 in the form

$$E|_{z=0}=E_0\exp(-r^2/2a^2), \tag{3}$$

this corresponds to a parallel Gaussian beam with the intensity concentrated in a cylinder of mean radius a incident from the region $z < 0$ onto the boundary[2)] at z = 0. Introducing the dimensionless variable $X = E/E_0$, we obtain the following equation for X:

$$\frac{\partial^2 X}{\partial r_1^2}+\frac{1}{r_1}\frac{\partial X}{\partial r_1}+2iN\frac{\partial X}{\partial z_1}+\frac{N^2|X|^2X}{1+\beta|X|^2}=0 \tag{4}$$

with the boundary condition

$$X|_{z_1=0}=\exp(-{}^1/_2 r_1^2). \tag{5}$$

Equation (4) with the boundary condition (5) was solved numerically with a BÉSM-6 computer, using an implicit difference scheme similar to the one employed in[1].

As an example, we present the solution to this problem for N = 6, which corresponds to an incident beam power P_0 equal to ten times the critical value $P_{cr}^{(1)}$, and for $\beta = 0.01$, which corresponds to a value of about 1/30 (see below) for the ratio of the diameters of the focal regions to the diameter of the incident beam (under typical experimental conditions, this ratio ranges from about 1/100 to 1/30). This solution is presented in Fig. 1 as a family of curves giving $|X|^2$ as a function of z_1 for various values of r_1 ($r_1 = k/18$, with k = 0, 1, ..., 9). Each curve has been drawn to its own scale, the several scales being so chosen as to reveal the over-all pattern most clearly. The lowest curve corresponds to $r_1 = 0$, i.e., it shows the axial intensity. We see that the axial intensity, as a function of z_1, has a number of sharp peaks, which correspond to a series of separate focal regions on the beam axis. The curves for $r_1 \neq 0$ reflect the process by which the foci are formed and the process by which the waves continue their propagation after passing through a focus (these processes are indicated schematically on Fig. 1 by dashed lines). We see that the waves that have passed through a focus leave the region of the initial beam on nearly straight trajectories.

We can judge the structure of the focal regions from the behavior of $|X|^2$ as a function of r_1 in the sections $z_1 = z_{f1}, z_{f2}, z_{f3}, \ldots$, where z_{fm} is the coordinate of the point at which $|X|^2$ reaches its maximum in the m-th focal region (we shall denote the value of $|X|^2$ at the center of the focal region, i.e., at the point $z_1 = z_{fm}$, $r_1 = 0$, by $|X_{fm}|^2$). The $|X|^2$ versus r_1 curves for the first three foci are presented in Fig. 2. These curves show that the ratio of the diameter of a focal region to the diameter of the initial beam is of the order of 1/30 for this example. On calculating the power P_{fm} flowing through the individual focal regions much as we did in [3], we obtained the following results:

$$P_{f1} \approx 1.8 P_{cr}^{(1)}, \quad P_{f2} \approx 2.4 P_{cr}^{(1)}, \quad P_{f3} \approx 2.5 P_{cr}^{(1)}, \quad P_{f4} \approx 2.1 P_{cr}^{(1)}, \quad P_{f5} \approx 1.2 P_{cr}^{(1)}.$$

Since $P_0 \approx 10 P_{cr}^{(1)}$, it is evident that only part of the initial beam power P_0 flows through any one of the focal regions; the rest of the power is carried by the part of the beam that bypasses the focal region in question and later forms the subsequent foci. Qualitatively, therefore, the present solution represents the multifocus structure predicted in [1]. We note that the inequality $|E_{fm}|^2/|E_x|^2 \equiv |X_{fm}|^2 > 1$ is satisfied for all the foci examined; this means that on approaching any of the foci, the intensity at the beam axis continues to increase until appreciable deviations from the weak-field Kerr nonlinearity of the medium have arisen[3)].

A quantitative comparison of the multifocus structure obtained with allowance for the attenuation of the Kerr nonlinearity of the medium in the absence of absorption with the structure obtained in the opposite case (i.e., with allowance for the nonlinear absorption by the medium in the absence of attenuation of the Kerr nonlinearity) would be of interest. First, we note that in the second case, according to [3], the power P_{fm} flowing through any of the focal regions is $\sim (2/3)P_{cr}^{(1)}$, i.e., it is smaller than the values given above for the first case. There are also quantitative differences between the two cases in the longitudinal structure of the focal regions. According to the bottom curve on Fig. 1, the z_1 dependence of $|X|^2$ in the vicinity of any of the foci is more symmetric about the corresponding focal point $z_1 = z_{fm}$ than in the case of nonlinear absorption (see [3]). To show the behavior of the field off the axis, we present in Fig. 3 (which is to be compared with Fig. 1) a family of curves showing the z_1 dependence of $|X|^2$ for the values $r_1 = k/18$ (k = 1, 2, ..., 9) for the case of three-photon absorption in a medium having Kerr nonlinearity. The curves were calculated for the case in which $P_0/P_{cr}^{(1)} \approx 10$ and the ratio of the diameters of the focal regions to the diameter of the initial beam is approximately 1/60 (in the notation of [3], this corresponds to the parameter values N = 6 and $\mu_4 = 0.05$). The dashed curves on Fig. 3, like those on Fig. 1, indicate schematically the processes by which the foci are formed and the annular waves leave them. It will be seen that there are quantitative differences between the two cases being compared as regards the relative positions of the foci on the beam axis and the interference of the waves that have passed through a focus with the "fresh" parts of the beam (i.e., the parts that have bypassed the foci).

Let us consider briefly the case in which nonlinear absorption and attenuation of the Kerr nonlinearity are both taken into account. Limiting ourselves for definiteness to the model defined by Eq. (1) as supplemented with two- and three-photon absorption, we obtain the following expression for the refractive index:

$$n = n_0 + \frac{{}^1/_2 n_0 n_2 |E|^2}{1 + |E|^2/|E_x|^2} + \frac{1}{2} n_0 [(n_2' + i m_2)|E|^2 + (n_4 + i m_4)|E|^4], \tag{6}$$

where m_2 and m_4 are real coefficients that specify the two- and three-photon absorption, respectively, while

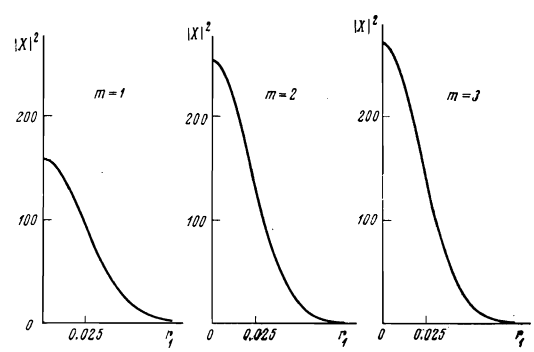

FIG. 2. Radial distribution of the intensity in the central cross sections of the first three focal regions for the case depicted in Fig. 1 ($r_1 = r/a$).

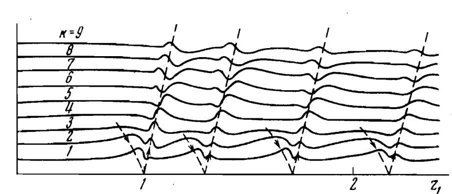

FIG. 3. The same as Fig. 1 but for the case of three-photon absorption by the medium.

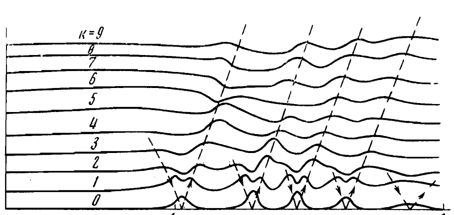

FIG. 1. A family of curves (drawn to different scales) showing the dependence on the longitudinal coordinate z_1 of the intensity ($|X|^2$) of a beam at various distances r from the axis propagating in a Kerr medium for the case of an initially parallel beam (i.e., a beam that is parallel on entering the medium) and attenuation of the nonlinearity of the medium in strong fields. The numbers at the curves give the values of $k = 18r/a$, where a is the diameter of the beam on entering the medium.

the coefficients n_2' and n_4 specify the changes in the real part of the refractive index associated with these types of absorption. Since the energy levels are usually substantially broadened in liquids and solids, we must have $|n_2'| < m_2$ and $|n_4| < m_4$ in Eq. (6). As was shown in[3], when $1/|E_x|^2 = n_2' = n_4 = m_2 = 0$, we have $m_4|E_{fm}|^2 \leq 0.15n_2$, where the $|E_{fm}|^2$ are the values of $|E|^2$ at the centers of the focal regions. Further, when $1/|E_x|^2 = n_2' = n_4 = m_4 = 0$, a well developed multifocus structure arises if $m_2 \leq 0.15n_2$. Therefore, we can approximate Eq. (6) (when $|E_{fm}|^2 << |E_x|^2$, for example) by the equation

$$n=n_0+\frac{{}^1\!/_2 n_0(n_2+n_2')|E|^2}{1+|E|^2(|E_x|^{-2}-n_4/n_2)}+\frac{i}{2}n_0(m_2|E|^2+m_4|E|^4), \qquad (7)$$

in which $|n_2'| < 0.15n_2$ and $|n_4|/n_2 < 0.15/|E_{fm}|^2$.

Thus, to take into account the corrections n_2' and n_4 in (7) we need merely make the following substitutions in (6): $n_2 \to n_2 + n_2'$ and $1/|E_x|^2 \to 1/|E_x|^2 - n_4/n_2$; moreover, the inequality $|E_{fm}|^2 << |E_x|^2$ is still valid for new value of $|E_x|^2$. From this it is evident (see footnote 3) that the correction to the real part of the refractive index due to three-photon absorption can lead only to small quantitative changes in the beam propagation pattern. As to the correction to the real part of the refractive index due to two-photon absorption, according to (7) it can lead only to a redefinition of the critical values $P_{cr}^{(m)}$ of the beam power and in this sense cannot change the propagation pattern under consideration even quantitatively (that is why these corrections were not taken into account in[3]). We made direct numerical calculations taking the corrections for three-photon absorption into account, and these calculations showed that the changes in the beam propagation pattern due to these corrections are indeed purely quantitative and, moreover, are small; for example, when $|n_4| \leq m_4$, the diameters of the focal regions are seldom changed by more than 15%. We also note that these numerical examples confirm the conclusion, which is evident from what has been said above, that the propagation pattern for a light beam in a medium having Kerr nonlinearity also has a multifocus structure when nonlinear absorption and attenuation of the Kerr nonlinearity in strong fields are both taken into account.

Now let us consider a focused Gaussian beam incident on the boundary $z = 0$ of a nonlinear medium. For such a beam the boundary condition takes the form

$$E|_{z=0}=E_0\exp\left(-\frac{r^2}{2a^2}-\frac{i}{2}\frac{kr^2}{R}\right). \qquad (8)$$

Here a is the radius of the beam as it enters the nonlinear medium, and R is the axial (z) coordinate of the focal point that the beam would reach in the medium if there were no nonlinearity. We have previously published the solution to this problem for the case of a medium having Kerr nonlinearity and negligible absorption[11]; this solution shows that a focused beam also acquires a multifocus structure. Here we present the solution to this problem for the case of a medium having two- and three-photon absorption and attenuation of the Kerr nonlinearity, i.e., for a medium having the refractive index (6) (for simplicity we set $n_2' = n_4 = 0$ at once). Introducing the dimensionless quantities

$$X=\frac{E}{E_0}\frac{N}{\nu},\quad N=E_0\sqrt{n_2(ka)^2},\quad \nu=\frac{ka^2}{R},\quad r_1=\frac{r}{a},$$

$$z_1=\frac{z}{R},\quad \bar\beta=\frac{\nu^2}{|E_x|^2 n_2(ka)^2},\quad \bar\mu_2=\frac{m_2\nu^2}{n_2},\quad \bar\mu_4=\frac{m_4\nu^4}{n_2^2(ka)^2}$$

we obtain the following equation for X:

$$\frac{\partial^2X}{\partial r_1^2}+\frac{1}{r_1}\frac{\partial X}{\partial r_1}+2i\nu\frac{\partial X}{\partial z_1}+\left(\frac{\nu^2|X|^2}{1+\bar\beta|X|^2}+i\bar\mu_2|X|^2+i\bar\mu_4|X|^4\right)X=0 \qquad (9)$$

with the boundary condition

$$X|_{z_1=0}=\frac{N}{\nu}\exp\left[-\frac{1}{2}(1+i\nu)r_1^2\right]. \qquad (10)$$

Let us first consider the case of two-photon absorption ($\bar\beta = \bar\mu_4 = 0$, $\mu_2 \neq 0$). As an example, the solution to (9) under the boundary condition (10) for $N = 3$, $\nu = 4$, and $\bar\mu_2 = 0.45$ is presented in Fig. 4 as a family of curves showing $|X|^2$ as a function of z_1 for the values $r_1 = k/18$ ($k = 1, 2, \ldots, 9$). In this example there is evidently just one definite focus, which lies in the region $z_1 < 1$ (i.e., in the region $z < R$). The quantity $F \equiv \nu^2|X|^2/N^2$ has the value $F_1 \approx 3000a$ at the center of this focal region. There is no corresponding focus in the region $z_1 > 1$; there the beam simply diverges, exhibiting no noticeable features. This result differs from the conclusion drawn in[12] that each focus in the region $0.5R < z < R$ has a corresponding focus in the region $z > R$. The error in[12] was explained in[13]: the solution found in[12] is discontinuous on the plane $z = R$. According to[11], subsequent foci of the multifocus structure will appear only when the initial power in the beam (or, what amounts to the same thing, N) is increased (the incipient formation of a second focus can be seen in Fig. 4; this focus also lies in the region $z_1 < 1$).

Now let us consider the case of three-photon absorption ($\bar\beta = \bar\mu_2 = 0$, $\mu_4 \neq 0$). As an example we present the solution for $N = 4$, $\nu = 1$, and $\bar\mu_4 = 3 \times 10^{-4}$ in Fig. 5, which shows the $|X|^2$ versus z_1 curves for $r_1 = k/18$ ($k = 1, 2, \ldots, 9$). In this case the multifocus structure has two definite foci. The quantity F has the values $F_1 \approx 62$ and $F_2 \approx 23$ at the centers of the focal regions.

Finally, let us consider the case in which the Kerr nonlinearity of the medium is attenuated in strong fields ($\bar\mu_2 = \bar\mu_4 = 0$, $\bar\beta \neq 0$). A family of curves showing $|X|^2$ as a function of z_1 at $r_1 = k/6$ ($k = 0, 1, \ldots, 9$) for the case $N = 3.5$, $\nu = 1$, and $\bar\beta = 2 \times 10^{-3}$ is presented in Fig. 6. There are four definite focal regions, at the centers of which F has the values $F_1 \approx 71$, $F_2 \approx 77$, $F_3 \approx 49$, and $F_4 \approx 30$.

Thus, the results obtained above show that a light beam propagating in a medium having Kerr nonlinearity develops a multifocus structure even in the presence

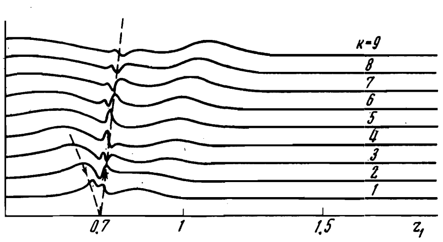

FIG. 4. The same as Fig. 1 but for the case of an initially convergent beam and two-photon absorption by the medium.

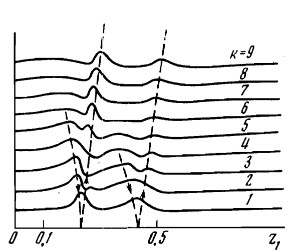

FIG. 5. The same as Fig. 1 but for the case of an initially convergent beam and three-photon absorption by the medium.

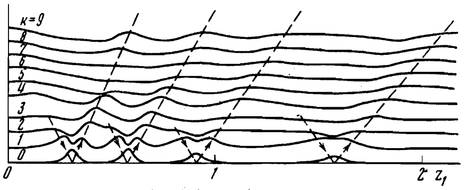

FIG. 6. The same as Fig. 1 but for the case of an initially convergent beam and attenuation of the nonlinearity of the medium. In this figure the numbers at the curves give the values of k = 6r/a.

of diverse processes that may take place in the focal regions, regardless of whether these processes affect the imaginary part of the refractive index of the medium, or its real part.

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1)The refractive index has been written in a similar form before, e.g., in [8-10], and $|E_x|$ has been assumed to be the characteristic saturation field of the orientational Kerr effect. It has been shown [3], however, that in that case one must ordinarily use Maxwell's equations directly, without transforming to the parabolic equation, and this was not taken into account in [8-10]. Since we use the parabolic equation in our calculations, we regard $|E_x|$ as a formal parameter, which is generally somewhat smaller than the characteristic saturation field of the orientational Kerr effect. Hence, in the subsequent discussions based on (1) we are attempting only to obtain a model description of those cases in which the attenuation of the Kerr nonlinearity is appreciable when the diameters of the focal regions are somewhat larger than the wavelength of the light (the parabolic equation is applicable under those conditions), disregarding the question of whether such cases can be realized (the other studies in which the parabolic equation was used [8-10] do not touch upon this question either).

2)We note that a similar problem has been treated in [8-10]; however, the results of numerical calculations presented in those papers, although they are consistent with the beam propagation pattern predicted in [1], are not complete enough to permit any definite conclusion to be drawn concerning the beam propagation process as a whole.

3)This result is in conformity with the results obtained in [8-10]. It is also evident from this that if the energy density at the focus is so limited by some other mechanism that $|E_{fm}|^2 \ll |E_x|^2$, the effect of the attenuation of the Kerr nonlinearity on the propagation of the beam in the medium will be quite negligible.

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