Magnetic properties of plane and layer systems

V. L. Pokrovskiĭ and G. V. Uĭmin

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
(Submitted June 20, 1973)
Zh. Eksp. Teor. Fiz. **65**, 1691–1703 (October 1973)

The behavior of degenerate two-dimensional systems below the phase-transition temperature T_c is investigated by employing scale-invariance considerations and a self-consistency technique. The order parameter is zero in these systems, but a "transverse rigidity" exists. The effect of an external field, anisotropy, and weak interplane interaction, which lead to the appearance of an order parameter below T_c, are considered. The results are compared with the experimental data of de Jongh, van Amstel, and Miedema.

1. SELF-CONSISTENCY METHOD FOR THE CALCULATION OF ρ_s

It is now firmly established that a phase transition at nonzero temperature exists in a planar degenerate system (PDS)[1-3]. By "degenerate," we mean a system whose order is described by an n-component order parameter $\varphi_\alpha(\mathbf{x})$ ($\alpha = 1, 2, \ldots n$). The Hamiltonian of a degenerate system is invariant with respect to rotations in the n-dimensional space of the φ_α, these rotations being independent of the real spatial coordinates $\mathbf{x}$. Examples of such systems are the Heisenberg magnet (n = 3), the two-component magnet (n = 2), and the Bose liquid (n = 2). Hohenberg[4] and Mermin and Wagner[5] have shown that long-range order is absent in a PDS:

$$\langle \varphi_\alpha \rangle = 0. \tag{1}$$

A new property appearing below the phase-transition point in a PDS is the "transverse stiffness" or "density of the superfluid component" ρ_s. This quantity was introduced by Berezinskiĭ[6], by analogy with superfluid helium. For a two-component PDS (n = 2), this quantity can be defined as follows. We shall characterize the two-component vectors φ_α by their modulus φ, which is assumed equal to 1 in the following, and by their angle of rotation ω. We shall consider a state in which the local values of ω vary slowly in space. Such a state is found to be metastable. Its free energy F differs from the equilibrium value by the quantity

$$F - F_0 = {}^1/_2 \int d^2x \rho_s v_s^2 \tag{2}$$

where $\mathbf{v_s} = \nabla\omega$.

Below the transition point in a three-dimensional system, in addition to (2) a term proportional to ω^2 appears in the free energy, where the angle ω is measured from the direction of the spontaneous order parameter $\langle \varphi_\alpha \rangle$. A specific feature of a two-dimensional system is the absence of such a term because of the fact that the spontaneous order is equal to zero (cf. (1)). For this reason, the correlators in a planar system fall off in accordance with a power law rather than exponentially (cf. Sec. 2 for more detail).

There exist different points of view with regard to the behavior near the transition point T_c. Thus, Berezinskiĭ[6] assumes that ρ_s goes to zero smoothly as T tends to T_c. Stanley and Kaplan[2] consider that the magnetic susceptibility χ becomes infinite at the transition point, and this, as will be shown below, is equivalent to asserting that ρ_s is finite at the transition point. Moreover, for χ to become infinite, ρ_s should be greater than a certain finite quantity. In order to elucidate this question and obtain a reasonable dependence for $\rho_s(T)$, we apply an approximation analogous to the Weiss self-consistent field approximation in the theory of magnetism.

The Hamiltonian of the system differs essentially from the Hamiltonian of the long-wavelength fluctuations (2) in that it is a periodic function of $\omega(\mathbf{x})$. We shall consider the simplest form of this Hamiltonian

$$\mathcal{H} = J \sum_{\mathbf{x},\mathbf{a}} \varphi(\mathbf{x}+\mathbf{a})\varphi(\mathbf{x}) = J \sum_{\mathbf{x},\mathbf{a}} \cos(\omega(\mathbf{x}+\mathbf{a}) - \omega(\mathbf{x})), \tag{3}$$

where $\mathbf{x}$ labels the lattice sites, and $\mathbf{x} + \mathbf{a}$ labels the site closest to $\mathbf{x}$. The Hamiltonian (3) is nonlinear, but in the limit of longwavelength fluctuations of $\omega(\mathbf{x})$ only the term proportional to $(\nabla\omega(\mathbf{x}))^2$ is important for us. First of all, we shall study the separation of this term from the Hamiltonian (3). We shall perform this separation by a self-consistent method. Namely, we shall assume that, in the final analysis, the Hamiltonian reduces to quadratic form:

$$\mathcal{H} = \frac{1}{2}\rho_s |J| \sum_{\mathbf{x},\mathbf{a}} (\omega(\mathbf{x}+\mathbf{a}) - \omega(\mathbf{x}))^2 = \frac{1}{2}\rho_s |J| \int d^2x (\nabla\omega(\mathbf{x}))^2. \tag{4}$$

The new definition of ρ_s given by formula (4) differs from the old (cf. (2)) by a constant factor. With this definition, ρ_s is a dimensionless quantity.

In the approximation used, the quantity $\omega(\mathbf{x})$ has a Gaussian distribution. Therefore, all averages of a product of a certain number of quantities $\omega(\mathbf{x_i})$ break down into products of all possible pair averages $\langle \omega(\mathbf{x})\omega(\mathbf{x}') \rangle$ (Wick's theorem). It is useful to introduce the concept of a normal product

$$:a_1(\mathbf{x}_1)a_2(\mathbf{x}_2)\ldots a_n(\mathbf{x}_n):$$

of quantities $a_i(\mathbf{x_i})$ that are functions of $\omega(\mathbf{x})$. The normal product is associated with the usual relation:

$$\begin{aligned} a_1 \ldots a_n = {} & :a_1 \ldots a_n: + \langle a_1 a_2 \rangle :a_3 \ldots a_n: + \ldots \\ & \ldots + \langle a_1 a_2 \rangle \langle a_3 a_4 \rangle :a_5 \ldots a_n: + \ldots \end{aligned} \tag{5}$$

It may be said that an ordinary product is equal to the sum of normal products with all possible pairings. By a "pairing," we mean the replacement of a pair of quantities by the average of their product.

We now expand the Hamiltonian (3) in a series in powers of $(\omega(\mathbf{x}+\mathbf{a}) - \omega(\mathbf{x}))^2$. Carrying out m − 1 pairings, from an arbitrary even power $(\omega(\mathbf{x}+\mathbf{a})$

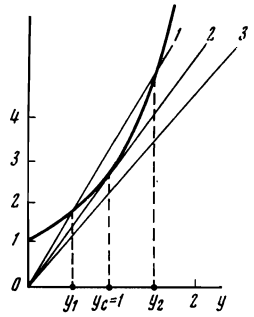

FIG. 1

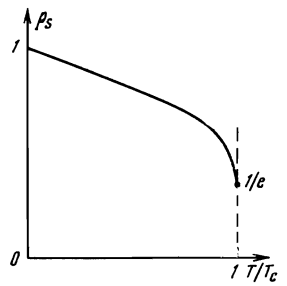

FIG. 2

FIG. 1. Graphical solution of Eq. (9). The three straight lines correspond to temperatures $T_1<T_c$, $T_2 = T_c$, and $T_3>T_c$.

$-\omega(x))^{2m}$ we separate out terms containing $:(\omega(x+a)-\omega(x)^2:$. Next, summing over m, we obtain

$$\rho_*=\langle\cos(\omega(x+a)-\omega(x))\rangle=\exp\left[-\tfrac{1}{2}\langle(\omega(x+a)-\omega(x))^2\rangle\right]. \quad (6)$$

We perform the calculation of the average using the Hamiltonian (4):

$$\tfrac{1}{2}\langle(\omega(x+a)-\omega(x))^2\rangle=T/4|J|\rho_*. \quad (7)$$

Substituting (7) into (6), we find

$$\rho_*=\exp(-\theta/\rho_*) \quad (8)$$

where $\theta = T/4|J|$. Eq. (8) determines the dependence $\rho_S(\theta)$. For $\theta = 0$, the quantity $\rho_S = 1$. We note that, for any nonzero θ, Eq. (8) has the solution $\rho_S = 0$. We introduce the variable $y = \theta/\rho_S$. Eq. (8) takes the form

$$e^y=y/\theta. \quad (9)$$

Its solution is depicted in Fig. 1. It can be seen that, for a certain $\theta = \theta_c$, the straight line is tangential to the graph of the exponential e^y. It is easy to find the values of θ_c, y_c and the other variables at this point:

$$\theta_c=\rho_{*c}=e^{-1}, \quad y_c=1. \quad (10)$$

For temperatures $\theta > \theta_c$, only the solution $\rho_S = 0$ exists. Thus, the self-consistency method leads to the conclusion that ρ_S takes a finite value at the transition point (cf. Fig. 2).

This conclusion is in agreement with the estimate of Josephson[7], according to which the scaling dimension $\Delta\rho_S$ in d-dimensional space is equal to $d - 2$, i.e., $\Delta\rho_S = 0$ in the two-dimensional case.[1)]

2. SCALE INVARIANCE OF THE PDS

We should like to point out that scale-invariance arguments in the case of a PDS are applicable not only at the actual transition point T_c but also at all temperatures $T < T_c$ in the region of large distances. In fact, below the transition point the Hamiltonian (4) can be assumed to be asymptotically exact in the limit of long-wavelength fluctuations of $\omega(x)$, since the extra terms contain higher powers of the gradient.

Of course, the calculation of ρ_S by means of Eq. (8) is not consistent, since the Hamiltonian (4) is used in the region of short distances where it is incorrect, but the existence of ρ_S and of the Hamiltonian (4) for $T < T_c$ is not in doubt. From this point of view, the long-wavelength fluctuations of $\omega(x)$ are a free field with a Gaussian distribution. Therefore,

$$\langle\varphi(x)\varphi(x')\rangle=\langle\cos(\omega(x)-\omega(x'))\rangle=\exp\{-\tfrac{1}{2}\langle(\omega(x)-\omega(x'))^2\rangle\}. \quad (11)$$

The usual Gibbs averaging with the energy (4) leads to the following result:

$$\langle(\omega(x)-\omega(x'))^2\rangle=\frac{2T}{|J|\rho_*(2\pi)^2}\int\frac{d^2k}{k^2}(1-\cos\mathbf{k}(\mathbf{x}-\mathbf{x}'))=\frac{T}{\pi|J|\rho_*}\ln|x-x'|. \quad (12)$$

Substituting (12) into (11), we obtain the result first found by Berezinskiĭ[6]:

$$\langle\varphi(x)\varphi(x')\rangle=|x-x'|^{-T/2\pi|J|\rho_*}. \quad (13)$$

The latter formula shows that, in the region of large distances, the correlations possess the property of scale invariance for all $T < T_c$. To the quantity φ we can assign the scaling dimension

$$\Delta_\varphi=\Delta=T/4\pi|J|\rho_*. \quad (14)$$

This implies that, under the scale transformation

$$\mathbf{x}\to l\mathbf{x}$$

the quantity φ transforms according to the law

$$\varphi(lx)=l^{-\Delta}\varphi(x).$$

The scaling dimension Δ varies continuously as a function of the temperature T (cf. (14)). This situation is exceptional in the theory of phase transitions. A continuous dependence of index on a parameter has been known before now, in the Baxter model[8]. Suppose that the exponent in the Gibbs distribution is changed by an amount

$$\lambda\int dx\,\varepsilon(x).$$

Polyakov[9] has shown that fulfilment of the following conditions is necessary and sufficient for a continuous dependence of index on the parameter λ:

1) the dimension $\Delta\epsilon$ of the quantity $\epsilon(x)$ is equal to the number of dimensions of space (in our case, $\Delta_\epsilon = 2$);

2) in the algebra of the fluctuating quantities, the coefficient b in the relation

$$\varepsilon(x)\varepsilon(0)=b\varepsilon(0)/x^2 \quad (15)$$

is equal to zero.

In our case (change of temperature),

$$\lambda=-\delta T\left(\frac{1}{T}-\frac{1}{\rho_*}\frac{\partial\rho_*}{\partial T}\right), \quad \varepsilon(x)=(\nabla\omega)^2.$$

Since ω is a free field, the dimension of $\epsilon = (\nabla\omega)^2$ is 2, and the first condition is fulfilled. More precisely, condition 1) is fulfilled for the quantity $:\epsilon(x):$.

We turn now to condition 2). In the product

$$:\varepsilon(x):\,:\varepsilon(x'):=:(\nabla\omega(x))^2:\,:(\nabla\omega(x'))^2:$$

we must separate out the terms proportional to $:\epsilon(x):$. We shall make use of the fact that the field is free and perform all possible pairings:

$$:\varepsilon(x):\,:\varepsilon(x'):=2\left\langle\frac{\partial\omega}{\partial x_\alpha}\frac{\partial\omega}{\partial x_\beta'}\right\rangle^2 +4\left\langle\frac{\partial\omega}{\partial x_\alpha}\frac{\partial\omega}{\partial x_\beta'}\right\rangle:\frac{\partial\omega}{\partial x_\alpha}\frac{\partial\omega}{\partial x_\beta'}:+:\left(\frac{\partial\omega}{\partial x_\alpha}\right)^2\left(\frac{\partial\omega}{\partial x_\beta'}\right)^2: \quad (16)$$

We shall be interested only in the second term in (16), since the others do not contain $:\epsilon:$. In order to obtain the coefficient b, defined by formula (15), in this term we must put $\alpha = \beta$ and $x = x'$ inside the symbol for the normal product, and sum over α:

$$b=4(x-x')^2\left\langle\frac{\partial\omega}{\partial x_\alpha}\frac{\partial\omega}{\partial x_\beta'}\right\rangle=-4(x-x')^2\frac{\partial^2}{\partial x_\alpha^2}\ln|x-x'|.$$

The reason for the slow (power-law) decay of the correlations at large distances and the consequent non-analyticity of the correlator (cf. (13)) is the strong fluctuations of the order parameter in the PDS. An analogous phenomenon (strong transverse fluctuations) also occurs in three-dimensional degenerate systems[10], where scale invariance also arises for all $T < T_c$. However, in this case long-range order exists ($\langle \varphi_\alpha \rangle \neq 0$), and the free field is the transverse component φ_α.

Mathematically, the theory of long-wavelength fluctuations is equivalent to the well-known so-called gauge model in quantum field theory (cf., e.g.,[11] (p. 40 in the Russian edition)). However, there is an important difference between the model under consideration and the gauge model: the Hamiltonian in the former model contains also nonquadratic terms of the type $(\nabla\omega)^4$, $(\nabla\omega)^6$, etc. These terms do not change the scaling dimensions of the principal quantities, since the dimensions of these corrections are greater than the dimensions of space d = 2 (cf.[9]). This is true so long as ρ_s is not equal to zero.

In the discussions in this section and the preceding section it has been assumed that the integration is performed over each of the $\omega(x)$ from $-\infty$ to $+\infty$ in the partition function. At the same time, the Hamiltonian is a periodic function of $\omega(x)$ with period 2π. If the integration is extended over m periods of each of the $\omega(x)$, this leads to the partition function multiplied by the quantity m^N, where N is the number of lattice sites, i.e., it leads to an extra quantity NTlnm in the free energy. This extra term does not change the distribution function of the quantities ($\omega(x)$, $\varphi(x)$, $\epsilon(x)$, etc.) of interest.

3. FIELD AND ANISOTROPY

We introduce the field h conjugate to the ordering field $\varphi(x)$ by changing the Hamiltonian (3) of the system by an amount

$$\mathcal{H}_h = -h\int \varphi(x)\, d^2x. \tag{17}$$

In accordance with the general relations of scaling theory (cf., e.g.,[12]), the scaling dimensions of the conjugate fields for d = 2 are connected by the relation

$$\Delta_\varphi + \Delta_h = 2. \tag{18}$$

Using (18), we can find the dependence of $\langle \varphi \rangle$ on h in the weak-field region (far from saturation):

$$\langle \varphi \rangle \propto h^{\Delta_\varphi/\Delta_h} = h^{\Delta/(2-\Delta)}. \tag{19}$$

As at the critical point, the dependence of $\langle \varphi \rangle$ on h is nonlinear at arbitrarily small h. Our estimates show that Δ increases monotonically with increasing temperature, and at the transition point takes the value

$$\Delta(T_c) = 1/\pi.$$

This means that, for all $T < T_c$, the susceptibility χ becomes infinite at h = 0.

We shall apply the self-consistency method for an approximate calculation of the coefficient in formula (19). For this, in the full Hamiltonian

$$\mathcal{H} = J\sum_{x,a} \cos(\omega(x+a) - \omega(x)) - h\sum_x \cos\omega(x) \tag{20}$$

we separate out the terms quadratic in ω, performing the necessary number of pairings. Then the Hamiltonian (20) is reduced to quadratic form:

$$\mathcal{H}_{\text{eff}} = \frac{1}{2}h\langle\varphi\rangle\int\omega^2 d^2x + \frac{1}{2}|J|\rho_s\int(\nabla\omega)^2 d^2x, \tag{21}$$

where

$$\langle\varphi\rangle = \langle\cos\omega\rangle = \exp(-\tfrac{1}{2}\langle\omega^2\rangle). \tag{22}$$

The self-consistent calculation of $\langle\varphi\rangle$ by means of (21) and (22) gives

$$\langle\varphi\rangle = (h/16|J|\rho_s)^{\Delta/(2-\Delta)}. \tag{23}$$

The field h has the meaning of an external magnetic field for a magnet with J < 0, which, by analogy with the three-dimensional case, we shall call a ferromagnet. For an antiferromagnet (J > 0) or a superfluid helium film, the field h has no simple physical meaning.

A real magnetic field H in an antiferromagnet leads to an effective change of the temperature

$$T \to T^* = T + aH^2 \quad (a>0).$$

The change of the indices in the magnetic field is determined by the relation

$$\Delta(T, H) = \Delta(T^*, 0).$$

We shall consider a magnet with weak anisotropy, which we shall describe by adding a term

$$\mathcal{H}_A = \lambda\int :\varphi_1^2(x): d^2x. \tag{24}$$

to the Hamiltonian (3). With no loss of generality, we can assume that $\lambda < 0$. Then the axis 1 becomes the easy-magnetization axis, and in a planar system a nonzero moment $\langle\varphi_1\rangle$ appears. We shall find the magnitude of the spontaneous moment for small values of λ. For this, we shall determine the scaling dimension Δ_λ of the quantity λ.

We shall make clear what we mean by this. So long as λ/J is small, the scaling indices of the quantities (ω, φ, ϵ, etc.) can be assumed to coincide with their values for $\lambda = 0$. Then the correction (24) to the Hamiltonian (3) can be regarded as the action of one of the external fields leading away from the critical curve, and λ can be regarded as this external field. The dimension of the field λ, like that of the field h, can be found:

$$\Delta_\lambda + \Delta(:\varphi_1^2:) = 2 \tag{25}$$

(cf. formula (18)). It remains to find the dimension $\Delta(:\varphi_1^2:)$. We shall use the fact that

$$:\varphi_1^2: = :\cos^2\omega: = \tfrac{1}{2}\cos 2\omega. \tag{26}$$

Next we can find the correlator $\langle\cos 2\omega(x)\cos 2\omega(x')\rangle$ in exactly the same way as the correlator (11) was found, and determine the scaling dimension $\Delta(:\varphi_1^2:)$. We give the result:

$$\Delta(:\varphi_1^2:) = 4\Delta. \tag{27}$$

In complete analogy with the case of the field h (cf. (19)), we find

$$\varphi_s \equiv \langle\varphi_1\rangle \propto |\lambda|^{\Delta_\varphi/\Delta_\lambda} = |\lambda|^{\Delta/(2-4\Delta)}. \tag{28}$$

Our estimates (cf. Sec. 1) show that the quantity $\Delta < \frac{1}{2}$ in the whole region $T < T_c$. Therefore, the inclusion of arbitrarily weak anisotropy leads to the appearance of a spontaneous moment. In the temperature region in which $T_c - T$ tends to zero with $\lambda \to 0$, the pattern of the phase transition changes sharply and it becomes isomorphous to an Ising phase transition.

A curious phenomenon arises in the case when the lattice possesses third-, fourth-, and sixth-order axes. In this case the anisotropy Hamiltonian has the form

$$\mathcal{H}_A=\lambda\int\cos(p\omega(\mathbf{x}))\,d^2x, \tag{29}$$

where p is the order of the symmetry axis. The analog of formula (28) is

$$\varphi_s\infty|\lambda|^{\Delta/(2-p^2\Delta)}. \tag{30}$$

The spontaneous moment φ_S vanishes at a temperature $T_1 < T_C$ determined by the condition

$$\Delta(T_1)=2/p^2.$$

An unusual situation arises, with two phase transitions: at $T = T_C$ a $\rho_S \neq 0$ appears, and at $T = T_1$ a spontaneous moment arises.

For an approximate calculation of the coefficient in formula (30), we apply the self-consistent field method. We reduce the Hamiltonian of the anisotropic system

$$\mathcal{H}=J\sum_{\mathbf{x},\mathbf{a}}\cos(\omega(\mathbf{x}+\mathbf{a})-\omega(\mathbf{x}))+\lambda\sum_{\mathbf{x}}\cos p\omega(\mathbf{x}) \tag{31}$$

to quadratic form, carrying out, as usual, the necessary number of pairings. Denoting

$$\psi=\langle\cos p\omega\rangle,$$

we obtain for this quantity the equation

$$\psi=(p^2|\lambda|/16|J|\rho_s)^{p^2\Delta/(2-p^2\Delta)} \tag{32}$$

(compare with formula (23)).

The quadratic effective Hamiltonian has the form

$$\mathcal{H}_{\text{eff}}=\frac{1}{2}\int(|J|\rho_s(\nabla\omega)^2+p^2|\lambda|\psi\omega^2)\,d^2x. \tag{33}$$

The calculation of $\langle\varphi\rangle = \langle\cos\omega\rangle$ using the Hamiltonian (33) and the expression (32) for ψ gives

$$\langle\varphi\rangle=(p^2|\lambda|/16|J|\rho_s)^{\Delta/(2-p^2\Delta)}. \tag{34}$$

In an anisotropic system the correlation length r_C below the transition point becomes a finite quantity, which can be determined from dimensionality arguments:

$$r_c\infty|\lambda|^{1/\Delta_\lambda},\quad \Delta_\lambda=2-p^2\Delta. \tag{35}$$

In connection with this, the magnetic susceptibility in zero field below the transition point also becomes finite:

$$\chi=\frac{\partial\langle\varphi\rangle}{\partial h}\infty|\lambda|^{(\Delta_\varphi-\Delta_h)/\Delta_\lambda}=|\lambda|^{-(2-2\Delta)/(2-p^2\Delta)}.$$

In a finite field h, the magnitude of the moment $\langle\varphi\rangle$ is determined both by the field h and by the anisotropy λ. The general form of the formula for $\langle\varphi\rangle$, satisfying the requirements of scale invariance, is:

$$\langle\varphi\rangle=h^{\Delta/\Delta_h}f(|\lambda|/h^{\Delta_\lambda/\Delta_h}).$$

The dimensionless function f(z) has the following asymptotic forms:

$$f(0)=\text{const}\neq0,\quad f(z)\infty z^{\Delta/\Delta_\lambda}\quad(z\to\infty).$$

The effect of the magnetic field becomes important when the field exceeds a certain characteristic value

$$h_a=|\lambda|^{\Delta_h/\Delta_\lambda}. \tag{36}$$

It is natural to call this quantity the anisotropy field. The anisotropy field, unlike that in a three-dimensional system, depends nonlinearly on λ.

In weak fields and for weak anisotropy, the mean order parameter $\langle\varphi\rangle$ is small and attains values of order unity (saturation) only at a temperature T_2 determined by the equation

$$\Delta(T_2)=\left|\ln^{-1}\max\left[\left(\frac{h}{16|J|\rho_s}\right)^{1/\Delta_h},\ \left(\frac{p^2|\lambda|}{16|J|\rho_s}\right)^{1/\Delta_\lambda}\right]\right|. \tag{37}$$

The dependence of $\langle\varphi\rangle$ on T is depicted schematically in Fig. 3.

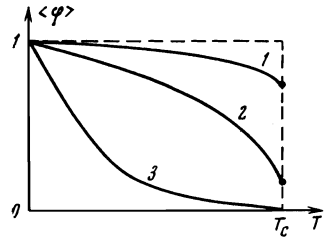

FIG. 3. Possible temperature dependences of the order parameter. For curve 1, $T_2>T_C$; for curve 2, $T_2\approx T_C$; for curve 3, $T_2<T_C$.

4. LAYER SYSTEMS

Experimenters have recently discovered and synthesized a number of three-dimensional magnets whose properties are close to those of a PDS. A bibliography referring to such magnets can be found in the review by de Jongh, Bloembergen and Colpa[13], in which experimental data on transition temperatures are also given.

All such magnets consist of a number of widely-spaced planes, such that the interaction between spins of magnetic ions lying in different planes is small compared with the interaction of spins within one plane. With good accuracy, the interaction of the spins can be assumed to be isotropic. In some of these compounds (Rb_2MnF_4, K_2MnF_4, etc.), the magnetic symmetry of the latter leads to the result that the effective interaction of neighboring magnetic layers vanishes. In such conditions, the role of anisotropy is dominant. In other compounds (e.g., $(C_nH_{2n+1}NH_3)_2CuCl_4$), the interplanar interaction is also important.

Thus, the problem of a theoretical description of magnetic layer structures becomes a real problem. A fairly large number of theoretical papers, of which we mention the paper by Berezinskiĭ and Blank[14], have been devoted to this problem. In[14], the low-temperature behavior of such systems is considered. Here we shall reproduce the results of the paper[14] by the methods of scaling theory, find their analogs at higher temperatures, and also demonstrate new results, such as a simple relation between the magnetic susceptibility and the scaling dimension $\Delta(T)$.

Let there be no anisotropy within a plane, and let the interplanar exchange integral J′ be small compared with the interplanar interaction J. The form of $\mathcal{H}_{\text{inter}}$, the correction to the Hamiltonian (3) due to the interplanar interaction, is

$$\mathcal{H}_{\text{inter}}=J'\sum_{\mathbf{x},l}\varphi_l(\mathbf{x})\varphi_{l+1}(\mathbf{x}), \tag{38}$$

where the integer index l labels the planes. For $J' \neq 0$ the system becomes three-dimensional, and therefore a spontaneous order parameter arises in it below the transition point. For small J′, the transition point of the three-dimensional system is only slightly displaced from T_C in the two-dimensional situation. We shall not study the calculation of this shift.

We shall study the behavior of φ_S sufficiently far from the transition point, when the properties of the

system are principally determined by the interaction within a plane. In the same way as in the preceding section, we shall determine the scaling dimension of the interplanar interaction constant J', regarding J' as an external field:

$$\Delta_{J'}+2\Delta_\varphi=2. \tag{39}$$

Hence,

$$\varphi_s=\langle\varphi\rangle\infty|J'|^{\Delta/(2-2\Delta)}. \tag{40}$$

Formula (40) shows that spontaneous order appears for any $J' \neq 0$ below T_c.

In the narrow region about the critical point in which the phase transition becomes three-dimensional, formula (40) is inapplicable. Here, φ_S tends to zero according to the law $(T_c - T)^\beta$, where the index $\beta \approx 1/3$.

Formula (40) was obtained by Berezinskiĭ and Blank[14] in the approximation $T/T_c \ll 1$.

As in the case of a field and anisotropy, we shall carry out a self-consistent procedure to determine the coefficient in formula (40). The full Hamiltonian of the layer system reduces to the effective quadratic Hamiltonian

$$\mathcal{H}_{\text{eff}}=\frac{1}{2}\int\left[|J|\rho_s\left(\left(\frac{\partial\omega}{\partial x}\right)^2+\left(\frac{\partial\omega}{\partial y}\right)^2\right)+|J'|\eta\left(\frac{\partial\omega}{\partial z}\right)^2\right]dx\,dy\,dz \tag{41}$$

(compare with formulas (21) and (33)). In the Hamiltonian (41) we have introduced the notation

$$\eta=\langle\cos(\omega_{l+1}(\mathbf{x})-\omega_l(\mathbf{x}))\rangle.$$

Calculation of this gives

$$\eta=(|J'|e/32|J|\rho_s)^{\Delta/(1-\Delta)} \tag{42}$$

The spontaneous moment of the layer system, calculated by means of the self-consistent method, is equal to

$$\varphi_s=(|J'|e^\Delta/32|J|\rho_s)^{\Delta/(2-2\Delta)}. \tag{43}$$

We now introduce anisotropy in the plane (cf. Sec. 3). Then the spontaneous moment φ_S is determined both by the anisotropy and by the interplanar interaction. The general form of φ_S satisfying the requirements of scale invariance is:

$$\varphi_s=|J'|^{\Delta_\varphi/\Delta_{J'}}g(|\lambda|/|J'|^{\Delta_\lambda/\Delta_{J'}}), \tag{44}$$

where g(z) is a standard dimensionless function with the following asymptotic forms:

$$g(0)=\text{const}\neq 0,\quad g(z)\infty z^{\Delta_\varphi/\Delta_\lambda}\quad(z\to\infty). \tag{45}$$

The dimension Δ_λ was determined in Sec. 3. The effect of the interplanar coupling becomes important when

$$|J'|\geqslant|\lambda|^{\Delta_{J'}/\Delta_\lambda}.$$

If we also introduce a field h, the magnitude of the moment $\langle\varphi\rangle$ can be determined as before by formula (44), in which, however, the function g depends also on the dimensionless argument

$$h/|J'|^{\Delta_h/\Delta_{J'}}.$$

We shall consider two possible cases.

1) $|J'|^{\Delta_\lambda} \gg |\lambda|^{\Delta_{J'}}$. In this case, the anisotropy is weak. The spontaneous moment φ_S is determined by the magnitude of the interplanar coupling (cf. (40) and (43)). The anisotropy becomes unimportant when $h > h_a$, where

$$h_a=\lambda\varphi_s. \tag{46}$$

This is the normal three-dimensional situation, when the anisotropy field is linearly related to the anisotropy constant λ.

2) $|J'|^{\Delta_\lambda} \ll |\lambda|^{\Delta_{J'}}$. Here the interplanar couplings are unimportant. In this case, the anisotropy field is a nonlinear function (cf. formula (36)).

With the appearance of interplanar coupling, the temperature T_2 at which saturation of the order parameter occurs can be changed (compare with formula (37)):

$$\Delta(T_2)=\left|\ln^{-1}\max\left[\left(\frac{h}{16|J|\rho_s}\right)^{1/\Delta_h},\ \left(\frac{p^2|\lambda|}{16|J|\rho_s}\right)^{1/\Delta_\lambda},\ \left(\frac{|J'|e^\Delta}{32|J|\rho_s}\right)^{1/\Delta_{J'}}\right]\right| \tag{47}$$

Of special interest are layer antiferromagnets in which the intraplanar couplings are ferromagnetic ($J < 0$) and the interplanar couplings antiferromagnetic ($J' > 0$). Such compounds include, e.g., $(C_2H_5NH_3)_2CuCl_4$, which has been thoroughly studied by de Jongh, van Amstel and Miedema[15].

The magnetic susceptibility χ of an antiferromagnet becomes finite below the transition point and at the critical point itself. A schematic arrangement of the spins of the magnetic ions is shown in Fig. 4. According to the estimates of the paper[15], the anisotropy in the plane is very small ($\sim 10^{-4}$). The anisotropy maintaining the spins in the plane is substantially greater ($\sim 10^{-3}$). The interplanar exchange interaction is also of the same order.

We shall calculate the magnetic susceptibility of a two-dimensional layer antiferromagnet with a two-component spin. We confine ourselves to the situation arising in the experiment of[15], when the anisotropy is small compared with the interplanar interaction.

Let φ_1 and φ_2 be the magnetizations of the different sublattices (planes). The free-energy density of each plane can be found from dimensionality arguments:

$$F_i=c|\varphi_i|^{2/\Delta}.$$

The free energy per unit volume of an antiferromagnet in an external field, with the interplanar coupling taken into account, has the form

$$F=c(|\varphi_1|^{2/\Delta}+|\varphi_2|^{2/\Delta})+J'\varphi_1\varphi_2-\mathbf{H}(\varphi_1+\varphi_2). \tag{48}$$

The moments of the sublattices can be found by minimizing the free energy (48):

$$\begin{aligned}J'\varphi_2+\frac{2c}{\Delta}\varphi_1|\varphi_1|^{(2-2\Delta)/\Delta}-\mathbf{H}=0,\\ J'\varphi_1+\frac{2c}{\Delta}\varphi_2|\varphi_2|^{(2-2\Delta)/\Delta}-\mathbf{H}=0.\end{aligned} \tag{49}$$

Solving the system (49) by a perturbation-theory method, we find the magnetic susceptibility in zero field:

$$\chi_\parallel=\Delta/2J',\quad \chi_\perp=1/2J'.$$

For a comparison with the experiment of[15], $\chi_\perp$ must refer to the b axis. Figure 5 shows the experimental graphs of the susceptibility, taken from the paper[15]. Our arguments refer to curves a and b. In agreement with the theory, the transverse susceptibility is practically constant and the longitudinal susceptibility tends to zero like $\sim\Delta$.

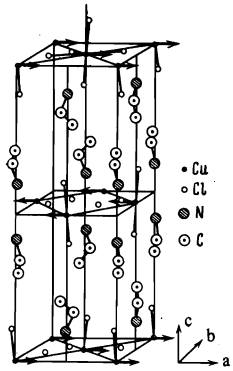

FIG. 4. Crystallographic and magnetic order in $(C_2H_5NH_3)_2CuCl_4$.

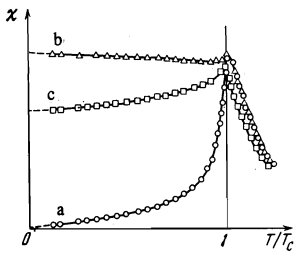

FIG. 5

We call attention to the fact that

$$\chi_{\parallel}/\chi_{\perp}=\Delta \tag{50}$$

Figure 6 juxtaposes the experimental values of the ratio $\chi_{\parallel}/\chi_{\perp}$ and the theoretical curve calculated from formulas (8) and (14).

The theory describes the experiment well at low temperatures. The discrepancy for T close to T_c can clearly be explained principally by the effects of the three-dimensional phase transition and by the inaccuracy of the self-consistent method of calculating ρ_s.

The susceptibility in zero field along the c axis can be calculated by an analogous method. We give the result:

$$\chi^b/\chi^c=1+\lambda'\varphi_s^2/2J', \tag{51}$$

where λ' is the anisotropy in the direction of the c axis. The quantities J', λ and λ' have been determined by a method independent of susceptibility measurements. Substitution of the values obtained in the experiment of[15] into formula (51) gives

$$\chi^b/\chi^c=0{,}62.$$

The experimental value is 0.66.

If the magnetic field is applied along the easy-magnetization axis, then when it reaches a certain critical value H_1 the well-known first-order phase transition occurs in which the sublattice moments are flipped perpendicularly to the field. The value of H_1 in our approximation is given by

$$H_1^2=2\lambda J'\varphi_s^2. \tag{52}$$

The dependence of φ_s on temperature is given by formula (43). Hence, we can find the temperature dependence of H_1^2:

$$H_1^2(T)/H_1^2(0)=\varphi_s^2(T). \tag{53}$$

Unfortunately, the experimental data at our disposal on H_1[15] are insufficient to check formula (53).

At a higher value $H = H_2(T)$ of the magnetic field a

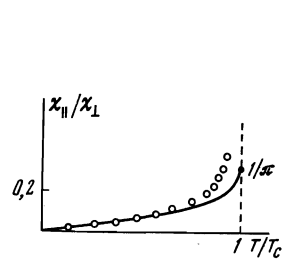

FIG. 6

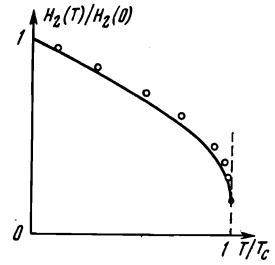

FIG. 7.

further phase transition, this time second-order, occurs: the antiferromagnet becomes a paramagnet. In this case, the antiferromagnetic couplings between the planes are effectively broken and the substance can be treated as a set of noninteracting planes placed in a magnetic field. The magnetic moment of such a system is determined by formula (19). The quantity $H_2(T)$ can itself be found by minimizing the free energy (48) with an additional condition: at the point of the minimum, the directions of φ_1 and φ_2 coincide. A simple calculation leads to the following result:

$$H_2(T)=2J'\varphi_s(T). \tag{54}$$

Comparing the formulas (52) and (54), we find

$$H_1(T)/H_2(T)=(\lambda/2J')^{1/2}. \tag{55}$$

We note that the ratio (55) does not depend on the temperature. From the experimental data given in[15], we have been able to establish that the ratio H_1/H_2 is constant in the temperature range $0.1 \lesssim T/T_c \lesssim 0.95$.

Formula (54) shows that measuring $H_2(T)$ is equivalent to measuring the spontaneous order $\varphi_s(T)$. In Fig. 7, the ratio $H_2(T)/H_2(0)$ found from the experimental curves of the paper[15] are juxtaposed with the theoretical curve for the ratio $\varphi_s(T)/\varphi_s(0)$ calculated by means of formula (43).

5. CONCLUSION

Throughout, we have considered two-component classical spins. However, the principal results of the paper do not depend on this assumption. For any n, a planar system is scale-invariant at large distances. Considerations of scale invariance have enabled us to progress from the region of low temperatures $T \ll T_c$ into the region $T \sim T_c$ and obtain a fairly complete description of the magnetic phenomena in the whole temperature range.

The theory of magnetic phenomena in layer magnets, based on scale-invariance considerations, leads to good agreement with experiment.

The region of temperatures close to the phase-transition point of a planar system, where the correlation length increases and the temperature becomes a dimensional quantity, has not been studied.

[1)]The authors are grateful to A. M. Polyakov, who directed their attention to this.

[1] G. S. Rushbrooke and P. J. Wood, Mol. Phys. 1, 257 (1958).
[2] H. E. Stanley and T. A. Kaplan, Phys. Rev. Lett. 17, 913 (1966).
[3] V. L. Berezinskiĭ, Zh. Eksp. Teor. Fiz. 59, 907 (1970) [Sov. Phys.-JETP 32, 493 (1971)].
[4] P. C. Hohenberg, Phys. Rev. 158, 383 (1967).

[5] N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966).
[6] V. L. Berezinskiĭ, Zh. Eksp. Teor. Fiz. 61, 1144 (1971) [Sov. Phys.-JETP 34, 610 (1972)].
[7] B. D. Josephson, Phys. Lett. 21, 608 (1966).
[8] R. J. Baxter, Ann. Phys. 70, 193 (1972).
[9] A. M. Polyakov, Zh. Eksp. Teor. Fiz. 63, 24 (1972) [Sov. Phys.-JETP 36, 12 (1973)].
[10] A. Z. Patashinskiĭ and V. L. Pokrovskiĭ, Zh. Eksp. Teor. Fiz. 64, 1445 (1973) [Sov. Phys.-JETP 37, 733 (1973)].
[11] A. S. Wightman, Introduction to some Aspects of the Relativistic Dynamics of Quantized Fields (Cargese Summer School of Theoretical Physics, July, 1964) (Russ. transl., Nauka, M., 1968).
[12] V. L. Pokrovskiĭ, Usp. Fiz. Nauk 94, 127 (1968) [Sov. Phys.-Uspekhi 11, 66 (1968)].
[13] L. J. de Jongh, P. Bloembergen, and J. H. Colpa, Physica 58, 305 (1972).
[14] V. L. Berezinskiĭ and A. Ya. Blank, Zh. Eksp. Teor. Fiz. 64, 725 (1973) [Sov. Phys.-JETP 37, 369 (1973)].
[15] L. J. de Jongh, W. D. van Amstel, and A. R. Miedema, Physica 58, 277 (1972).

Translated by P. J. Shepherd
175