The three-photon interaction in intense fields and scale invariance

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We derive the three-photon vertex function (the polarization tensor of rank three) describing the interaction of three real or virtual photons in an intense electromagnetic field of the form $\mathbf{E} \perp \mathbf{H}$, $E = H$. The limiting cases of low energies and virtualities and of large energies are considered. In the latter case the vertex function exhibits scale invariance, the role of the mass scale being played by the variable $L^{1/3}$, related to the photon momentum and the field by: $L_\mu = eF_{\mu\nu}l_\nu$. It is shown that the probability for the splitting of the photon $\gamma \to \gamma' + \gamma''$ is sensitive to the real and imaginary parts of the photon masses in the field owing to the nearness of the boundary of the spectrum of final-state photons to its maximum.

1. INTRODUCTION

It is well known that in an intense field three photons can interact, leading to the splitting of one photon into two others, with a change of the energy and the polarization (cf. [1–4], where further references to earlier works, which have turned out to be incorrect, can be found.

In an earlier paper [4] (which will be quoted below as I) we have considered the general structure of the polarization operator of three photons (or the three-photon vertex function) in a constant crossed field ($\mathbf{E} \perp \mathbf{H}$, $E = H$) of arbitrary intensity, we have obtained its exact expression for real photons, and we have found the probability for the splitting $\gamma \to \gamma' + \gamma''$, including the limiting cases of small and large energies or fields. For large energies the result turns out to be scale-invariant, i.e., independent of the electron mass.

In the present paper we obtain the three-photon polarization operator off the mass shell, i.e., for virtual photons. The purpose of considering such a vertex is dictated first of all by the uniqueness of the three-photon interaction in vacuum. Moreover, this vertex function: 1) can be part of diagrams describing higher-order radiative effects, 2) describes with certain simplification such processes as the scattering of photons by a Coulomb center in an intense field, etc., 3) allows one to take into account in the splitting amplitude the mass acquired by the photon as a result of its motion through an external field.

The three-photon vertex off the mass shell retains the property of scale invariance and in the limit $m = 0$ depends nontrivially on five dimensionless variables, of which only one survives on the mass shell, cf. I.

The interaction of the photon with an external field becomes important when in the frame where the photon has an energy of the order of mc^2 the field strength attains the value characteristic of quantum electrodynamics[1)]

$$F_0 = m^2c^3/e\hbar = 4.4 \cdot 10^{13}\ \text{Oe}. \qquad (1)$$

Since all known fields are much weaker than F_0, noticeable effects appear only for photons with energies much larger than m. For such photons any external field, in a reference frame where the photons have energies of the order of m, becomes very close to a plane-wave field. If, in addition, the characteristic wavelengths and the period of the field are large compared with m/eF (the length and time interval of formation of the process in a field of intensity F), then such a field can be considered as a constant crossed field. For example, the field of a ruby laser of field-strength $F \sim 4 \times 10^8$ Oe can be considered constant, since the buildup (formation) time of the process in it is $m/eF \sim 1.3 \times 10^{-16}$ s, which is one-third the field period $1/\omega \sim 3.7 \times 10^{-16}$ s.

Thus, a process caused by a high-energy photon in an external field is determined essentially by a single invariant parameter

$$\varkappa = \sqrt{(eFl)^2}/m^3, \qquad (2)$$

since the pure field invariants

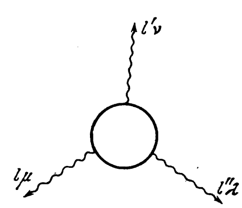
$$\mathcal{F} = e^2F_{\mu\nu}^2/m^4, \quad \mathcal{G} = e^2F_{\mu\nu}F^*_{\mu\nu}/m^4 \qquad (3)$$

are, under the given conditions, very small compared with unity and with the variable κ. Neglecting these invariants is equivalent to a passage to the crossed field, for which $\mathcal{F} = \mathcal{G} = 0$ rigorously.

The main contents of this paper are concentrated in Secs. 2–5. In Sec. 2 we discuss briefly the kinematic structure of the three-photon polarization operator, which is characterized by ten invariant functions; we discuss methods for calculating them and in the Appendix we list explicit expressions for all ten functions.

In Sec. 3 the polarization operator is considered in the region of small energies $\kappa \ll 1$ and small virtualities of the photons. In this case all proper time integrals can be calculated and the simple formulas obtained for the scalar functions show clearly the dynamical structure of the polarization operator. In addition, selection rules are given for Delbrück scattering with different special directions of the photon momenta with respect to electric and magnetic field vectors.

In the limit of large energies $\kappa \gg 1$, considered in Sec. 4, the three-photon vertex function becomes scale-invariant, i.e., does not depend on the electron mass. The mass scale is now $L^{1/3}$, where $L = \sqrt{(eFl)^2}$. The property of scale invariance distinguishes the three-photon vertex from the two- and four-photon vertices, which are not scale-invariant owing to vacuum contribu-



tions (i.e., parts which do not vanish in the limit F = 0).

Finally, in Sec. 5, we show that in the region $\kappa \gtrsim 1$ the splitting probability is quite sensitive to the real and imaginary parts of the masses of the photon in the field, since the boundary of the spectrum of the final-state photons is within the limits of the natural width of the spectral line.

2. THE STRUCTURE OF THE POLARIZATION OPERATOR. EXACT RESULTS

We consider the interaction of three photons to lowest order in the radiation field and exactly in the external field. Such an interaction is described by two diagrams, one of which is represented in the figure, the other differing from it by transposition of two of the photons $l'\nu \rightleftarrows l''\lambda$. In vacuum these two diagrams cancel each other (Furry's theorem). In an external field, the interaction with which is represented in the diagram by the boldface electron loop, the sum of such diagrams does not vanish and gives rise (cf. I) to a diagonal tensor of rank three

$$(2\pi)^4\delta(l+l'+l'')\,\Pi_{\mu\nu\lambda}(ll'l''). \qquad (4)$$

The momenta l, l', l'' belong in general to virtual photons (i.e., their squares are not zero), and the polarization operator defined in this manner may be considered as part of more complicated diagrams.

The polarization vector of a photon of momentum l_μ can be decomposed in terms of three independent vectors (this is a consequence of the transversality of the photon):

$$L_\mu=eF_{\mu\nu}l_\nu,\ L_\mu^*=eF_{\mu\nu}^*l_\nu,\ G_\mu=L^{-2}l^2e^2F_{\mu\nu}F_{\nu\lambda}l_\lambda+l_\mu, \qquad (5)$$

these three vectors are orthogonal to l_μ and for $FF^* = 0$ they are mutually orthogonal.

In I, starting from the transversality, charge and space parity and the Bose statistics of the photons we have obtained a representation of $\Pi_{\mu\nu\lambda}(ll'l'')$ in a crossed electromagnetic field in terms of ten invariant functions $c_{n_1}c_{n_2}c_{n_3}$ (n_1, n_2, n_3 are respectively the photon numbers in the polarization states L, L*, G; $n_1 + n_2 + n_3 = 3$):

$$\Pi_{\mu\nu\lambda}(ll'l'')=\sum\psi_{n_1n_2n_3}(l\mu,l'\nu,l''\lambda), \qquad (6)$$

where

$$\psi_{300}=c_{300}^{++}(ll'l'')L_\mu L_\nu' L_\lambda''/LL'L'', \qquad (7.1)$$

$$\psi_{210}=\sum_{\text{sym}}c_{210}^{+-}(ll'l'')L_\mu L_\nu' L_\lambda^{*\prime\prime}/LL'L^{*\prime\prime}, \qquad (7.2)$$

$$\psi_{201}=\sum_{\text{sym}}c_{201}^{-+}(ll'l'')L_\mu L_\nu' G_\lambda''/LL'G'', \qquad (7.3)$$

$$\psi_{120}=\sum_{\text{sym}}c_{120}^{++}(ll'l'')L_\mu L_\nu^{*\prime} L_\lambda^{*\prime\prime}/LL^{*\prime}L^{*\prime\prime}, \qquad (7.4)$$

$$\psi_{111}=\sum_{\text{sym}}c_{111}^{--}(ll'l'')L_\mu L_\nu^{*\prime} G_\lambda''/LL^{*\prime}G'', \qquad (7.5)$$

$$\psi_{102}=\sum_{\text{sym}}c_{102}^{++}(ll'l'')L_\mu G_\nu' G_\lambda''/LG'G'', \qquad (7.6)$$

$$\psi_{030}=c_{030}^{+-}(ll'l'')L_\mu^* L_\nu^{*\prime} L_\lambda^{*\prime\prime}/L^*L^{*\prime}L^{*\prime\prime}, \qquad (7.7)$$

$$\psi_{021}=\sum_{\text{sym}}c_{021}^{-+}(ll'l'')L_\mu^* L_\nu^{*\prime} G_\lambda''/L^*L^{*\prime}G'', \qquad (7.8)$$

$$\psi_{012}=\sum_{\text{sym}}c_{012}^{+-}(ll'l'')L_\mu^* G_\nu' G_\lambda''/L^*G'G'', \qquad (7.9)$$

$$\psi_{003}=c_{003}^{-+}(ll'l'')G_\mu G_\nu' G_\lambda''/GG'G''. \qquad (7.10)$$

The plus or minus signs of the $c_{n_1n_2n_3}$ denote their spatial and charge parities. The symbol $\sum_{\text{sym}}$ denotes symmetrization with respect to all possible permutations of the photons of different types, and the functions $c_{n_1n_2n_3}$ with any index $n_i \geq 2$ are already symmetric with respect to permutations of the photon of type i. In Eqs. (7) the vectors (5) for the momenta l'_α, l''_α are marked by one, respectively two dashes, and the invariants L, L*, G are defined by the relations[2)]

$$L=L^*=m^3\varkappa=l_-eF,\ G=\sqrt{-l^2}, \qquad (8)$$

i.e., the signs of κ, L, and L* are determined by the sign of l.

From the tensor of the crossed constant field $F_{\mu\nu}$ and the momenta of the three photons one can form, in addition to the three squared momenta, five other invariants: three charge-symmetric scalars κ, κ', κ'' (or L, L′, L″), one charge-antisymmetric scalar $\rho = eF_{\alpha\beta}l'_\alpha l_\beta$ and one charge-antisymmetric pseudoscalar τ = eF $\tau = eF^*_{\alpha\beta}l'_\alpha l_\beta$, related by the conservation laws

$$\varkappa+\varkappa'+\varkappa''=0, \qquad (9)$$

$$\rho^2+\tau^2+l^2L'L''+l'^2L''L+l''^2LL'=0. \qquad (10)$$

The invariants ρ and τ are antisymmetric with respect to permutations of any two photons and their product $\rho\tau$ forms the unique charge-symmetric pseudoscalar which is symmetric under permutations of the photons. Therefore, from parity considerations c_{201}^{-+}, c_{121}^{-+}, c_{003}^{-+} are proportional to ρ, the function c_{111}^{--} is proportional to τ, and c_{210}^{+-}, c_{030}^{+-}, c_{012}^{+-} are proportional to $\rho\tau$ or to odd functions of the appropriate variables[3)]. The expressions (21) obtained below by a direct calculation for the invariant functions $c_{n_1n_2n_3}$ have just such a general structure, and the role of the odd function is played either by the sine or by an odd power of ρ or τ.

In order to compute the polarization operator we use the Green's function of the electron in a constant homogeneous field in the proper time representation

$$G(x_2,x_1)=\exp\left(ie\int_{x_1}^{x_2}dx'\,A(x')\right)S(x_2-x_1). \qquad (11)$$

The diagonal part, which depends only on the coordinate difference, has the following form in crossed fields:

$$S(x)=\frac{-i}{(4\pi)^2}\int_0^\infty\frac{ds}{s^2}\exp\left[i\frac{x^2}{4s}-is\left(m^2+\frac{(eFx)^2}{12}\right)\right](m+i\gamma V+i\sigma T+\gamma_5\gamma A), \qquad (12)$$

where

$$V_\alpha=-{}^1/_2x_\alpha/s+{}^1/_3se^2F_{\alpha\beta}F_{\beta\gamma}x_\gamma,\ T_{\alpha\beta}={}^1/_2mseF_{\alpha\beta},\ A_\alpha={}^1/_2eF_{\alpha\beta}^*x_\beta. \qquad (13)$$

Further, following I, we write out the matrix element of the diagram (cf. the figure) and go over to momentum space, by integrating over the coordinates of one of the photons (this integral yields the delta-function expressing 4-momentum conservation) and over the differences of the coordinates x′ and x″. The result is the matrix element $I_{\mu\nu\lambda}$ in the form of a triple integral over the proper times of the photons s, s′, s″:

$$I_{\mu\nu\lambda}(ll'l''F)=\frac{ie^3}{(2\pi)^2}\iiint_0^\infty\frac{ds\,ds'\,ds''}{(s+s'+s'')^2}e^{-i\Phi}Q_{\mu\nu\lambda}, \qquad (14)$$

where the phase of the integrand is

$$\Phi=m^2(s+s'+s'')+\beta\left(\frac{l^2}{s}+\frac{l'^2}{s'}+\frac{l''^2}{s''}\right)+2\beta\rho+\frac{\beta}{3}(L^2\gamma+L'^2\gamma'+L''^2\gamma''),\tag{15}$$

with $\beta = ss's''/(s + s' + s'')$, $\gamma = s' + s'' - s + s's''/s$, and γ' and γ'' are obtained from γ and γ', respectively, by the cyclic permutation $s \to s' \to s'' - s$. The tensor $Q_{\mu\nu\lambda}$ can be represented as the result of applying the differential operator $\hat{Q}_{\mu\nu\lambda}$ to the exponential $e^{-i\Phi}$: $\hat{Q}_{\mu\nu\lambda}e^{-i\Phi} = e^{-i\Phi}Q_{\mu\nu\lambda}$. This operator is determined by the trace

$$\hat{Q}_{\mu\nu\lambda}={}^1/_4\mathrm{Sp}[(m+i\gamma\hat{V}'+i\sigma T'+\gamma_5\gamma\hat{A}')\gamma_\lambda(m+i\gamma\hat{V}+i\sigma T+\gamma_5\gamma\hat{A})\times\gamma_\nu(m+i\gamma\hat{V}''+i\sigma T''+\gamma_5\gamma\hat{A}'')\gamma_\mu],\tag{16}$$

where $\hat{V}$ and $\hat{A}$ are given by the equations (13) with the substitution of the differential operators $\hat{x}$, $\hat{x}'$, $\hat{x}''$ for x, x', x'':

$$\hat{x}_\alpha''=i\partial/\partial l_\alpha',\ \hat{x}'_\alpha=-i\partial/\partial l_\alpha'',\ \hat{x}+\hat{x}'+\hat{x}''=0,\quad \hat{V}_\alpha=V_\alpha(\hat{x},s),\ \hat{V}_\alpha'=V_\alpha(\hat{x}',s')\tag{17}$$

etc. The explicit expression for $\hat{Q}_{\mu\nu\lambda}$ is given in the appendix to I and contains terms which are linear and cubic in x. Replacing these terms by their "eigenvalues" (given by Eqs. (16), (17) in I), one can obtain the numerical tensor $Q_{\mu\nu\lambda}$.

The polarization tensor $\Pi_{\mu\nu\lambda}$ is a sum of matrix elements from two diagrams, which on account of charge symmetry differ by a sign and the substitution $F_{\mu\nu} \to -F_{\mu\nu}$, i.e.,

$$\Pi_{\mu\nu\lambda}(ll'l''F)=I_{\mu\nu\lambda}(ll'l''F)-I_{\mu\nu\lambda}(ll'l''-F)=\frac{ie^3}{(2\pi)^2}\iiint_0^\infty\frac{ds\,ds'\,ds''}{(s+s'+s'')^2}[e^{-i\Phi(F)}Q_{\mu\nu\lambda}(F)-e^{-i\Phi(-F)}Q_{\mu\nu\lambda}(-F)].\tag{18}$$

It follows from this expression that when the external field is switched off the three-photon polarization operator vanishes.

However, the matrix element of the separate three-photon diagrams do not vanish when the field is switched off:

$$I_{\mu\nu\lambda}\ (F=0)=\sum_{\text{antisym}}\{a(ll'l'')\delta_{\mu\nu}l_\lambda'+b(ll'l'')({}^1/_3l_\mu'l_\nu''-\delta_{\mu\nu}l'l'')l_\lambda+c(ll'l'')[l_\mu(l_\nu l_\lambda'+l_\lambda l_\nu'-\delta_{\mu\nu}ll')-{}^1/_2l^2(\delta_{\mu\nu}l_\lambda'-\delta_{\nu\lambda}l_\mu'+\delta_{\lambda\mu}l_\nu')]\}.\tag{19}$$

It is antisymmetric with respect to the permutation of any pair of photons (i.e., of the variables $l\mu$, $l'\nu$, $l''\lambda$) and is determined by three invariant functions of the momenta:

$$a(ll'l'')=\frac{ie^3}{4\pi^2}\int_0^1du\int_0^1dv\,u^2(\Lambda^{-1}-1-\ln\Lambda),$$
$$b(ll'l'')=\frac{ie^3}{2\pi^2m^2}\int_0^1du\int_0^1dv\,u^3v(1-u)(1-v)\Lambda^{-1},\tag{19'}$$
$$c(ll'l'')=\frac{ie^3}{2\pi^2m^2}\int_0^1du\int_0^1dv\,u^4v(1-v)^2\Lambda^{-1};$$
$$\Lambda=1+\frac{l^2}{m^2}u^2v(1-v)+\frac{l'^2}{m^2}u(1-u)(1-v)+\frac{l''^2}{m^2}uv(1-u).$$

Of these b is symmetric with respect to any permutation of the arguments, and a is symmetric with respect to a permutation of the last two arguments. In Eq. (19′) we have used the variables (22) and have carried out a regularization of the logarithmic divergence.

Since the explicit expression of the tensor $Q_{\mu\nu\lambda}$ is rather complicated it is more convenient to construct the polarization operator directly in the representation (7) of the invariant functions $c_{n_1n_2n_3}$. For this purpose we write $\Pi_{\mu\nu\lambda} = \delta_{\mu\alpha}\delta_{\nu\beta}\delta_{\lambda\gamma}\Pi_{\alpha\beta\gamma}$ and expand the Krönecker symbols in terms of the vectors (5):

$$\delta_{\mu\alpha}=\frac{l_\mu l_\alpha}{l^2}+\frac{L_\mu L_\alpha}{L^2}+\frac{L_\mu^*L_\alpha^*}{L^{*2}}+\frac{G_\mu G_\alpha}{G^2}.\tag{20}$$

$\delta_{\nu\beta}$ and $\delta_{\lambda\gamma}$ are expanded in terms of the same system of vectors, but constructed respectively out of the momenta l' and l''. Transversality implies

$$l_\alpha\Pi_{\alpha\beta\gamma}=l_\beta'\Pi_{\alpha\beta\gamma}=l_\gamma''\Pi_{\alpha\beta\gamma}=0$$

and in the product of three δ-symbols there remain 27 terms which make up the sum (6). The coefficients $c_{n_1n_2n_3}$ appear then as contractions of the tensor $\Pi_{\alpha\beta\gamma}$ with the vectors L_α, L^*_α, G_α, e.g.,

$$c_{120}(ll'l'')=\Pi_{\alpha\beta\gamma}L_\alpha L_\beta^{*\prime}L_\gamma^{*\prime\prime}/LL^{*\prime}L^{*\prime\prime}.$$

As a result we obtain the following representation of the coefficients $c_{n_1n_2n_3}$

$$c^{\pm P}_{n_1n_2n_3}(ll'l'')=\frac{ie^2}{(2\pi)^2}\iiint_0^\infty\frac{ds\,ds'\,ds''}{(s+s'+s'')^2}[e^{-i\Phi(F)}q_{n_1n_2n_3}(F)\pm e^{-i\Phi(-F)}q_{n_1n_2n_3}(-F)],\tag{21}$$

where $q_{n_1n_2n_3}(F)$ denotes the contraction of the tensor $Q_{\mu\nu\lambda}(F)$ with the vectors (5) for the state $n_1n_2n_3$, and the upper and lower signs correspond to positive and negative charge-conjugation parity of the functions $c_{n_1n_2n_3}$. In the expressions $\Phi(F)$ and $q_{n_1n_2n_3}(F)$, which are functions of the parameters L, L', L'', l^2, l'^2, l''^2, ρ, τ, a change of the sign of the field, $F_{\mu\nu} \to -F_{\mu\nu}$ signifies a change of sign only of ρ and τ, cf. (8).

Explicit expressions for the ten contractions $q_{n_1n_2n_3}$, which together with the phase $\Phi(F)$, cf. (21), determine the invariant functions $c_{n_1n_2n_3}$, are listed in the Appendix.

Going onto the mass shell $l^2 = l'^2 = l''^2 = 0$, it follows from the conservation law of four-momentum that the four-momenta of the photons become parallel. This means that ρ and τ vanish (cf. also the identity (10)). Then it can be seen from the equations of the Appendix that all $q_{n_1n_2n_3}$ vanish, with the exception of q_{120}, where the second and last bracket survive, and q_{300}, which reduces to the second terms in the first square bracket and in the last square bracket. Going over to dimensionless integration variables

$$\eta=m^2(s+s'+s''),\quad u=\frac{s'+s''}{s+s'+s''},\quad v=\frac{s'}{s'+s''},\tag{22}$$

we obtain exactly the results (19)—(24) of I.

It should be remarked that as one of the L, L′ or L″ tends to zero, some terms in $q_{n_1}q_{n_2}q_{n_3}$ (usually the first ones written in the curly brackets of the equations listed in the Appendix), and together with these also $c_{n_1n_2n_3}$, tend to infinity. However the tensor $\Pi_{\mu\nu\lambda}$ remains finite, since the increasing terms in the different polarization states $c_{n_1n_2n_3}$ compensate one another. Such a phenomenon arises also in the decomposition of the Krönecker delta with respect to the vectors, where the separate terms tend to infinity for $L \to 0$, whereas $\delta_{\mu\alpha}$ remains, of course, finite. Thus, for $L \to 0$ both the coefficient-functions $c_{n_1n_2n_3}$ and their integrands $q_{n_1n_2n_3}$ are related among themselves. For example,

$$[l_1 q_{300}(ll'l'') - l_2 q_{210}(l''l'l) - \sqrt{-l^2}\, q_{201}(l''l'l)]_{L\to 0}=0. \qquad (23)$$

For $L \to 0$ there appear six such relations; they are useful for a control of the calculations.

3. SMALL VALUES OF THE PARAMETERS

The structure of the expressions given in the Appendix simplifies considerably and becomes more transparent in the lowest approximation in κ and in the virtuality of the photons, in other words, when all κ are small compared to one and when the squares of the photon four-momenta are small compared to m^2. This approximation is also interesting from the experimental point of view. In this limit one can integrate the expressions (21) for the coefficient-functions by making the substitution (22) in the triple integral. We then obtain up to terms of the order κ and $(l^2/m^2)^2$ (according to (10) the quantities ρ^2 and τ^2 are also small, of the order $l^2\kappa^2$):

$$c_{300}=\frac{e^3 m}{9\pi^2}\left\{-\frac{l^2\varkappa+l'^2\varkappa'+l''^2\varkappa''}{20m^2}+\frac{2\rho^2}{5m^8}\left(\frac{1}{\varkappa}+\frac{1}{\varkappa'}+\frac{1}{\varkappa''}\right)-\frac{\rho^4}{70m^{16}\varkappa\varkappa'\varkappa''}+\frac{\rho^2}{56m^{10}\varkappa\varkappa'\varkappa''}\left[l^2\left(\frac{\varkappa^2}{5}-\frac{\varkappa'^2+\varkappa''^2}{2}\right)+l'^2\left(\frac{\varkappa'^2}{5}-\frac{\varkappa''^2+\varkappa^2}{2}\right)+l''^2\left(\frac{\varkappa''^2}{5}-\frac{\varkappa^2+\varkappa'^2}{2}\right)\right]\right\},$$

$$c_{210}=\frac{e^3 m}{3\pi^2}\frac{\tau\rho}{m^8}\left[\frac{2}{15\varkappa''}+\frac{\varkappa''}{20\varkappa\varkappa'}-\frac{\rho^2}{210m^8\varkappa\varkappa'\varkappa''}-\frac{1}{56m^2\varkappa''}\left(\frac{l^2+l'^2}{10}+\frac{l''^2}{3}\right)\right],$$

$$c_{201}=\frac{e^3 m}{45\pi^2}\frac{\rho\sqrt{-l''^2}}{m^5}\left(\frac{\varkappa-\varkappa'}{\varkappa''}-\frac{\rho^2(\varkappa-\varkappa')}{28m^4\varkappa\varkappa'\varkappa''}-\frac{5l''^2(\varkappa-\varkappa')-l^2(11\varkappa+14\varkappa')+l'^2(11\varkappa'+14\varkappa)}{112m^2\varkappa''}\right)$$

$$c_{120}=\frac{e^3 m}{2\pi^2}\left[\frac{7\rho^2}{45m^6\varkappa}+\frac{\tau^2\varkappa}{30m^8\varkappa'\varkappa''}-\frac{\tau^2\rho^2}{315m^{14}\varkappa\varkappa'\varkappa''}-\frac{\rho^2}{84m^{10}\varkappa}\left(\frac{l^2}{3}+\frac{l'^2+l''^2}{10}\right)-\frac{l^2\varkappa}{40m^2}+\frac{l'^2(3\varkappa-16\varkappa'')+l''^2(3\varkappa-16\varkappa')}{360m^2}\right],$$

$$c_{111}=\frac{e^3 m}{180\pi^2}\frac{\tau\sqrt{-l'^2}}{m^5}\left(\frac{7\varkappa+4\varkappa'}{\varkappa''}-\frac{\rho^2(\varkappa-\varkappa')}{7m^8\varkappa\varkappa'\varkappa''}\right), \qquad (24)$$

$$c_{102}=\frac{e^3 m}{45\pi^2}\frac{\sqrt{l'^2 l''^2}}{m^2}\left(\varkappa+\frac{\rho^2(3\varkappa'^2+3\varkappa''^2-2\varkappa^2)}{28m^8\varkappa\varkappa'\varkappa''}\right),$$

$$c_{030}=\frac{e^3 m}{18\pi^2}\frac{\tau\rho}{m^8}\left\{\frac{7}{5}\left(\frac{1}{\varkappa}+\frac{1}{\varkappa'}+\frac{1}{\varkappa''}\right)-\frac{\tau^2}{35m^8\varkappa\varkappa'\varkappa''}+\frac{1}{28m^2\varkappa\varkappa'\varkappa''}\left[l^2\left(\frac{\varkappa^2}{5}-\frac{\varkappa'^2+\varkappa''^2}{2}\right)+l'^2\left(\frac{\varkappa'^2}{5}-\frac{\varkappa''^2+\varkappa^2}{2}\right)+l''^2\left(\frac{\varkappa''^2}{5}-\frac{\varkappa^2+\varkappa'^2}{2}\right)\right]\right\},$$

$$c_{021}=\frac{e^3 m}{180\pi^2}\frac{\rho\sqrt{-l''^2}}{m^5}\left(7\frac{\varkappa-\varkappa'}{\varkappa''}+\frac{\tau^2(\varkappa-\varkappa')}{7m^8\varkappa\varkappa'\varkappa''}-\frac{5l''^2(\varkappa-\varkappa')-l^2(11\varkappa+14\varkappa')+l'^2(11\varkappa'+14\varkappa)}{28m^2\varkappa''}\right)$$

$$c_{012}=\frac{e^3 m}{1260\pi^2}\frac{\tau\rho\sqrt{l'^2 l''^2}}{m^{10}}\frac{3\varkappa'^2+3\varkappa''^2-2\varkappa^2}{\varkappa\varkappa'\varkappa''}$$

$$c_{003}=\frac{e^3 m}{420\pi^2}\frac{\rho\sqrt{-l^2 l'^2 l''^2}}{m^7}\frac{(\varkappa-\varkappa')(\varkappa'-\varkappa'')(\varkappa''-\varkappa)}{\varkappa\varkappa'\varkappa''}$$

A distinguishing feature of the expressions obtained here is their linearity with respect to the weak external field F. This property is exhibited by the real part of the off-mass-shell polarization operator for $F \to 0$. On the mass shell the terms which are linear in the external field will be absent and the expansion of the real part of the polarization operator for $\kappa \to 1$ starts with cubic terms[1-4], since the quadratic ones are absent owing to charge-conjugation selection rules. The physical meaning of this property of the one-shell polarization operator is discussed at the end of I, where it is also pointed out that the imaginary parts of the functions $c_{n_1n_2n_3}$ behave for $\kappa \to 0$ like $\exp(-1/\kappa)$, and thus cannot be expanded in a perturbation theory series. In general, the point $\kappa = 0$ is an essential singularity for the polarization operator, so that the expansion of the real part with respect to the field is an asymptotic expansion.

As we already recalled, the polarization operator off the mass shell describes the scattering of a photon on a Coulomb center in an intense field. In this case two photons are real $l^2 = l'^2 = 0$, and one is virtual, with $l''^2 > 0$. If the three-momenta $\mathbf{l}$ and $\mathbf{l}'$ of the incident and scattered photon are coplanar with $\mathbf{H}$ and $\mathbf{E}\times\mathbf{H}$ the polarization states ψ_{201}, ψ_{021}, ψ_{003}, ψ_{210}, ψ_{030}, ψ_{012} are absent. If, however, the momenta $\mathbf{l}$ and $\mathbf{l}'$ are coplanar with $\mathbf{E}$ and $\mathbf{E}\times\mathbf{H}$, then the states ψ_{111}, ψ_{210}, ψ_{030}, ψ_{012} are absent. These selection rules follow directly from the vanishing of the invariants ρ or τ, which in a special coordinate system have the form

$$\rho=eF(l_1'l_- - l_-'l_1),\quad \tau=-eF(l_2'l_- - l_-'l_2). \qquad (25)$$

We stress that these selection rules have a general character, not related to the approximation made in this section, and follow from considerations of charge- and space-parity (cf. Sec. 2) and the expressions (24) are only an intuitive illustration thereof.

4. SCALE INVARIANCE

Let us consider the polarization operator in the limit of high energies or strong fields $\kappa \gg 1$, which is equivalent to going to the limit of an infinitesimal electron mass $m^2 \to 0$. For this purpose it is convenient to replace the proper times by the dimensionless integration variables

$$u=\frac{s'+s''}{s+s'+s''},\quad v=\frac{s'}{s'+s''},\quad z=L^{2/3}(s+s'+s''), \qquad (26)$$

which differ from (22) by the use of $L^{2/3}$ instead of m^2 for making time dimensionless. Then, the polarization operator (more precisely, its invariant functions $c_{n_1n_2n_3}$) is naturally expressed in terms of six independent dimensionless invariant parameters[4)]

$$\varkappa=\frac{L}{m^3},\quad \theta=-\frac{L'}{L},\ \frac{l^2}{L^{2/3}},\ \frac{l'^2}{L^{2/3}},\ \frac{l''^2}{L^{2/3}},\ \frac{\rho}{L^{4/3}},\ \frac{\tau}{L^{4/3}}, \qquad (27)$$

where the electron mass enters only into κ.

The dependence of the polarization operator on κ is such that the limit $m \to 0$ exists and is different from zero. This can be seen from the concrete representation

$$\Pi_{\mu\nu\lambda}=\frac{ie^3L^{-2/3}}{(2\pi)^2}\int_0^1 u\,du\int_0^1 dv\int_0^\infty dz\,[e^{-i\Phi(F)}Q_{\mu\nu\lambda}(F)-e^{-i\Phi(-F)}Q_{\mu\nu\lambda}(-F)], \qquad (28)$$

where the phase has the form

$$\Phi=(\varkappa^{-2/3}+\lambda)z+\sigma z^2+\omega z^3/3,$$
$$\lambda=\frac{l^2}{L^{2/3}}u^2v(1-v)+\frac{l'^2}{L^{2/3}}u(1-u)(1-v)+\frac{l''^2}{L^{2/3}}u(1-u)v,$$
$$\sigma=2u^2(1-u)v(1-v)\frac{\rho}{L^{4/3}},$$
$$\omega=u^2\{[v(1-uv)-\theta(1-u)]^2+4\theta u(1-u)v(1-v)^2\}, \qquad (29)$$

and the tensor $Q_{\mu\nu\lambda}$ is defined by Eqs. (18) and (21). The phase Φ depends on m only through the additive parameter $\kappa^{-2/3}$, which vanishes in the limit $m = 0$, whereas Φ remains different from zero and finite. This is also valid on the mass shell, when $\lambda = \sigma = 0$ and the phase retains only the terms cubic in z. Similarly the tensor $Q_{\mu\nu\lambda}$ depends on m only through the additive parameter $\kappa^{-2/3}$ which vanishes in the limit m = 0 (cf. the Appendix). As a result of this, in the limit m = 0 the functions $c_{n_1n_2n_3}$ depend nontrivially only on the five invariant parameters:

$$c_{n_1n_2n_3}|_{m=0}=L^{1/3}\tilde{c}_{n_1n_2n_3}(\theta,\ \psi,\ l^2/L^{2/3},\ l'^2/L^{2/3},\ l''^2/L^{2/3}), \qquad (30)$$

and the dimensional parameter $L^{1/3}$ fixes the scale of these functions ($\tilde{c}$ is a dimensionless function). On the mass shell only the functions c_{300} and c_{120} are different from zero; these functions depend here only on one variable θ and have a constant ratio between their imaginary and real parts:

$$c_{n_1n_2n_3}=\frac{e^3(3L)^{1/3}}{12\pi^2}\Gamma\left(\frac{1}{3}\right)(1+i\sqrt{3})\,g_{n_1n_2n_3}(\theta). \qquad (31)$$

The real functions g_{300} and g_{120} are given in I and their graphs are represented there in Fig. 2. If the squares of the photon momenta are nonzero, but satisfy the conditions $|l^2|$, $|l'^2|$, $|l''^2| \ll L^{2/3}$ then for $\kappa \gg 1$ the polarization operator is again determined by two functions only, namely c_{300} and c_{120}, cf. Eq. (31), since the other terms will be of the order $|l^2|^{1/2}/L^{1/3}$ compared with these. The corrections to Eq. (31) will be of the same order.

One may call the disappearance of the dependence on the electron mass for $\kappa \gg 1$ scale invariance, since the mass of the electron is no longer the scale of the energy variables, but one of the energy variables itself, in our case $L^{1/3}$. As was made clear in[6,7], scale invariance is a property of the simplest vertex, describing the photon-electron interaction in an intense field at large energies, and is physically related to the fact that for $\kappa \lesssim 1$ the interaction happens over a length ~m/eF depending on the electron mass, whereas for $\kappa \gg 1$ it happens over a length $(m/eF)\kappa^{1/3} = L^{1/3}/eF$ which does not depend on m. Whereas over the first length the work done by the field is of the order of m, over the second length it is of the order of $L^{1/3}$. In the proper coordinate frame this latter length is small compared to the Compton wavelength. Thus, the process builds up over a length determined by the field in the proper reference frame, and not by the characteristic dimensions of the electron.

This paper shows that similar properties are exhibited by the more complicated interaction of three photons among themselves.

The dimensions of the region where the three-photon interaction is formed follow directly from an estimate of the proper times which give the main contribution to the integral (28). It is obvious that all three proper times play the same role and therefore the effective values of u and v are ~1. We obtain the effective values of the variable z by an estimate for the case when

$$l^2/L^{2/3},\ l'^2/L^{2/3},\ l''^2/L^{2/3}\leqslant 1. \qquad (32)$$

Then $\sigma^2 \lesssim \lambda \lesssim 1$ and the effective values of z are determined by the quantity κ. If $\kappa \ll 1$, the integral with respect to z can be estimated by means of the method of steepest descent. The saddle point z_0 is in the complex plane, far from the real axis at a distance

$$\mathrm{Im}\, z_0=[\omega(\varkappa^{-2/3}+\lambda)-\sigma^2]^{1/2}/\omega\sim\varkappa^{-1/3}\gg 1,$$

which is much larger than the size of the saddle

$$[2|\Phi''(z_0)|^{-1}]^{1/2}=[\omega(\varkappa^{-2/3}+\lambda)-\sigma^2]^{-1/4}\sim\varkappa^{1/6}\ll 1.$$

Therefore, along the real axis the effective values of z are near Re z_0 in the interval $\Delta z \sim \mathrm{Im}\, z_0$. This means that the effective proper times $\Delta s \sim L^{-2/3}\kappa^{-1/3} = m/L$. Since a change in the proper time is related to a change of the coordinate and the momentum by the relation $dx_\mu/ds = 2\pi_\mu$ (cf.[5]), the spatial region of formation of the process is $\Delta x \sim l_- m/L = m/eF$.

For $\kappa \gg 1$ and the condition (32) values $\Delta z \sim 1$ will be important in the integral with respect to z, i.e., $\Delta s \sim L^{-2/3}$ and the spatial region of formation is $\Delta x \sim l_- L^{-2/3} = L^{1/3}/eF$.

It should be noted that the usual polarization operator of second rank does not exhibit scale invariance in an intense field. This operator has the structure

$$\Pi_{\mu\nu}(l,F)=\pi_1\frac{L_\mu L_\nu}{L^2}+\pi_2\frac{L_\mu^* L_\nu^*}{L^2}+\pi_3\left(\delta_{\mu\nu}-\frac{l_\mu l_\nu}{l^2}\right) \qquad (33)$$

and is characterized by three invariant functions $\pi_{1,2,3}$, depending on κ and l^2/m^2. Utilizing for these functions the representation given in[8] it is easy to see that for $\kappa \gg 1$ or $m \to 0$ the first two functions become scale invariant[5)]:

$$\pi_{1,2}=\frac{2\alpha L^{2/3}}{3\pi}\int_4^\infty dv\,\frac{2v+1\mp 3}{v^{5/3}[v(v-4)]^{1/2}}f'(\xi v^{-2/3}),\ \xi=\frac{l^2}{L^{2/3}}, \qquad (34)$$

whereas the third one does not have this property:

$$\pi_3=\frac{4\alpha l^2}{\pi}\int_4^\infty\frac{dv}{v^2[v(v-4)]^{1/2}}\left[f_1(\xi v^{-2/3})-\ln\frac{l^2}{m^2 v}\right], \qquad (35)$$

but vanishes on the mass shell. The logarithmic divergence of π_3 for $m^2 \to 0$ is due to the vacuum part of the polarization operator. The field part (which vanishes when the field is switched off) is scale invariant. In this sense the properties of the polarization operators of the second and third ranks are similar, since the operator $\Pi_{\mu\nu\lambda}$ does not have a vacuum part on account of the Furry theorem, but its individual constituents are scale invariant.

5. ON THE INFLUENCE OF THE PHOTON MASS ON THE PROBABILITY OF PHOTON SPLITTING

In the papers of Adler et al.[2,3] polarization selection rules have been derived for the splitting of photons which propagate through a magnetic field and have energies below the pair production threshold, $\omega < 2m$. We would like to point out that for photons in a crossed field (as well as in a magnetic field but at energies larger than the pair production threshold) these rules are, in general, not valid, but the mass which the photons acquire influences the splitting probability. We recall that the polarization selection rules appear when one considers the four-dimensional delta function $\delta(l' + l'' - l)$ contained in the amplitude for the splitting of the photon of momentum l into two photons of momenta l' and l''. For massless photons the argument of this delta function vanishes only for collinear photons, i.e., just on the boundary of the physical region of variation of the variables. Therefore in the integration over the final states there appears an indeterminacy as to whether the delta function has "fired" or not. In order to remove this indeterminacy the authors of[2,3] have taken into account the fact that photons in an external field acquire a small "mass," depending on the polarization of the photon. For some polarization channels the argument in the delta function does not vanish at all in the physical region of variation of the variables, and these channels are forbidden.

In the crossed field the photon mass has always a negative imaginary part, which for $\kappa \gtrsim 1$ is of the order of the real part. The damped photon wave is (approximately[6]) described by the function e^{ilx} where in a special coordinate system the four-vector l is characterized by three independent real components l_1, l_2, l_- and its complex square $l^2 = -\mu_i^2(\kappa)$, which depends only on κ (i.e., in the special reference frame, depends only on l_- and the field F) and, in addition, on the polarization of the photons relative to the field (the subscript i, which will be understood in the sequel), cf.[8,9]. Using the coordinates[7] $x_- = x_0 - x_3$, $x_+ = (x_0 + x_3)/2$, one can represent the photon wave in the form

$$e^{ilx} = \exp\{i(l_1x_1 + l_2x_2 - l_-x_+ - l_+x_-)\}, \qquad (36)$$

where the complex μ^2 enters only in the component

$$l_+ = (l_0 + l_3)/2 = (\mu^2 + l_1^2 + l_2^2)/2l_-, \qquad (37)$$

so that the damping goes in the direction of increasing x_-.

Owing to the complex character of the l_+-components the splitting amplitude will no longer contain $(2\pi)^4\delta(l' + l'' - l)$ but (for sufficiently large size of the field)

$$(2\pi)^3\delta(l_1' + l_1'' - l_1)\,\delta(l_2' + l_2'' - l_2)\,\delta(l_-' + l_-'' - l_-)\frac{t}{l_+'^* + l_+''^* - l_+}. \qquad (38)$$

The splitting probability per unit volume and unit time, after integration over the final states[8], takes the form

$$W = \frac{n_-}{16 l_- L_-}\int \frac{dl_1'\,dl_2'\,dl_-'}{(2\pi)^3 l_-' l_-''}\,\frac{|c|^2}{\beta^2 + \gamma^2/4}, \qquad (39)$$

where in the right hand side we have used the notation

$$\beta = \mathrm{Re}\,(l_+'^* + l_+''^* - l_+), \quad \gamma/2 = \mathrm{Im}\,(l_+'^* + l_+''^* - l_+)$$

and have assumed the conservation law $l''_{1,2,-} = (l - l')_{1,2,-}$. In addition, for the 4-volume in which the process occurs, we have used the relation $VT = L_1L_2L = L_1L_2L_+L_-$, denoting by L_+ and L_- the intervals of variation of the coordinates x_+ and x_-.

The probability of splitting and the functions c, β, γ in (39) refer to a specific polarization channel, which is uniquely characterized by the index $n_1n_2n_3$ and the ordering of the momenta l, l', and l'' according to Eq. (7). Thus to the channel $L \to L^{*\prime}L^{*\prime\prime}$ corresponds $c_{120}(-ll'l'')$, and to the channel $L^* \to L'L^{*\prime\prime}$ corresponds $c_{120}(l' - ll'')$.

We note that

$$\frac{\gamma}{2} = \mathrm{Im}\,(l_+'^* + l_+''^* - l_+) = -\left(\frac{\mathrm{Im}\,\mu^2}{2l_-} + \frac{\mathrm{Im}\,\mu'^2}{2l_-'} + \frac{\mathrm{Im}\,\mu''^2}{2l_-''}\right) > 0 \qquad (40)$$

is always positive, since it is the sum of the half-widths for the damping of photons. The quantity β can be represented in the form (compare with the left-hand side of (10))

$$\beta = \frac{(l_-l_1' - l_-'l_1)^2 + (l_-l_2' - l_-'l_2)^2}{2l_-l_-'l_-''} + \beta_{min}, \quad \beta_{min} = \frac{\mathrm{Re}\,\mu'^2}{2l_-'} + \frac{\mathrm{Re}\,\mu''^2}{2l_-''} - \frac{\mathrm{Re}\,\mu^2}{2l_-}, \qquad (41)$$

from where it can be seen that for fixed l_- and l_-', and other variables changing, β varies in the interval $\beta_{min} \le \beta < \infty$, attaining a maximum value $\beta = \beta_{min}$ at the collinearity point

$$l_1'/l_1 = l_2'/l_2 = l_-'/l_-. \qquad (42)$$

Thus, the function $(\beta^2 + \gamma^2/4)^{-1}$ describes the natural form of the spectral distribution with respect to $l'_{1,2}$ and has an absolute maximum for $\beta = 0$, situated either within the physical region or outside it, depending on the sign of the shift $\beta_{min} \lessgtr 0$. In the latter situation the physical distribution attains a maximum at the boundary $\beta = \beta_{min}$; for $\kappa \gtrsim 1$ this value is of the order of the absolute maximum, since $\beta_{min} \sim \gamma$.

For the fundamental polarization channels $L \to L'L''$, $L \to L^{*\prime}L^{*\prime\prime}$, $L^* \to L'L^{*\prime\prime}$, $L^* \to L^{*\prime}L''$ the polarization operator in (39) can be taken on the mass shell $l^2 = l'^2 = l''^2 = 0$. The integration over l_1' and l_2' can be carried out explicitly and choosing $L_- = 1/\gamma$ (the damping length in x) for the correct definition of the differential probability per unit volume and unit time, we obtain:

$$W = \frac{n_-}{16l_-}\int_0^1 d\theta\,\mathrm{arc\,ctg}\,\frac{2\beta_{min}}{\gamma}\,\frac{|c|^2}{2\pi^2}. \qquad (43)$$

The distribution in $\theta = l_-'/l_-$ depends on radiative effects, owing to the factor arc $\cot(2\beta_{min}/\gamma)$, i.e., depends on the real and imaginary parts of the photon masses, which can thus be measured. Although for some polarization channels we have $\beta_{min} > 0$, i.e., the distribution with respect to $l'_{1,2}$ is displaced into the unphysical region, there is no selection rule forbidding these channels, since for $\kappa \gtrsim 1$ the shift is of the order of the width of the distribution, $\beta_{min} \sim \gamma$. For a shift deep into the unphysical region ($\beta_{min} < 0$, $|\beta_{min}| \gg \gamma$), Eq. (43) goes over into the expression (30) of I. The ratio $2\beta_{min}/\gamma$ for different polarization channels can be found from the graphs of[9].

If in distinction from the case considered here the size of the region occupied by the field is small compared to the damping length $L_-\gamma \ll 1$ (but is, of course, large compared to the formation length), then the natural lineshape $(\beta^2 + \gamma^2/4)^{-1}$ in (39) is replaced by the function $2\beta^{-2}(1 - \cos\beta L_-)$, which behaves like $2\pi L_-\delta(\beta)$, but with a width $\sim L_1^{-1}$ much larger than the natural line width γ. If this width is large compared to the shift $L_-\beta_{min} \ll 1$ only half of this delta function operates in the integration with respect to $l'_{1,2}$ and there arises Eq. (43), with $\cot^{-1}(2\beta_{min}/\gamma)$ replaced by $\pi/2$. Again there are no interdictions on the polarization channels.

APPENDIX

Here we list the explicit expressions for the ten contractions $q_{n_1n_2n_3}$ which together with the phase $\Phi(F)$, cf. (21), determine the invariant functions $c_{n_1n_2n_3}$:

$$q_{300} = \frac{\beta^2(1 + i\eta - i\beta\lambda)}{LL'L''ss's''}\left[\rho\left(\frac{L^2}{st} + \frac{L'^2}{s't'} + \frac{L''^2}{s''t''}\right) + LL'L''\left(\frac{Lw}{t} + \frac{L'w'}{t'} + \frac{L''w''}{t''}\right)\right] + \frac{i\beta^3}{LL'L''ss's''}$$
$$\times\left\{\frac{4\rho^3}{ss's''} + \rho^2(L'L''c + L''Lc' + LL'c'') + \frac{\rho}{3}\left[\frac{3}{\beta}\left(\frac{l^2L^2}{s} + \frac{l'^2L'^2}{s'} + \frac{l''^2L''^2}{s''}\right)\right.\right.$$
$$\left.\left. - L'^2L''^2\frac{a}{t} - L''^2L^2\frac{a'}{t'} - L^2L'^2\frac{a''}{t''} - 2LL'L''(Lb + L'b' + L''b'')\right]\right\}$$
$$+ \frac{i\beta^2}{2ss's''}\left\{l^2\left[s(1-w)\left(\frac{L'}{t''} + \frac{L''}{t'}\right) - 4L\right] + l'^2\left[s'(1-w')\left(\frac{L''}{t} + \frac{L}{t''}\right) - 4L'\right]\right.$$
$$+ l''^2\left[s''(1-w'')\left(\frac{L}{t'} + \frac{L'}{t}\right) - 4L''\right] + \frac{2\beta}{3}\left[L'^2L''a\frac{s''-s'}{t}\right.$$
$$\left.\left. + L''^2La'\frac{s-s''}{t'} + L^2L'a''\frac{s'-s}{t''} + LL'L''(k + k' + k'')\right]\right\},$$

$$q_{210}(ll'l'') = -\frac{\tau\beta(1 + i\eta)}{L''ss's''}\left(1 + \frac{\beta}{ss'}\right) + \frac{i\tau\beta^3}{LL'L''ss's''}\left[\frac{4\rho^2}{ss's''}\right.$$
$$+ LL'\left(\frac{\lambda}{ss'} + \frac{l''^2}{\beta s''}\right) - \rho(LL'c'' + L'L''d_{ss's''} + L''Ld_{s'ss''}) -$$
$$\left. - \frac{LL'}{3}\left(LL'\frac{a''}{t''} + L'L''h_{ss'} + L''Lh_{s's}\right)\right],$$

$$q_{201}(ll'l'')=\frac{iG''\beta^2}{L''ss's''}\Big\{(i-\eta+\beta\lambda+2\beta\rho)\left(\frac{L}{s't'}-\frac{L'}{st}\right)+l^2\left(\frac{L''}{s''}-\frac{L}{s}\right)$$
$$+l'^2\left(\frac{L'}{s'}-\frac{L''}{s''}\right)+\frac{2\rho^2\beta}{LL'ss's''}\left(L''\frac{s'-s}{s''}+L-L'\right)-\frac{\rho\beta}{LL'}\Big[L\left(\frac{L''}{s''}-\frac{L}{s}\right)$$
$$\times\left(2L\frac{s''+s-s'}{s''s}+L'\frac{(s'-s'')(3s-s'-s'')}{ss's''}\right)+L'\left(\frac{L'}{s'}-\frac{L''}{s''}\right)$$
$$\times\left(2L'\frac{s'+s''-s}{s's''}+L\frac{(s-s'')(3s'-s''-s)}{ss's''}\right)\Big]+LL'L''\frac{s'-s}{3}(2+5w'')$$
$$-\frac{\beta}{3}\left(\frac{L}{s}-\frac{L'}{s'}\right)\Big[LL'a''+\frac{L'L''}{t}\left(4s-2s'-7s''-8\frac{ss'}{s''}+5\frac{s'^2}{s''}\right)$$
$$+\frac{L''L}{t'}\left(4s'-2s-7s''-8\frac{ss'}{s''}+5\frac{s^2}{s''}\right)\Big]\Big\},$$

$$q_{120}(ll'l'')=-\frac{\rho\beta(1+i\eta)}{Lss's''}\left(1+\frac{\beta}{s's''}\right)+\frac{\beta^2}{ss's''}\Big[L'\left(\frac{2}{t''}+\frac{s''}{ss'}\right)$$
$$+L''\left(\frac{2}{t'}+\frac{s'}{s''s}\right)-i\eta\left(L'\frac{w''}{t''}+L''\frac{w'}{t'}\right)\Big]+\frac{i\beta^2}{2ss's''}\Big\{l'^2\Big[L\frac{ss'}{t''^2}(1-w')$$
$$+L''\left(4+\frac{s-s''-sw'}{s'}+\frac{sw''}{t''}\right)\Big]+l''^2\Big[L\frac{ss''}{t'^2}(1-w'')$$
$$+L'\left(4+\frac{s-s'-sw''}{s''}+\frac{sw'}{t'}\right)\Big]-\lambda\Big[L'\left(\frac{2\beta}{t''}+s''-s'+sw'\right)$$
$$+L''\left(\frac{2\beta}{t'}+s'-s''+sw''\right)\Big]\Big\}+\frac{i\beta^3}{ss's''}\Big\{\frac{1}{LL'L''}\Big[\frac{4\rho\tau^2}{ss's''}+\rho^2L'L''r$$
$$+\tau^2L(L'd_{s''ss'}+L''d_{s'ss''})+\rho L'L''\left(\frac{\lambda}{s's''}+\frac{l^2}{\beta s}-\frac{L'L''}{3t}p-\frac{LL'}{3t''}g_{s's''}\right.$$
$$\left.-\frac{LL''}{3t'}g_{s''s'}\right)\Big]+LL'L''\left[(2+w)^2+\frac{s's''}{3}\left(\frac{s'-s''}{\beta}\right)^2\right]+\frac{1}{3}\left(\frac{L'}{s'}-\frac{L''}{s''}\right)$$
$$\times\Big[L'L''(s''-s')p+LL'ss'^2s''\frac{2+w''}{t''^3}+L''Lss's''^2\frac{2+w'}{t'^3}\Big]\Big\},$$

$$q_{111}(ll'l'')=\frac{iG''\tau\beta^3}{LL'L''ss's''}\Big\{\frac{2\rho}{ss's''}\left(L''\frac{s'-s}{s''}+L-L'\right)+L\Big[-LL'd_{s''ss'}$$
$$+L'L''\left(\frac{4}{s''}+\frac{4}{ts}+\frac{w'}{tt'}-\frac{s'}{tss''}\right)+2L''L\frac{s''+s-s'}{s''st'}\Big]\Big\},$$

$$q_{102}(ll'l'')=\frac{2iG'G''\beta^3}{LL'L''ss's''}\left(\frac{L'}{s'}-\frac{L''}{s''}\right)\left(L\frac{s'-s''}{s}-L'+L''\right)$$
$$\times\left[\frac{\rho}{s's''}+L\left(\frac{L}{s}-\frac{L'}{s}-\frac{L''}{s''}\right)\right],$$

$$q_{030}=\frac{\tau\beta^2(1+i\eta-i\beta\lambda)}{LL'L''ss's''}\left(\frac{L^2}{st}+\frac{L'^2}{s't'}+\frac{L''^2}{s''t''}\right)+\frac{i\tau\beta^3}{LL'L''ss's''}\Big\{\frac{4\tau^2}{ss's''}$$
$$+\frac{1}{\beta}\left(\frac{l^2L^2}{s}+\frac{l'^2L'^2}{s'}+\frac{l''^2L''^2}{s''}\right)+\rho(LL'r''+L'L''r+L''Lr')$$
$$-\frac{2}{3}LL'L''\Big[L\left(b+\frac{3w}{s}\right)+L'\left(b'+\frac{3w'}{s'}\right)+L''\left(b''+\frac{3w''}{s''}\right)\Big]$$
$$-\frac{1}{3}\left(L^2L'^2\frac{p''}{t''}+L'^2L''^2\frac{p}{t}+L''^2L^2\frac{p'}{t'}\right)\Big\}$$

$$q_{021}(ll'l'')=\frac{G''\beta^2(1+i\eta-i\beta\lambda)}{L''ss's''}\left(\frac{L'}{st}-\frac{L}{s't'}\right)+\frac{iG''\beta^3}{L''ss's''}\Big\{\frac{2\tau^2}{LL'ss's''}$$
$$\times\left(L''\frac{s'-s}{s''}+L-L'\right)+\frac{l^2}{\beta}\left(\frac{L''}{s''}-\frac{L}{s}\right)+\frac{l'^2}{\beta}\left(\frac{L'}{s'}-\frac{L''}{s''}\right)$$
$$+\rho\Big[\frac{L}{t'}\left(\frac{2}{s'}+\frac{s''-s+s'}{\beta}\right)-\frac{L'}{t}\left(\frac{2}{s}+\frac{s-s'+s''}{\beta}\right)\Big]+\frac{1}{3}\left(\frac{L}{s}-\frac{L'}{s'}\right)$$
$$\left(L'L''ss'\frac{w+2}{t^2}+L''Lss'\frac{w'+2}{t'^2}-LL'p''\right)+LL'L''\frac{s-s'}{3\beta}(w''-2)\Big\},$$

$$q_{012}(ll'l'')=\frac{2iG'G''\tau\beta^3}{LL'L''ss's''}\left(\frac{L'}{s'}-\frac{L''}{s''}\right)\left(L\frac{s'-s''}{s}-L'+L''\right),$$

$$q_{003}=-\frac{4iGG'G''\beta^3}{LL'L''ss's''}\left(\frac{L}{s}-\frac{L'}{s'}\right)\left(\frac{L'}{s'}-\frac{L''}{s''}\right)\left(\frac{L''}{s''}-\frac{L}{s}\right).$$

Here we have used the notations:

$$\lambda=\frac{l^2}{s}+\frac{l'^2}{s'}+\frac{l''^2}{s''},\quad \beta=\frac{ss's''}{s+s'+s''},\quad \eta=m^2(s+s'+s''),$$
$$\frac{1}{t}=\frac{1}{s'}+\frac{1}{s''},\quad w=\frac{s'+s''}{s},\quad a=\frac{s^2}{s's''}(3w^2-4w+4),$$
$$b=\frac{3}{s}-\frac{2s}{s's''}+\frac{(s'-s'')^2}{s^2t},\quad c=\frac{2}{s's''}+\frac{(s-s'-s'')^2}{ss's''t},$$
$$d_{ss's''}=\frac{(s'+s'')^2(s''-s'-2s)+s^2(s''-s')}{s(s's'')^2},$$
$$g_{s's''}=7+w+\frac{5s''+6s}{s'}+2\frac{s^3+s'^3+s''^3}{ss's''},$$
$$h_{ss'}=\frac{1}{t}\left(1-6w+7w''+\frac{5s-6s''}{s'}+2\frac{s^3+s'^3+s''^3}{ss's''}\right)-\frac{12s}{s's''},$$
$$k=3-12\frac{s}{s'}-2\frac{s'}{s}-3\frac{s^2}{s'^2}-4\frac{s'^2}{s^2}+4\frac{ss'}{s''^2}-2\frac{s''^2}{ss'},$$
$$p=\frac{s^2}{s's''}(3w^2+8w+4),\quad r=\frac{2}{s's''}+\frac{(s+s'+s'')^2}{ss's''t}.$$

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1)In the sequel we use units with $\hbar=c=1$, $\alpha=e^2/4\pi=1/137$, and the notations $l_\mu=(1, il_0)$, $ll'=\mathbf{l}\cdot\mathbf{l}-l_0l_0'$, $F^*=(i/2)\mathcal{E}_{\mu\nu\lambda\sigma}F_{\lambda\sigma}$ is the tensor dual to $F_{\mu\nu}$.

2)The noncovariant notation always refers to a special coordinate system with the axes 1, 2, and 3 along **E**, **H**, and **E** × **H**, respectively; $l_-=l_0-l_3$, F = E = H is the magnitude of the field in this system.

3)In I the absence of the states ψ_{201}, ψ_{021}, and ψ_{003} was asserted erroneously; this did not influence the results obtained there, since that paper was dedicated to obtaining the polarization operator on the mass shell where only ψ_{300} and ψ_{120} are nonzero.

4)Eq. (10) relates the latter two parameters to the four preceding ones and are therefore equivalent to one independent parameter, e. g., the angle ψ, according to the equations $\rho=(\rho^2+\tau^2)^{1/2}\cos\psi$, $\tau=(\rho^2+\tau^2)^{1/2}\sin\psi$.

5)The definition and properties of the special function f(z) can be found in[8]; they are not essential here.

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