Influence of proximity effect on the thermodynamic and magnetic properties of multilayer superconductors

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Formulas are obtained for the critical temperature of a superconducting sample consisting of alternating thin ($d < \xi_0$) films of two metals that have different electron-electron interaction constants and different electron densities. Under the same conditions, equations of the Ginzburg–Landau type are obtained for the order parameter. The inhomogeneity of the sample leads to the onset of effective anisotropy of the superconducting properties, as a result of which the latter are described with the aid of tensorlike quantities.

A characteristic feature of the superconducting state of metals is the macroscopically large correlation length of the superconducting electrons, $\xi_0 \sim 10^{-4}$ cm. Consequently, in systems whose microscopic properties vary over distances smaller than ξ_0, specific effects are produced as the result of nonlocality over distances on the order of ξ_0. An example of such a system can be a multilayer sandwich, consisting of different superconducting films with thicknesses smaller than ξ_0. The mutual influence of the films makes it possible to regard this system as a new superconductor which is generally speaking anisotropic, with a certain average critical superconducting-transition temperature.

The effects of mutual penetration of the superconducting properties (proximity effects) has been the subject of a number of studies[1-5], in which different models of layered structures were considered. De Gennes[1], who initiated the investigations of the proximity effects, has proposed a very simple model, constituting a junction of two superconducting films differing only in the electron-electron interaction constants. As a result he obtained a formula relating the critical temperature of a two-layer superconductor with the critical temperatures of each of the films separately, and elucidated the character of the averaging of the interaction constant.

From a more general point of view, interest attaches to a study of the influence of the proximity effect on the magnetic properties of an inhomogeneous superconducting system, i.e., a clarification of the character of the spatial variation of the order parameter $\Delta(\mathbf{r})$ in the presence of magnetic fields and currents. It is clear that the dimensions of the system should be large in comparison with the characteristic linear superconductor parameters $\lambda(T)$ and $\xi(T)$; under this condition, the behavior of systems in a magnetic field will be analogous in many respects to the behavior of a bulky homogeneous superconducting sample.

We have investigated in this connection the superconducting properties of a sandwich consisting of a large number of alternating flat films of two different metals and constituting thus a macroscopic sample (in comparison with λ and ξ). It is assumed that the films differ not only in their interaction constants, but also in the electron densities (i.e., in the Fermi energies). We consider the temperature region near the critical temperature T_e (see Sec. 1 of the present article), so that we can obtain a closed system of equations of the Ginzburg-Landau type[6] for the order parameter.

1. CRITICAL TEMPERATURE OF SUPERCONDUCTING TRANSITION

The problem of finding the critical temperature of an inhomogeneous superconductor reduces mathematically to a solution of the Gor'kov linear integral equation[7] for the order parameter $\Delta(\mathbf{r})$:

$$\Delta(\mathbf{r}) = |g(\mathbf{r})| T_c \sum_{\omega_c} \int dV' \, G(\mathbf{r}, \mathbf{r}'; \omega_c) G(\mathbf{r}', \mathbf{r}; -\omega_c) \Delta(\mathbf{r}'), \tag{1}$$

where $g(\mathbf{r}) < 0$ is the interaction constant; $\omega_c = \pi T_c(2n+1)$, $n = 0, \pm 1, \pm 2, \ldots$; $G(\mathbf{r}, \mathbf{r}'; \omega_c)$ is the Green's function of the electrons in the normal metal and satisfies the equation

$$(\hat{\mathbf{p}}^2/2m - \mu(\mathbf{r}) - i\omega_c) G(\mathbf{r}, \mathbf{r}'; \omega_c) = \delta(\mathbf{r} - \mathbf{r}').$$

Here $\hat{\mathbf{p}} = -i\nabla$ ($\hbar = 1$), and $\mu(\mathbf{r})$ is the local value of the Fermi energy. In the model in question, $g = g(x)$ and $\mu = \mu(x)$ (the x axis is perpendicular to the layers) are specified in the following manner:

$$kd < x < kd + d_1, \quad \mu(x) = \mu_1 = p_{F1}^2/2m, \quad g(x) = g_1,$$
$$kd + d_1 < x < (k+1)d, \quad \mu(x) = \mu_2 = p_{F2}^2/2m, \quad g(x) = g_2,$$

where d is the period of the layered structure ($d \ll \xi_0$), k is the number of the layer, while d_1 and $d_2 = d - d_1$ are the thicknesses of the alternating films. For the sake of argument, we assume that $\mu_1 > \mu_2$.

The critical temperature T_c is defined as the maximum eigenvalue of the integral equation (1). Substituting in (1) the spectral expansion of the Green's function

$$G(\mathbf{r}, \mathbf{r}'; \omega_c) = \sum_\lambda \psi_\lambda(\mathbf{r}) \psi_\lambda^*(\mathbf{r}') / [\varepsilon(\lambda) - i\omega_c],$$
$$(\hat{\mathbf{p}}^2/2m - \mu(x)) \psi_\lambda(\mathbf{r}) - \varepsilon(\lambda) \psi_\lambda(\mathbf{r})$$

and separating the free motion of the electrons with momentum p along the layers, we obtain after summation over ω_c

$$\Delta(x) = |g(x)| \int dx' \, \Delta(x') \int \frac{d^2p}{(2\pi)^2} \iint d\varepsilon \, d\varepsilon' \, F(\varepsilon, \varepsilon', T_c) X_{p,p;\varepsilon,\varepsilon'}(x, x'), \tag{2}$$

where

$$F(\varepsilon, \varepsilon'; T) = \frac{\operatorname{th}(\varepsilon/2T) + \operatorname{th}(\varepsilon'/2T)}{2(\varepsilon + \varepsilon')}$$

$$X_{p,p';\varepsilon,\varepsilon'}(x, x') = f_{p\varepsilon}(x) f_{p\varepsilon}^*(x') f_{p'\varepsilon'}^*(x) f_{p'\varepsilon'}(x'),$$

and the wave functions of the electrons $f_{p\epsilon}(x)$ satisfy the equation

$$[(\hat{p}_x^2 + p^2)/2m - \mu(x)] f_{p\varepsilon}(x) = \varepsilon f_{p\varepsilon}(x). \tag{3}$$

The investigation of Eqs. (2) and (3) can be carried out by relatively simple methods in the case when the reflection of the electrons from the layer interface can be neglected[1]. The corresponding solution of (3) is ($kd < x < (k+1)d$)

$$f_{p\varepsilon}(x)=\frac{\exp[i\sigma x\lambda(\mu(x))]}{(2\pi v(x))^{1/2}}(\chi_1^{(k)}(x)e^{-i\sigma\lambda kd_2}$$

$$+\chi_2^{(k)}(x)e^{i\sigma\lambda(k+1)d_1})\quad \text{if}\quad 0<p^2<2m\mu_2, \tag{4}$$

$$f_{p\varepsilon}(x)=(2/d_1)^{1/2}\chi_1^{(k)}(x)\sin(x-kd)\lambda(\mu_1)\quad\text{if}\quad 2m\mu_2<p^2<2m\mu_1,$$

$$f_{p\varepsilon}(x)=0\quad\text{if}\quad p^2>2m\mu_1,$$

where

$$mv(x)=\lambda(\mu(x))=[2m(\mu(x)+\varepsilon)-p^2]^{1/2},\quad \lambda=\lambda(\mu_1)-\lambda(\mu_2),\quad \sigma=\pm1,$$

$$\chi_1^{(k)}(x)=1,\quad \chi_2^{(k)}(x)=0,\quad kd<x<kd+d_1,$$

$$\chi_1^{(k)}(x)=0,\quad \chi_2^{(k)}(x)=1,\quad kd+d_1<x<(k+1)d.$$

We note that at $0 < p^2 < 2m\mu_2$ the energy spectrum of the electrons is continuous, and the corresponding wave function in (4) is normalized to a δ-function with respect to the energy ϵ. At $2m\mu_2 < p^2 < 2m\mu_1$ and $d_2 \gg 1/\lambda_1$ the electrons can be regarded as trapped in the layers $kd < x < kd + d_1$, and consequently can be assumed to have a quasidiscrete spectrum:

$$\varepsilon_n(p)=(\pi v_1/d_1)(n+\alpha(p)),\quad n=0,\ \pm1,\ \pm2,\ldots$$

$$\alpha(p)=\{d_1\lambda_1/\pi\},\quad 0<\alpha\leqslant1,$$

where the symbol $\{x\}$ denotes the fractional part of x. The wave function of these electrons is normalized to the δ symbol with respect to the quantum number n, and the integration with respect to the energy, which is contained in formula (2), is replaced in the case of $2m\mu_2 < p^2 < 2m\mu_1$ by summation over n.

Using the equation of the BCS theory[8] to determine the critical temperature of a bulky superconductor

$$1=|g(x)|\nu(x)\int\limits_{-\omega_D}^{\omega_D}\operatorname{th}\frac{\varepsilon}{2T_c(x)}\frac{d\varepsilon}{2\varepsilon}$$

(ω_D is the Debye temperature in energy units) and taking into account the completeness relation for the wave function $f_{p\epsilon}(x)$

$$\int d\varepsilon\, f_{p\varepsilon}^*(x)f_{p\varepsilon}(x')=\delta(x-x'),$$

we transform Eq. (2) in the following manner:

$$\int dx'\,Q(x,x')\Delta(x')=\frac{1}{2}\nu(x)\Delta(x)\ln\frac{T_c}{T_c(x)}, \tag{5}$$

where

$$Q(x,x')=\int\frac{d^2p}{(2\pi)^2}\iint d\varepsilon\, d\varepsilon'\,\Phi(\varepsilon,\varepsilon';T_c)X_{p,p;\varepsilon,\varepsilon'}(x,x'),$$

$$\Phi(\varepsilon,\varepsilon';T)=F(\varepsilon,\varepsilon';T)-F(\varepsilon,\varepsilon;T). \tag{6}$$

Substituting in (6) the expressions for the wave functions (4) and discarding terms of oscillate over the wavelength of the electron $\lambda_F \ll d$, we obtain the explicit form of the kernel $Q(x, x')$. It is the sum of two terms, $Q^{(c)} + Q^{(d)}$, corresponding respectively to the continuous and discrete spectra:

$$Q=Q^{(c)}+Q^{(d)},$$

$$Q^{(c)}(x,x')=\int\limits_{0<p^2<2m\mu_2}\frac{d^2p}{(2\pi)^2}\iint d\varepsilon\, d\omega\,\Phi(\varepsilon,\varepsilon-\omega;T_c)$$

$$\times\sum_{\sigma=\pm1}\frac{Y_\sigma(x)Y_\sigma^*(x')}{(2\pi)^2v(x)v(x')}\exp(i\sigma\omega(k-k')\overline{dv^{-1}(x)}), \tag{7}$$

$$Q^{(d)}(x,x')=\int\limits_{2m\mu_2<p^2<2m\mu_1}\frac{d^2p}{(2\pi)^2}\int d\varepsilon\sum_n\Phi\left(\varepsilon,\varepsilon-\frac{\pi v_1}{d_1}n;T_c\right)$$

$$\times\sum_m\frac{\chi_1^{(m)}(x)\chi_1^{(m)}(x')}{\pi v_1 d_1}\cos\left(\frac{\pi nd}{d_1}\left\{\frac{x}{d}\right\}\right)\cos\left(\frac{\pi nd}{d_1}\left\{\frac{x'}{d}\right\}\right).$$

Here and henceforth, the superior bar denotes averaging over x within the limits of the period d of the layered structure;

$$\overline{v^{-1}(x)}=\frac{1}{d}\left(\frac{d_1}{v_1}+\frac{d_2}{v_2}\right),\quad k=\left[\frac{x}{d}\right],\quad k'=\left[\frac{x'}{d}\right],$$

the symbol $[x]$ denotes the integer part of x, and

$$Y_\sigma(x)=\chi_1^{(k)}(x)\exp\left(i\sigma\omega\frac{d}{v_1}\left\{\frac{x}{d}\right\}\right)$$

$$+\chi_2^{(k)}(x)\exp\left(i\sigma\omega\frac{d}{v_2}\left(\left\{\frac{x}{d}\right\}-1\right)+i\sigma\omega\,\overline{dv^{-1}(x)}\right),$$

$$kd<x<(k+1)d,\quad Y_\sigma(x+d)=Y_\sigma(x).$$

Since the system in question is macroscopically homogeneous, and there is neither magnetic field nor current, the coordinate dependence of $\Delta(x)$ is determined completely by the coordinate dependences of $g(x)$ and $\mu(x)$. It follows therefore that the solution of (5) should be sought in the class of functions $\Delta(x) = \Delta(x + d)$ that are periodic in x. Then the integral term contained in (5) can be transformed as follows:

$$\int dx'\,Q(x,x')\Delta(x')=\sum_{k'}\int\limits_{k'd}^{(k'+1)d}dx'\,Q(x,x')\Delta(x')=\int\limits_0^d dx'\,Q_R(x;x')\Delta(x'),$$

where

$$Q_R(x,x')=\sum_{k'}Q(x,x'+k'd).$$

Let us investigate the properties of the "reduced" kernel $Q_R(x, x')$; to this end we sum in (7) over k, using the Poisson formula:

$$\sum_k\exp(i\sigma\omega\,\overline{dv^{-1}(x)}k)=\frac{2\pi}{\overline{dv^{-1}(x)}}\sum_{l=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi l}{\overline{dv^{-1}(x)}}\right). \tag{8}$$

Combining formulas (8) and (7) and integrating in (7) with respect to ω, we note that the term with $l = 0$ drops out of the sum over l (since $\Phi(\epsilon, \epsilon; T) = 0$), and the expression for $Q^{(c)}$ acquires a structure similar to $Q^{(d)}$. In particular, $Q^{(c)}$ and $Q^{(d)}$ contain integrals of the type

$$I(\omega_n)=\int\limits_{-\infty}^{+\infty}d\varepsilon\,\Phi(\varepsilon,\varepsilon-\omega_n;T_c),\quad \omega_n\sim n\frac{v_F}{d}\gg T_c.$$

It is easy to verify that the integrals $I(\omega_n)$, and with them also the kernel $Q_R(x, x')$, are asymptotically large ($\sim\ln(v_F/dT_c) \sim (\ln \xi_0/d) \gg 1$). Thus, the kernel $T(x, x')$ is logarithmically large in the class of functions that are periodic in x. This makes it possible to carry out all the subsequent calculations with logarithmic accuracy in the parameter $d/\xi_0 \ll 1$. In the case when the dimensionless coupling constant $g(x)\nu(x)$ does not vary too strongly on going from one layer to the other (i.e., $[\ln(T_c/T_c(x))| \ll \ln(\xi_0/d))$, Eq. (5) contains in first-order approximation only the integral operator

$$\int\limits_0^d dx'\,\Delta_0(x')Q_R(x,x')=0. \tag{9}$$

Taking into account the orthogonality of the functions $f_{p\epsilon}(x)$ contained in the kernel $Q_R(x, x')$, we can show that the only solution of (9) is $\Delta_0(x) = \Delta_0 = \text{const}$. The condition under which the next-approximation equation

$$\int\limits_0^d dx'\,Q_R(x,x')\Delta_1(x')=\frac{1}{2}\nu(x)\Delta_0\ln\frac{T_c}{T_c(x)}$$

can be solved is that its right-hand side be orthogonal to the solution Δ_0 of the homogeneous equation, i.e.,

$$\frac{1}{d}\int_0^d dx\,\nu(x)\ln\frac{T_c}{T_c(x)}=\overline{\nu(x)\ln\frac{T_c}{T_c(x)}}=c_1\nu_1\ln\frac{T_c}{T_{c1}}+c_2\nu_2\ln\frac{T_c}{T_{c2}}=0, \quad (10)$$

where $c_1 = d_1/d$ and $c_2 = 1 - c_1$ are the concentrations of the superconductors. Relation (10) is the sought equation for T_c. Solving it with respect to T_c, we obtain

$$T_c=T_{c1}^{c_1\nu_1/(c_1\nu_1+c_2\nu_2)}\,T_{c2}^{c_2\nu_2/(c_1\nu_1+c_2\nu_2)}.$$

To clarify the physical meaning of this result, we turn to the BCS-theory formula for the critical temperature:

$$T_c\sim\omega_D\exp(-2/|g|\nu),\quad T_c(x)\sim\omega_D(x)\exp(-2/|g(x)|\nu(x)).$$

Neglecting the logarithmically insignificant differences between the Debye frequencies, we obtain

$$\overline{\nu(x)\ln\frac{T_c}{T_c(x)}}=2\left(\overline{\frac{1}{|g(x)|}}-\frac{\overline{\nu(x)}}{|g|\nu}\right)=0,$$

whence

$$\frac{1}{|g|\nu}=\frac{1}{\overline{\nu(x)}}\left(\overline{\frac{1}{|g(x)|}}\right).$$

Thus, in the case $|\ln(T_c/T_c(x))| \ll \ln(\xi_0/d)$ an independent averaging of the quantities $g^{-1}(x)$ and $\nu(x)$ takes place in the superconducting system. It appears that this conclusion is general in character and does not depend significantly on the shapes of the layers.

In the opposite limiting case (e.g., if $g_2 \to 0$ or $g_1 \to 0$) the logarithmically exact solution of (5) is the periodic function

$$\Delta_1\chi_1^{(k)}(x)+\Delta_2\chi_2^{(k)}(x)\quad(kd<x<(k+1)d,\ \Delta_1\neq\Delta_2).$$

The integral equation (5) is then transformed into a system of homogeneous algebraic equations in Δ_1 and Δ_2, and the conditions under which it can be solved is the formula for the determination of T_c. In particular, as $g \to 0$ ($|g_1| \gg |g_2|$) the expression for T_c takes the form

$$T_c=T_{c1}\left(\frac{T_{c1}}{\tau_0}\right)^{c_2\nu'/(\nu_1-c_2\nu')}, \quad (11)$$

where

$$\tau_0\sim\frac{v_F}{d}\gg T_{c1},\quad \frac{1}{2}\nu'=\int\limits_{0<p^2<2m\mu_2}\frac{d^2p}{(2\pi)^2}\frac{1}{\pi v_1v_2\overline{v^{-1}(x)}}$$

If $g_1 \to 0$ ($|g_2| \gg |g_1|$) it is necessary to make the substitutions $T_{c1} \to T_{c2}$, $\nu_1 \to \nu_2$, $c_2 \to c_1$, in (11). Obviously, at $v_1 = v_2 = v$ we have $\nu_1 = \nu' = \nu$; thus, the relation (11) is a generalization of the known McMillan formula[5] for metals with different Fermi energies.

2. THE GINZBURG-LANDAU EQUATIONS

The general scheme for the derivations of the Ginzburg-Landau equations for the investigated model is perfectly analogous to the known procedure for expanding the Gor'kov equations in the order parameter Δ (see, e.g.,[7]) with account taken of the expansion terms nonlinear in Δ. Use is made here of a gauge-invariant formulation of superconductivity theory, with the superfluid velocity v_s separated in such a way that the order parameter is real. The integral equation obtained for Δ as a result of the expansion is

$$\Delta(\mathbf{r})=|g(x)|\int dV'\,Q(\mathbf{r},\mathbf{r}')\Delta(\mathbf{r}')+|g(x)|O(\Delta^3),$$
$$Q(\mathbf{r},\mathbf{r}')=T\sum_\omega G(\mathbf{r},\mathbf{r}';\omega)G(\mathbf{r}',\mathbf{r};-\omega)\cos(2\mathbf{p}_s(\mathbf{r})(\mathbf{r}-\mathbf{r}')), \quad (12)$$
$$\mathbf{p}_s(\mathbf{r})=m\mathbf{v}_s(\mathbf{r})={}^1/_2\nabla\chi-e\mathbf{A}(\mathbf{r}).$$

The symbol $O(\Delta^3)$ denotes here the term nonlinear (cubic) in Δ, $\chi(\mathbf{r})$ is the phase of the order parameter, and $\mathbf{A}(\mathbf{r})$ is the vector potential of the magnetic field ($\operatorname{curl}\mathbf{A} = \mathbf{H}$) and is taken into account, as usual[7], in the "quasiclassical" approximation. To obtain a gauge-invariant system of equations, it is necessary to supplement the integral equation (12) by the condition that Δ be real:

$$T\sum_\omega\int dV'\,\Delta(\mathbf{r}')G(\mathbf{r},\mathbf{r}';\omega)G(\mathbf{r}',\mathbf{r};-\omega)\sin(2\mathbf{p}_s(\mathbf{r})(\mathbf{r}-\mathbf{r}'))=0 \quad (13)$$

and by a relation for the connection between the current j and the order parameter:

$$\mathbf{j}(\mathbf{r})=\frac{ie}{m}T\sum_\omega\iint dV_1\,dV_2\,\Delta(\mathbf{r}_1)\Delta(\mathbf{r}_2)\sin(2\mathbf{p}_s(\mathbf{r}_1)(\mathbf{r}_1-\mathbf{r}_2))\cdot$$
$$\times(\hat{\mathbf{p}}-\hat{\mathbf{p}}')G(\mathbf{r},\mathbf{r}_1;\omega)G(\mathbf{r}_1,\mathbf{r}_2;-\omega)G(\mathbf{r}_2,\mathbf{r}';\omega)_{\mathbf{r}'=\mathbf{r}}. \quad (14)$$

It is convenient to carry out the subsequent calculations by transforming Eq. (12) in the same manner as Eq. (2):

$$\int dV'\,Q_0(\mathbf{r},\mathbf{r}')\Delta(\mathbf{r}')=\frac{1}{2}\Delta(\mathbf{r})\nu(x)\ln\frac{T}{T_c(x)}-O(\Delta^3), \quad (15)$$

where

$$Q_0(\mathbf{r},\mathbf{r}')=K_0(\mathbf{r},\mathbf{r}')\cos(2\mathbf{p}_s(\mathbf{r})(\mathbf{r}-\mathbf{r}')),$$
$$K_0(\mathbf{r},\mathbf{r}')=\iint\frac{d^2p\,d^2p'}{(2\pi)^4}e^{i(\mathbf{p}-\mathbf{p}')(\rho-\rho')}. \quad (16)$$
$$\times\iint d\varepsilon\,d\varepsilon'\,\Phi(\varepsilon;\varepsilon';T)\,X_{\mathbf{p},\mathbf{p}';\varepsilon,\varepsilon'}(x,x'),$$

where ρ is the component of the vector $\mathbf{r}$ in the yz plane.

We note that the procedure customarily used in the derivation of the Ginzburg-Landau equations, that of expanding $\Delta(\mathbf{r})$ and $\chi(\mathbf{r})$ in the gradients of these quantities, cannot be applied directly to (15) because, as already noted above, the inhomogeneity of the system over distances on the order of $d \ll \xi_0$ generates a rapid variation of the functions $\Delta(\mathbf{r})$ and $\chi(\mathbf{r})$ along the x axis. Since, however, the principal physical interest attaches to relatively slow changes of Δ and χ (over distances of the order of $\xi(T)$) and not the small scale oscillations of these quantities, which are connected with the initial inhomogeneity of the system, it is advantageous to establish the hierarchy of the characteristic scales of the functions that enter in (15). Namely, the largest scale over which a smooth variation of Δ takes place is the effective coherence length near T_c: $\xi(T) \sim \xi_0/\tau$ ($\tau = (T_c - T)/T_c \ll 1$). The radius of the kernel Q_0, the order of magnitude of which is $\xi_0 \sim v_F/T_c \ll \xi(T)$, is the next smaller characteristic scale; finally, the dimension d of the inhomogeneity of the considered system is the smallest linear parameter: $d \ll \xi_0 \ll \xi(T)$. Since the spatial dependence of $\Delta(\mathbf{r})$ has the character of small-scale oscillations along the x axis, "modulated" by a smooth function of the coordinates, it is convenient to separate formally the dependence of Δ on $n = [x/d]$ and $\xi = \{x/d\}$:

$$\Delta(\mathbf{r})=\Delta(x,\rho)=\Delta(n;\xi;\rho).$$

The dependence of Δ on n and ρ is determined only by the external conditions, and is quite slow in the scale of ξ_0 (and all the more of d). This circumstance enables us to represent Δ in the form of a sum of a slowly varying function $\Delta(n, \rho)$ and a small rapidly oscillating (over distances on the order of d) term $\delta\Delta(n, \xi; \rho)^{2}$

$$\Delta(n, \xi; \rho) \approx \Delta(n, \rho) + \delta\Delta(n, \xi; \rho).$$

In the first nonvanishing approximation we have

$$\int dV' Q_0(\mathbf{r}, \mathbf{r}')\delta\Delta(n', \xi'; \rho') \approx \int dV' Q_0(\mathbf{r}, \mathbf{r}')\delta\Delta(n, \xi'; \rho)$$

$$= \int dx' Q(x, x')\delta\Delta(n, \xi'; \rho) = \int_0^d dx' Q_R(x, x')\delta\Delta(n, x'; \rho),$$

where

$$Q(x, x') = \int d\rho' Q(\mathbf{r}, \mathbf{r}'),$$

and account is taken of the fact that the dependence of $\delta\Delta$ on n and ρ, like the $\Delta(n, \rho)$ dependence, is weak in the scale of the radius of the kernel Q_0.

Since the kernel Q_R is logarithmically large (in the class of functions periodic in x) (see Sec. 1), Eq. (15) takes in the principle approximation the form

$$\int_0^d dx' Q_R(x, x')\delta\Delta(n, x'; \rho) \approx \frac{1}{2}\nu(x)\Delta(n, \rho)\ln\frac{T}{T_c(x)} - O(\Delta^3) - \int dV' Q_0(\mathbf{r}, \mathbf{r}')\Delta(n', \rho'), \tag{17}$$

i.e., it is an inhomogeneous integral equation for $\delta\Delta$. In the condition for the solvability of (17)

$$\frac{1}{2}\Delta(n, \rho)\overline{\nu(x)\ln\frac{T}{T_c(x)}} = \overline{O(\Delta^3)} + \int dV' \overline{Q_0(\mathbf{r}, \mathbf{r}')}\Delta(n', \rho')$$

we can replace, with the assumed degree of accuracy, the slowly-varying "step" function $\Delta(n, \rho)$ by the "smooth" function $\Delta(\mathbf{r})$:

$$\frac{1}{2}\Delta(\mathbf{r})\overline{\nu(x)\ln\frac{T}{T_c(x)}} = \overline{O(\Delta^3(\mathbf{r}))} + \int dV' \overline{Q_0(\mathbf{r}, \mathbf{r}')}\Delta(\mathbf{r}'), \tag{18}$$

where

$$\overline{Q_0(\mathbf{r}, \mathbf{r}')} = \overline{K_0(\mathbf{r}, \mathbf{r}')}\cos(2\mathbf{p}_s(\mathbf{r})(\mathbf{r}-\mathbf{r}')),$$

$$\overline{K_0(\mathbf{r}, \mathbf{r}')} = \int\limits_{0<p^2<2m\mu_2} \frac{d^2p}{(2\pi)^2}\int\frac{d^2q}{(2\pi)^2}\exp(iq(\rho-\rho'))\int\int d\varepsilon\, d\omega \times\Phi\left(\varepsilon, \varepsilon-\frac{\mathbf{pq}}{m}-\omega; T\right)\frac{2(\overline{v^{-1}(x)})^2}{(2\pi)^2 d}\cos(\omega(x-x')\overline{v^{-1}(x)}). \tag{19}$$

In the derivation of (19) we used the fact that $\Delta(\mathbf{r})$ varies over distances $\sim\xi(T)$, and consequently the characteristic frequencies $\omega \lesssim T_c \ll v_F/d$. We note that a contribution to the integral with respect to p in (19) is made only by the interval $0 < p^2 < 2m\mu_2$; this corresponds physically to the fact that the electrons "trapped" in the layer $kd < x < kd + d_1$ make no contribution to the essentially nonlocal properties of the superconducting system. Taking the definition of T_c (see (10)) and the condition $\tau \ll 1$ into account, we transform Eq. (8) in the following manner:

$$\int dV' \overline{Q_0(\mathbf{r}, \mathbf{r}')}\Delta(\mathbf{r}') + \frac{1}{2}\tau\overline{\nu(x)}\,\Delta(\mathbf{r}) + \overline{O(\Delta^3(\mathbf{r}))} = 0. \tag{20}$$

Equation (20) contains only slowly varying functions of the coordinates $\Delta(\mathbf{r})$, and therefore the integrand in (20) can be expanded in powers of $\nabla\Delta(\mathbf{r})$ and p_s. Confining ourselves to the quadratic terms of the expansion, we have

$$\int dV' \overline{Q_0(\mathbf{r}, \mathbf{r}')}\,\Delta(\mathbf{r}') = L_{\alpha\beta}(\nabla_\alpha\nabla_\beta - 4p_{s\alpha}p_{s\beta})\Delta(\mathbf{r}), \tag{21}$$

where

$$L_{\alpha\beta} = \frac{1}{2}\int dV' \overline{K_0(\mathbf{r}, \mathbf{r}')}(\mathbf{r}-\mathbf{r}')_\alpha(\mathbf{r}-\mathbf{r}')_\beta. \tag{22}$$

The calculation of $L_{\alpha\beta}$ with the aid of formulas (22) and (19) is elementary and yields

$$L_{\alpha\beta} = (2\pi\eta T_c^2)^{-1} l_{\alpha\beta}, \quad \eta = 8\pi^2/7\zeta(3),$$

where $l_{\alpha\beta}$ is a diagonal tensor with components

$$l_{xx} = \int\limits_{0<p^2<2m\mu_2} \frac{d^2p}{(2\pi)^2}(\overline{v^{-1}(x)})^{-1},$$

$$l_{yy} = l_{zz} = \int\limits_{0<p^2<2m\mu_2} \frac{d^2p}{(2\pi)^2}v_y^2(\overline{v^{-1}(x)}). \tag{23}$$

As is well known[7], we can neglect the spatial dependences of Δ and χ in the term of (18) cubic in Δ, and consider this term in the local approximation:

$$\overline{O(\Delta^3(\mathbf{r}))} \approx -\frac{1}{2\eta}\frac{\overline{\nu(x)}}{T_c^2}\Delta^3(\mathbf{r}).$$

A similar procedure of separating the "smooth" functions Δ and χ can be carried out also in (13) and (14). The condition (13) that the order parameter be real reduces in this case to the current-conservation law

$$L_{\alpha\beta}\nabla_\alpha v_{s\beta}\Delta^2(\mathbf{r}) = 0, \tag{24}$$

and the connection between (14) the current and v_s takes the form

$$j_\alpha = 2en_{\alpha\beta}(\mathbf{r})v_{s\beta}, \quad n_{\alpha\beta}(\mathbf{r}) = 4mL_{\alpha\beta}\Delta^2(\mathbf{r}). \tag{25}$$

It is easily seen that Eqs. (20) and (24) are respectively the real and imaginary parts of the equation

$$\{L_{\alpha\beta}(\nabla+2i\mathbf{p}_s)_\alpha(\nabla+2i\mathbf{p}_s)_\beta + {}^1/_2\overline{\nu(x)}[\tau - \Delta^2(\mathbf{r})/\eta T_c^2]\}\Delta(\mathbf{r}) = 0, \tag{26}$$

and form together with (25) a complete system of equations, which are a generalization of the usual Ginzburg-Landau equations[6]. It can be verified that $L_{\alpha\beta}$ is proportional to $\delta_{\alpha\beta}$ in the homogeneous case ($\nu(x) = \nu$ = const). Thus, the superconducting properties of the considered layered structure exhibit an effective anisotropy described by the tensor $l_{\alpha\beta}$.

The physical meaning of the quantities in (25) and (26) becomes clear if (26) is rewritten in the more complete form

$$[\Xi_{\alpha\beta}(T)(\nabla+2i\mathbf{p}_s)_\alpha(\nabla+2i\mathbf{p}_s)_\beta + 1 - \Delta_0^{-2}\Delta^2(\mathbf{r})]\Delta(\mathbf{r}) = 0. \tag{27}$$

Here $\Delta_0 = \Delta_0(T) = T_c\sqrt{\eta\tau}$ is the equilibrium value of the order parameter in the absence of external fields and currents, while the tensor $\Xi_{\alpha\beta}$ differs from $L_{\alpha\beta}$ only in the coefficient

$$\Xi_{\alpha\beta}(T) = (\pi\Delta_0^2\overline{\nu(x)})^{-1}l_{\alpha\beta}.$$

Simple calculations show that in the homogeneous case we have $\Xi_{\alpha\beta}(T) \to \xi^2(T)\delta_{\alpha\beta}$, where $\xi(T)$ is the effective coherence length near T_c: $\xi^2(T) = v_F^2/6\Delta_0^2$. It follows therefore that $\Xi_{\alpha\beta}$ has the meaning of the tensor of the squares of the coherence length.

Using Maxwell's equation curl $\mathbf{H} = 4\pi\mathbf{j}/c$ and relation (25), we can introduce in similar fashion the tensor of the squares of the magnetic-field penetration depth:

$$\Lambda_{\alpha\beta}(T) = \frac{c^2}{16e^2\tau}l_{\alpha\beta}^{-1}$$

and the Ginzburg-Landau tensor parameter

$$\varkappa_{\alpha\beta} = \frac{c}{4e}[\eta\overline{\nu(x)}]^{1/2}T_c l_{\alpha\beta}^{-1}.$$

Direct calculation of the components of the tensor $l_{\alpha\beta}$ in accordance with formulas (23) leads to expressions that contain a rather complicated dependence on v_{F1}, v_{F2}, c_1 and c_2, and are too complicated to present here. It is easy to show that at $\mu_1 > \mu_2$ we have $l_{xx} > l_{yy}$, l_{zz}, and consequently $\kappa_{xx} < \kappa_{yy}$, κ_{zz}. The asymptotic form of $l_{\alpha\beta}$ at $\epsilon = (\mu_1 - \mu_2)/\mu_2 \ll 1$ is[3)]

$$l_{xx}=\frac{m^2}{6\pi}v_{F2}^3\left(1+\frac{3}{2}c_1\varepsilon\right),$$
$$l_{yy}=l_{zz}=\frac{m^2}{6\pi}v_{F2}^3\left(1-\frac{3}{2}c_1\sqrt{\varepsilon}\right). \tag{28}$$

Thus, if the properties of the superconducting films are such that κ is near its "critical" value $1/\sqrt{2}$, then in principle there can be realized a situation in which the sandwich behaves as a type-I superconductor in a direction perpendicular to the layers and as a type-II superconductor in the plane of the layers. In particular, an intermediate state with normal and superconducting layers parallel to the film in the sandwich can be produced in such a superconductor, since the surface energy of the interface between the phases is positive in this case. At the same time, if the magnetic field is perpendicular to the layers, quantized Abrikosov vortices can appear[9] and a mixed-state structure can consequently be produced.

As seen from the foregoing, a distinguishing feature of the considered system[4)] is the noticeable anisotropy of its magnetic properties. This aspect of the behavior of a layered-periodic superconductor can be greatly enhanced by interlining the superconducting films with dielectrics, and the latter can be made of the same material in this case. A detailed theoretical calculation of this system is now underway, and we note here only some physical features of this case. The presence of multiple reflection of the electrons from the dielectric layers leads to the appearance, in the electron wave functions, of random phase factors that oscillate over the wave length of the electron. As a result, a unique problem arises, that of the averaging of the physical quantities over the positions of the dielectric layers, analogous to the averaging over the random position of the impurity[7]. The role of the scattering "centers" is played here by two-dimensional "defects," which are the dielectric layers. Another feature is that in the case of low transparency of the dielectric layers, this system constitutes a superconductor with weak coupling, consisting of a set of parallel Josephson junctions. The Josephson jumps in the phase of the superconducting-ordering parameter, which occur in this case, should lead to a number of specific electrodynamic phenomena.

1)From the point of view of the study of proximity effect, it is precisely this case which is of greatest interest, since the mutual influence of the films is maximal when the electrons are weakly reflected.

2)In accordance with the results of Sec. 1, the separation of the small increment $\delta\Delta$ is possible only if the difference between $g_1\nu_1$ and $g_2\nu_2$ is small enough.

3)Only the limiting transition $c_1 \to 0$ is possible in formulas (23) and (28), but not $c_1 \to 1$. The latter is due to the fact that expressions (4) are valid only for a sufficiently large thickness d_2, when the electrons with longitudinal momenta $2m\mu_2 < p^2 < 2m\mu_1$ can be regarded as "trapped" in the layers $kd < x < kd + d_1$.

4)From the point of view of technology, it appears that the construction of such a multilayered sandwich is not a complicated problem and can be effected, e.g., by successive sputtering of superconducting films made of different materials.

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210