Bresmstrahlung effect in a strong radiation field

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(Submitted June 9, 1973)
Zh. Eksp. Teor. Fiz. 65, 2196–2202 (December 1973)

The cross sections of multiquantum emission and absorption are calculated in the presence of a strong electromagnetic field in scattering by a Coulomb potential. Estimates are obtained for the plasma absorption coefficient in a strong field.

1. The problem of determining the cross sections for bremsstrahlung or absorption in a monochromatic field beyond the limits of applicability of perturbation theory is of interest, for example, in connection with the study of plasma heating by laser radiation. Numerous attempts were made to calculate the cross sections by using non-relativistic wave functions that are exact in terms of the field and the first Born approximation with respect to the potential of the scatterer[1–4]. The purpose of the present paper is to calculate by this method the cross sections for a multiquantum stimulated bremsstrahlung effect in the case of scattering by a Coulomb potential $V(r) = Ze^2r^{-1}$.

We call the field p-strong if

$$p=eEv/\hbar\omega^2\gg 1,$$

where E and ω are the intensity and frequency of the electromagnetic field, and e and v are the charge and velocity of the electron; p^2 is the parameter of the perturbation-theory expansion. The character of the cross sections in a p-strong field depends on the classical parameter of the field strength

$$q=2eE/mv\omega=2v_e/v.$$

Here m is the electron mass and v_e is the amplitude of the velocity oscillations of the free electron in the classical case.

We introduce a radiation-intensity scale that is independent of the electron velocity

$$I_0=m\omega^3/16\pi\alpha,$$

where α is the fine-structure constant; $I_0 = 1.4 \times 10^{12}$ W/cm^2 for the emission of a neodymium laser. Then

$$p^2=\frac{I}{I_0}\xi^{-1},\quad q^2=\frac{I}{I_0}\xi,$$

where $\xi=2\hbar\omega m^{-1}v^{-2}$ is the ratio of the quantum energy to the initial electron energy. Inasmuch as $\xi = 10^{-2}$ for a typical laser plasma, and I/I_0 can reach 10^4, it is possible to have either $q \ll 1$ or $q \gg 1$ in a p-strong field. We confine ourselves to the case $\xi \ll 1$, so that a p-weak field is also q-weak.

The expression for the differential cross section of multiquantum transitions of an electron when scattered in the presence of a homogeneous field $\mathbf{A}(t) = ec\omega^{-1}\mathbf{E} \times \sin\omega t$ is given by (see[1–4])

$$\frac{d\sigma^{\pm n}}{d\Omega}=\frac{d\sigma_0}{d\Omega}(\mathbf{n}_0-\lambda\mathbf{n})\lambda J_n^2[p\mathbf{e}(\mathbf{n}_0-\lambda\mathbf{n})] \quad (1)$$

where $\lambda = \sqrt{1 \pm n\xi}$, n is the number of quanta (the upper sign pertains throughout to absorption and the lower to emission), $\mathbf{n}_0$ and $\mathbf{n}$ are unit vectors in the directions of the initial and final electron momenta, and J_n is a Bessel function;

$$\frac{d\sigma_0(\mathbf{n}_0-\lambda\mathbf{n})}{d\Omega}=\left(\frac{m}{2\pi\hbar^2}\right)^2\left|\int Ze^2r^{-1}\exp[i\hbar^{-1}p_0(\mathbf{n}_0-\lambda\mathbf{n})r]dr\right|^2. \quad (2)$$

We put

$$\mathbf{n}_0=(0,0,1),\ \mathbf{e}=(\sin\psi,0,\cos\psi),$$
$$\mathbf{n}=(\sin\theta\cos\varphi;\ \sin\theta\sin\varphi;\ \cos\theta);$$

and obtain

$$\frac{d\sigma^{\pm n}}{d\Omega}=s\cdot 4\lambda\frac{J_n^2\{p[\cos\psi(1-\lambda\cos\theta)-\lambda\sin\psi\sin\theta\cos\varphi]\}}{(1+\lambda^2-2\lambda\cos\theta)^2} \quad (3)$$

Thus, the calculation of the total cross section reduces to integration of expression (3) over the solid angle (see the Appendix) at fixed ψ. We confine ourselves henceforth to the cases $\psi = 0$ and $\psi = \pi/2$; the factor $s=Z^2e^4m^{-2}v^{-4}$ which is common to all the cross sections will be omitted in Secs. 2 and 3.

2. In a p-weak field, only single-quantum cross sections have significant values; we present the expressions calculated for them by perturbation theory[1] (the cross sections in [1] are given not relative to the electron flux but to the radiation flux, so that all the formulas for the cross sections contain an extra factor $4\pi\hbar\omega E^{-2}c^{-1}$.) For the electrons whose initial momenta are parallel to the polarization of the field we have

$$\sigma^{\pm 1}=\pi p^2[2\mp 4\xi\ln 4\xi^{-1}\pm 4\xi], \quad (4)$$

and for the electrons whose momenta are perpendicular to the polarization of the field

$$\sigma^{\pm 1}=\pi p^2[\ln 4\xi^{-1}-1\pm 2\xi\ln 4\xi^{-1}]. \quad (5)$$

We emphasize the following phenomenon, first noticed by Marcuse[5]: at $\mathbf{n}_0 \parallel \mathbf{e}$ the emission cross section exceeds the absorption cross section.

In a p-strong field at $q \ll 1$ and $\mathbf{n}_0 \parallel \mathbf{e}$, the cross sections decrease in power-law fashion at

$$\delta<2q\pm q^2, \quad (6)$$

where $\delta = n\xi$ is the ratio of the electron energy increment due to scattering to its initial energy. The cross sections of the processes at $n > 1$ are given by

$$\sigma^{+n}=\frac{8}{p\delta}\left[\frac{1}{(1+\sqrt{1+\delta})^2}-\frac{q}{\delta(q+2)}+\frac{1}{\delta}\ln\frac{2+q}{1+\sqrt{1+\delta}}\right],$$
$$\sigma^{-n}=\frac{8}{p\delta}\left[\frac{q}{\delta(2-q)}-\frac{1}{(1+\sqrt{1-\delta})^2}+\frac{1}{\delta}\ln\frac{2-q}{1+\sqrt{1-\delta}}\right]. \quad (7)$$

Expanding in powers of q up to third order, we obtain

$$\sigma^{\pm n}=p(1\mp q)/n^2. \quad (8)$$

The single quantum case is an exception: its cross section is

$$\sigma^{\pm 1}=\pi p^2\left[1+\frac{1}{2p}\mp\frac{1}{2}\xi(\ln 2q^{-1}+\ln\xi^{-1})\right]. \quad (9)$$

Thus, in a q-weak field the Marcuse effect is preserved also at $p \gg 1$ for all the multiquantum processes. The absorption coefficient is given by

$$\Delta=\sum_{n=1}^{N}n(\sigma^{+n}-\sigma^{-n}). \quad (10)$$

At $\mathbf{n}_0 \parallel \mathbf{e}$ we have

$$\Delta=-p^2\xi\left[\pi\left(\ln\frac{2}{q}+\ln\xi^{-1}\right)+2\ln 2p\right]<0. \quad (11)$$

If $\mathbf{n}_0 \perp \mathbf{e}$, then the cross sections decrease in power-law fashion at

$$\delta<{}^1/_2q(\sqrt{4+q^2}\pm q), \quad (12)$$

i.e., the effective quantum multiplicity has in this case half the value as at $\mathbf{n}_0 \parallel \mathbf{e}$. The cross sections of the processes at $n > 1$ are

$$\sigma^{\pm n}=\frac{16}{p\delta^2}\left(q\lambda-\frac{1+\lambda^2}{\sqrt{1+4q^{-2}}}\right). \qquad (13)$$

Expanding in powers of q up to third order, we obtain

$$\sigma^{\pm n}=\frac{2p^2}{n^3}\left(1\pm\frac{n\xi}{2}\right). \qquad (14)$$

The single-quantum cross sections are exceptions also in this case:

$$\sigma^{\pm 1}=\pi p^2\left(1\pm\frac{\xi}{2}\right)\left[\ln\frac{4}{\xi}-1+\frac{1}{2}\left(1-\frac{1}{p^2}\right)^{1/2}-\operatorname{arch}p\right]. \qquad (15)$$

The integral difference between the cross sections, which determines the absorption coefficient, is

$$\Delta=p^2\xi\left[\pi\left(\ln\frac{4}{\xi}-1+\frac{1}{2}\left(1-\frac{1}{p^2}\right)^{1/2}-\operatorname{arch}p\right)+2\ln p\right]>0. \qquad (16)$$

We note that the main contribution to the integral difference of the cross sections at $q \ll 1$ is made by single-quantum processes. Thus, in a q-weak field the cross section increases with increasing p also at $p \gg 1$. The single-quantum cross sections are then overwhelmingly large in comparison with the multiquantum processes: this "single-photon anomaly" is typical of scattering by a Coulomb potential and is connected with the predominant role played by small-angle scattering.

We have confined ourselves above to cross sections that make noticeable contributions to the change in the electron energy, and decrease in power-law fashion with n. At $\mathbf{n}_0 \parallel \mathbf{e}$ and $\delta > 2q \pm q^2$, the cross sections are given by

$$\begin{aligned}\sigma^{-n}&=\frac{2}{pn(1-\sqrt{1-\delta})^2}\left\{\exp\left[-2n\operatorname{arch}\frac{\delta}{q(1-\sqrt{1-\delta})}\right]-\exp\left[-2n\operatorname{arch}\frac{\delta}{q(1+\sqrt{1-\delta})}\right]\right\},\\ \sigma^{+n}&=\frac{2}{pn(\sqrt{1+\delta}-1)^2}\left\{\exp\left[-2n\operatorname{arch}\frac{\delta}{q(\sqrt{1+\delta}-1)}\right]+\exp\left[-2n\operatorname{arch}\frac{\delta}{q(1+\sqrt{1+\delta})}\right]\right\}.\end{aligned} \qquad (17)$$

These expressions are suitable at $n-(2p\pm pq)\gg n^{1/3}$. In accordance with the hypothesis advanced in [2], the cross sections decrease exponentially. In the case of the emission cross section, an additional cutoff, of the root type, of the order of ξ^{-1}, appears as $\delta \to 1$.

3. In a q-strong field at $\mathbf{n}_0 \parallel \mathbf{e}$, the emission cross section decrease in power-law fashion at $\delta < 1$:

$$\sigma^{-n}=\frac{8}{p}\left[\frac{4\sqrt{1-\delta}}{\delta^3}+\frac{1}{\delta^2}\ln\frac{1-\sqrt{1-\delta}}{1+\sqrt{1-\delta}}\right]. \qquad (18)$$

We represent the emission cross section in the form of the sum $\sigma^{+n} = \mu_n + \nu_n$, where μ_n is used when $\delta < q^2-2q$ and ν_n at $\delta<q^2+2q$:

$$\begin{aligned}\mu_n&=\frac{8}{p}\left[\frac{1}{\delta(\sqrt{1+\delta}-1)^2}-\frac{q}{\delta^2(q-2)}+\frac{1}{\delta^2}\ln\frac{q-2}{\sqrt{1+\delta}-1}\right],\\ \nu_n&=\frac{8}{p}\left[\frac{1}{\delta(\sqrt{1+\delta}+1)^2}-\frac{q}{\delta^2(q+2)}+\frac{1}{\delta^2}\ln\frac{q+2}{\sqrt{1+\delta}+1}\right].\end{aligned} \qquad (19)$$

At small δ we have

$$\begin{aligned}\sigma^{-n}&=\frac{8}{p}\left[\frac{4}{\delta^3}-\frac{2}{\delta^2}-\frac{1}{2\delta}+\frac{1}{\delta^2}\ln\frac{\delta}{4}\right],\\ \mu_n&=\frac{8}{p}\left[\frac{4}{\delta^3}-\frac{2}{\delta^2}-\frac{1}{\delta^2}\frac{q}{q-2}+\frac{1}{\delta^2}\ln(q-2)-\frac{1}{\delta^2}\ln\frac{\delta}{2}-\frac{1}{4\delta}\right],\\ \nu_n&=\frac{8}{p}\left[-\frac{1}{\delta^2}\frac{q}{q+2}+\frac{1}{\delta^2}\ln\frac{q+2}{2}\right].\end{aligned} \qquad (20)$$

The cross section difference is

$$\sigma^{+n}-\sigma^{-n}\approx\frac{16}{p\delta^2}\ln\frac{\delta q}{2\sqrt{2}}. \qquad (21)$$

Thus, in a q-strong field the Marcuse effect is retained only at $\delta < 2\sqrt{2}q^{-1}$.

For processes with a large number of quanta, the absorption begins to prevail over emission, and to determine whether the electron energy increases or decreases (in the mean) in such a field it is necessary to consider the integral emission coefficient

$$E=\sum_{n=1}^{[\xi^{-1}]}n\sigma^{-n}\approx\xi^{-2}\int_{\xi}^{1}\sigma^{-}(\delta)\,\delta\,d\delta \qquad (22)$$

and the absorption coefficient

$$A\approx\xi^{-2}\left[\int_{\xi}^{q^2-2q}\mu(\delta)\,\delta\,d\delta+\int_{\xi}^{q^2+2q}\nu(\delta)\,\delta\,d\delta\right]. \qquad (23)$$

The general (exceedingly cumbersome) formula shows that the integral cross-section difference (together with the absorption coefficient) is positive when q exceeds the root q_0 of the equation

$$\frac{q}{q-2}\ln\frac{\xi}{q(q-2)}+\ln^2\frac{\xi}{2}=0 \qquad (24)$$

(at $\xi\ll 1$ $(q_0-2)\ll 1$). At $q \gg 1$, the integral emission and absorption coefficients, accurate to terms that are quadratic in the large logarithms, are given by

$$\begin{aligned}E&=\frac{8}{\xi q}\left(\frac{4}{\xi}-\frac{1}{2}\ln^2\xi\right),\\ A&=\frac{8}{\xi q}\left(\frac{4}{\xi}+\frac{1}{2}\ln^2\xi+2\ln^2 q-2\ln\xi\ln q\right).\end{aligned} \qquad (25)$$

The integral cross section difference is therefore

$$\Delta=\frac{16}{\xi q}\left(\ln^2 q-\ln\xi\ln q+\frac{1}{2}\ln^2\xi\right) \qquad (26)$$

If $\mathbf{n}_0 \perp \mathbf{e}$, then we have for the cross sections of the processes at $q \gg 1$

$$\sigma^{\pm n}=\frac{8}{p\delta^2}\left[\left(\frac{2\pm\delta}{\delta}\right)\left(\operatorname{arch}\frac{q\lambda}{\sqrt{1+\lambda^4}}-\frac{1}{\sqrt{1+4q^{-2}}}\right)+1\right]. \qquad (27)$$

The cross sections decrease in power-law fashion for an emission at $\delta < 1 - q^2$ and for absorption at $\delta < q^2$, i.e., in a q-strong field the quantum multiplicity is the same as at $\mathbf{n}_0 \parallel \mathbf{e}$. The single-photon anomaly vanishes on going to the q-strong field. We note that at any value of q the cross sections of the processes with $\mathbf{n}_0 \perp \mathbf{e}$ are larger than the corresponding cross sections at $\mathbf{n}_0 \parallel \mathbf{e}$.

The integral emission and absorption coefficients are given in this case by

$$\begin{aligned}E&=\frac{16}{\xi q}\left[\frac{2(\ln q-1)}{\xi}+\ln q\ln\xi\right],\\ A&=\frac{16}{\xi q}\left[\frac{2(\ln q-1)}{\xi}+\ln^2 q-\ln q\ln\xi\right].\end{aligned} \qquad (28)$$

The integral difference of the cross sections is

$$\Delta=\frac{16}{\xi q}[\ln^2 q-2\ln q\ln\xi] \qquad (29)$$

and differs from (26) only in the form of the logarithmic factor. This result can be easily understood: in a q-strong field the predominant role is played by absorption processes with $\delta \gg 1$, and the direction of the initial momentum for such processes is immaterial. An analysis of the integrand shows that at any direction of the initial momentum the most probable directions of the final momentum lie in the vicinity of the field-polarization vector.

4. The expressions given for the cross sections yield

semiquantitative estimates for the absorption coefficient of the electron magnetic radiation of a fully ionized plasma with an isotropic velocity distribution function normalized by the condition

$$4\pi \int v^2 f(v)\,dv = N_e.$$

In a q-weak field, according to Sec. 2, the main contribution to the absorption coefficient is produced by the single-quantum processes:

$$\varkappa = N_i \hbar\omega I^{-1} \langle v(\sigma^{+1} - \sigma^{-1})\rangle. \quad (30)$$

At $\xi \ll 1$ this expression reduces to

$$\varkappa = N_i \hbar\omega I^{-1} \int_0^\infty v^3 S^{+1} \left[f\left(v^2 + \frac{2\hbar\omega}{m}\right)^{1/2} - f(v) \right] dv, \quad (31)$$

where s^{+1} is the single-quantum absorption cross section averaged over the angle ψ. From a comparison of (9) and (15), we see that the largest contribution in the integration with respect to ψ is given by the region $n_0\mathbf{e} \ll 1$, due to the large logarithm $\ln 4\xi^{-1}$. Thus we represent S^{+1} in the form $\pi p^2 s\eta \ln(4/\xi)$, where η is a numerical factor on the order of unity.

Returning to the initial variables, we obtain

$$\varkappa = \frac{N_i 16\pi^2 e^6}{\omega^2 c m^3} \int_0^\infty 4\pi\eta \ln\frac{4}{\xi} \frac{\partial f}{\partial v} dv. \quad (32)$$

This expression coincides with the result of the classical calculation [2], the only difference being that in place of $\eta \ln^4 \xi^{-1}$ as in the classical case we have under the integral sign $L(v)/3$, where the Coulomb logarithm $L(v)$, like $\ln 4\xi^{-1}$, is a monotonically increasing function of v. The difference between the logarithmic factors is due to the fact that in the quantum calculation there is no divergence at small scattering angles. The approximate character of the calculations does not permit a determination of the corrections to the plasma absorption coefficient in a p-strong field; with increasing p, however, the ratio of the cross sections $\sigma^{\pm 1}$ to the field intensity increases more rapidly than the increase of the contribution from the multiquantum processes, both when $\mathbf{n}_0 \parallel \mathbf{e}$ and when $\mathbf{n}_0 \perp \mathbf{e}$, so that the absorption coefficient decreases with increasing p.

The rate of increase of the electron energy in a q-strong field

$$\alpha(\psi, v, p) = N_i \hbar\omega v \Delta \quad (33)$$

depends only logarithmically on the magnitude and direction of the initial momentum. For the absorption coefficient of a plasma with average velocity w, such that $q(w) \gg 1$, we can therefore write

$$\varkappa = N_e N_i \hbar\omega I^{-1} \frac{16}{\xi q} (\ln q + \zeta \ln \xi^{-1}) \ln q, \quad (34)$$

where ζ is a numerical factor on the order of unity. This formula differs from the classical expression for the absorption coefficient of an isotropic Maxwellian plasma in a q-strong field [6] only by the logarithmic term in the parentheses; in the classical case this term is replaced by the Coulomb logarithm $L(v)$.

Thus, at $p \lesssim 1$ the behavior of the coefficient of absorption of electromagnetic radiation by a plasma is determined not by the quantum parameter p of the field strength, but by the classical parameter q. Expressions for the absorption coefficient in both a q-weak and q-strong field differ from the corresponding classical formulas at most in the forms of the logarithmic factors. The approximate character of the calculations does not permit a definite conclusion to be drawn concerning the existence and magnitude of the quantum corrections to the classical expressions.

The author thanks L. V. Keldysh for useful discussions.

APPENDIX

The asymptotic form of the Bessel functions $J_n(z)$ depends on the relation between n and z [7]. The well known formula

$$J_n(z) \approx \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\pi}{2} n - \frac{\pi}{4}\right) \quad (A.1)$$

is valid only at $z > n$ ($z - n \gg n^{1/3}$). In the calculations for $z \geq n$, we use the asymptotic form averaged over the oscillations

$$J_n^2(z) = \frac{1}{\pi z} \quad (A.2)$$

In the region $z < n$ $(n - z \gg n^{1/3})$ at large n, we can use the Meissel formula

$$J_n^2(z) \approx \exp\left[-2n\left(\operatorname{arch}\frac{n}{z} - \left(1 - \frac{z^2}{n^2}\right)^{1/2}\right)\right] \Big/ 2\pi(n^2 - z^2)^{1/2}. \quad (A.3)$$

Comparison of (A.2) and (A.3) shows that the contribution from the region $z < n$ at $n \gg 1$ is negligibly small in comparison with the contribution from the region $z > n$. It is natural to define the effective quantum multiplicity N as the integer part of the maximum value of the argument of the Bessel function at a given p. Then at $n > N$ the cross sections decrease in power-law fashion, and at $n > N$ they decrease exponentially. The power-law expansions of the Bessel functions were used at small n to calculate the integrals in the region $z < n$.

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Translated by J. G. Adashko
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