Collective interaction between a monoenergetic beam and a plasma layer

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The stability of a monoenergetic electron beam that is continuously injected into a plasma layer whose thickness is small compared to the relaxation length of the beam in an infinite plasma is investigated theoretically. The problem of finding the spectra of the natural oscillations of the system is reduced to that of solving a transcendental equation whose coefficients are determined by the transformation tensors for the natural waves of the system at the entrance and exit boundaries of the layer. These tensors are computed analytically by solving integral equations for the field in the corresponding plasma-beam half-spaces, thereby generalizing the well-known Landau results to the case of the nonequilibrium plasma. Analytic solutions to the equation for the spectrum are obtained: The excitable frequencies and the increments are found in the limiting cases of large and small differences between the beam velocity and the phase velocity of the plasma waves in the layer. The physical mechanisms underlying the instability under consideration are discussed.

1. The study of the interaction of high-frequency fields with a bounded plasma is important in connection with the many applications of the theory of collective processes in a plasma: beam and high-frequency heating, plasma methods of generation and amplification of microwaves, acceleration of charged particles, etc. The presence of plasma boundaries leads not only to the discontinuity of the wave-number spectrum and the distortion of the field pattern of the natural and forced oscillations owing to the elastic scattering of the corresponding waves by the plasma boundary, but also to the appearance of new (surface) types of waves, as well as to the appearance of a unique absorption, due, in particular, to the processes of inelastic scattering of these waves by the boundary with the generation of new types of oscillations (the interconversion of the waves corresponding to the various branches of the spectrum).

The penetration of a bounded plasma by a high-frequency longitudinal field was first investigated theoretically by Landau[1], while the kinetic theory of transverse-wave scattering by the boundary of a semi-infinite plasma was developed by Silin and Fetisov[2]. The results obtained in the indicated papers pertain to the equilibrium plasma with a Maxwellian electron distribution function; they have recently been successfully generalized to the case of finite values of the high-frequency-field amplitude[3-5].

In many applications, the source of the plasma instability is an electron beam injected from without. The presence of the beam in the bounded plasma leads to the appearance among the possible channels for the inelastic wave scattering by the plasma boundary of beam-type oscillations—charge-density waves. The decisive role played by these types of waves in the collective-interaction processes under the conditions of applicability of the hydrodynamic approximation ($\delta \gg k_{\parallel} v_{Tb}$) follows from the fact that the increment δ (or the amplification factor α) is positive only for the slow charge-density wave. Therefore, the extraction of the energy of the oscillations excited by the beam requires, in essence, the conversion of this wave into waves of other types (in particular, electromagnetic waves) when the charge-density wave is scattered by the plasma inhomogeneities[1)]. The effects of the interconversion of the charge-density waves of the beam and the natural waves of the plasma essentially influence the natural-oscillation spectra of the bounded plasma-beam systems (cf. [10]). In the present paper we investigate the collective interaction of a monoenergetic beam with a plasma layer whose thickness is small compared to the beam relaxation length in an infinite plasma:

$$L < l_\infty = \frac{V_0}{\omega_p}\left(\frac{n_p v_{Tp}^2}{n_b V_0^2}\right)^{1/3},$$

and show that the indicated effects ensure the development of an instability in the system under consideration.

2. Let us consider a plane layer (of thickness L) of an equilibrium plasma of density n_p into which a monoenergetic electron beam with the equilibrium velocity and density values V_0 and $n_b \ll n_p$, respectively, is continuously injected perpendicularly to the plane of the layer. The self-consistent system of equations describing the interaction of the beam with such a plasma consists of the kinetic equations for the beam and plasma particles and the Poisson equation for the field:

$$\begin{gathered} \frac{\partial f_s}{\partial t} + v\frac{\partial f_s}{\partial z} + \frac{e}{m}E\frac{\partial f_{s0}}{\partial v} = 0 \qquad (s = p, b), \\ \frac{\partial E}{\partial z} = 4\pi e \int dv\,(f_p + f_b), \qquad \int dv\, f_{s0} = n_s, \end{gathered} \tag{1}$$

where n_+ is the density of the ion background ($n_+ = n_b + n_p$).

As boundary conditions for the plasma-particle distribution function, we use the condition for specular reflection from the boundaries of the layer; as to the beam, in the most interesting case when $V_0 \gg v_{Tb}$ (v_{Tb} and v_{Tp} are the thermal velocities of the beam and plasma particles, respectively) we can neglect its interaction with the layer boundaries[11], and consider the layer to be transparent to the beam particles.

Under the conditions of applicability of the hydrodynamic description:

$$\frac{v_{Tb}}{V_0} \ll \left(\frac{n_b}{n_p}\frac{V_0^2}{v_{Tp}^2}\right)^{1/3} \ll 1,$$

the spectrum of the natural oscillations of the corresponding infinite plasma, given by the dispersion equation

$$D(k) \equiv 1 - K_p(k) - \frac{\omega_b^2}{(\omega - kV_0)^2} = 0,$$

$$K_p(k)=\frac{4\pi e^2 i}{\omega m}\int_{-\infty}^{+\infty}\frac{v\,dv}{i(kv-\omega)+\nu_p}\frac{df_{p0}}{dv},\qquad \omega_b^2=\frac{4\pi e^2 n_b}{m},\qquad (2)$$

consists of two plasma and two beam waves (charge density waves).

For a fixed frequency ω ($\mathrm{Im}\,\omega>0$), the dependence of the wave numbers of these waves on the frequency ω is determined by the relations

$$k_b^{\pm}\equiv k_{1,2}=\frac{\omega}{V_0}\pm\frac{\omega_b}{V_0\sqrt{\varepsilon(0)}},\qquad \varepsilon(0)\equiv 1-K_p(0),$$
$$k_p^{\pm}\equiv k_{3,4}=\pm\frac{\omega\,({}^2/{}_3\varepsilon(0))^{-1/2}}{v_{Tp}},\qquad \frac{3}{2}v_{Tp}^2\ll V_0^2|\varepsilon(\theta)|;\qquad (3a)$$

$$k_b^{\pm}\equiv k_{1,2}=\frac{\omega_0}{V_0}[1+\delta_0 e^{\pm\pi i/6}];$$
$$k_p^{+}\equiv k_3=\frac{\omega_0}{V_0}[1-\delta_0],\qquad k_p^{-}\equiv k_4=-\frac{\omega_0}{V_0},$$
$$\omega_0\equiv\omega_p(1-\mu)^{-1/2},\qquad \delta_0\equiv\left(\frac{n_b}{2\mu n_p}\right)^{1/3}\ll 1,\qquad \mu\equiv\frac{3}{2}\frac{v_{Tp}^2}{V_0^2}\ll 1.\qquad (3b)$$

The equation for the spectrum of the layer under consideration can be obtained from the relations between the complex amplitudes of the waves scattered by the boundaries of the layer:

$$A_\alpha^{(-)}(0)=T_{\alpha\beta}^{(-)}A_\beta^{(+)}(0),\qquad A_\beta^{(+)}(L)=T_{\beta\gamma}^{(+)}A_\gamma^{(-)}(L),$$
$$A_\sigma^{(\pm)}(L)=A_\sigma^{(\pm)}(0)\exp(ik_\sigma L),\qquad (4a)$$

here the tensors $T_{\alpha\beta}^{(\pm)}$ describe the transformation processes at the walls, where the beam respectively enters (–) and emerges from (+) the plasma.

The relations (4a) are equivalent to a system of homogeneous algebraic equations for the amplitudes $A^-(0)$:

$$A_\alpha^{(-)}(0)=T_{\alpha\beta}^{(-)}T_{\beta\gamma}^{(+)}A_\gamma^{(-)}(0)\exp[i(k_\gamma-k_\alpha)L],\qquad (4b)$$

from which the required dispersion equation can be found by equating its determinant to zero[10]:

$$\|\delta_{\alpha\gamma}-T_{\alpha\beta}^{(-)}T_{\beta\gamma}^{(+)}\exp[i(k_\gamma-k_\alpha)L]\|=0.\qquad (4c)$$

Thus, the problem of finding the spectrum of the natural oscillations of the system under consideration reduces to that of finding the tensors (the scattering matrices) $T_{\alpha\beta}^{(\pm)}$, which can be computed on the basis of the solution of the scattering problems for the corresponding semi-infinite systems. Since the system is in a nonequilibrium state, the conditions of applicability of the reciprocity theorem are not fulfilled in the present case, and therefore the tensors $T_{\alpha\beta}^{(-)}$ and $T_{\alpha\beta}^{(+)}$ cannot be obtained from each other, which leads to the necessity for a self-consistent treatment of the problems of scattering at the entrance and exit boundaries of the layer.

3. Let us consider the plasma half-space ($z<0$), from which the beam emerges into a vacuum ($z>0$). The natural waves $\mathscr{E}_\alpha\exp(ik_\alpha z)$ ($\alpha=1, 2, 3$; see (3)) of the system are incident on the plasma-vacuum interface from the $z\to-\infty$ side. From the plasma boundary, only one wave can propagate in the direction opposite to that of the beam's motion (let us call this wave the counter plasma wave); therefore, the tensor $T_{\beta\alpha}^{(+)}$ has only three nonvanishing components $T_{4\alpha}^{(+)}$ ($\beta=4$).

We shall seek the resultant field $E_+(z)$ in the plasma in the form of the sum:

$$E_+(z)=\mathscr{E}_+(z)+\sum_{\alpha=1}^{3}\mathscr{E}_\alpha\exp(ik_\alpha z),\qquad (5a)$$

in which the first term describes the field that decreases as we go to infinity, while the second describes the waves incident on the boundary. If the operating frequency ω is such that the conditions for the amplification of the slow charge-density waves ($\alpha=2$, $\mathrm{Im}\,k_2<0$) are fulfilled, then the amplitude of this wave also vanishes at infinity, and the corresponding term in the sum over α in (5a) drops out. In that case the amplifiable charge-density wave contains the term $\mathscr{E}_+(z)$.

Similarly, we shall seek the high-frequency corrections f_S to the equilibrium distribution functions f_{S0} in the form of a sum of two parts which differ in their behavior as $z\to-\infty$:

$$f_s(v,z)=F_s(v,z)+\frac{e}{m}\sum_\alpha\frac{\mathscr{E}_\alpha\exp(ik_\alpha z)}{i(k_\alpha v-\omega)+\nu_s}\frac{df_{s0}}{dv}.\qquad (5b)$$

Taking into account the vanishing of the total plasma current across the plasma boundary and the continuity of the beam current at this boundary, as well as the boundedness of the function $F_S(v, z)$ for $z\to-\infty$, we obtain for the determination of the field $\mathscr{E}_+(z)$ the following integral equation:

$$\mathscr{E}_+(z)-\int_{-\infty}^{z}\mathscr{E}_+(z')K_b(z-z')dz'-\int_{-\infty}^{0}\mathscr{E}_+(z')dz'[K_p(|z-z'|)$$
$$-K_p(|z+z'|)]+g_+(z)=0,\qquad z<0,$$
$$g_+(z)=\frac{4\pi e^2}{i\omega m}\int_0^\infty v\,dv\frac{df_{p0}}{dv}\sum_\alpha\mathscr{E}_\alpha\exp\left(-\frac{i\omega-\nu_p}{v}z\right)\qquad (6)$$
$$\times\left[\frac{1}{i(\omega+k_\alpha v)-\nu_p}-\frac{1}{i(k_\alpha v-\omega)+\nu_p}\right],$$
$$K_s(x)=\frac{4\pi e^2 i}{\omega m}\int_0^\infty v\,dv\frac{df_{s0}}{dv}\exp\left[\frac{i\omega-\nu_s}{v}x\right].$$

Equation (6) does not possess the symmetry that would allow us, in solving it by the Fourier method, to directly continue the functions in an even or odd manner into the region $z>0$ (see [1,2]); therefore, to solve it, we use a generalized Fourier method that allows us to reduce this equation to a boundary-value problem of conjugation of analytic functions [12,13]. With that end in view, let us continue $\mathscr{E}_+(z)$ to the semiaxis $z>0$ by zero, and let us introduce the auxiliary function $H_+(z)$ in the following manner:

$$H_+(z)=0\qquad z<0,$$
$$H_+(z)=-\int_{-\infty}^{z}\mathscr{E}_+(z')K_b(z-z')dz'\qquad (7)$$
$$-\int_{-\infty}^{0}\mathscr{E}_+(z')[K_p(|z-z'|)-K_p(|z+z'|)]-g_+(z),\qquad z>0.$$

Then Eq. (6) assumes a form valid for all z:

$$\mathscr{E}_+(z)-\int_{-\infty}^{z}\mathscr{E}_+(z')K_b(z-z')dz'-\int_{-\infty}^{0}\mathscr{E}_+(z')dz'[K_p(|z-z'|)$$
$$-K_p(|z+z'|)]-G_+(z)=H_+(z),\qquad -\infty<z<+\infty;\qquad (8)$$
$$G_+(z)\equiv g_+(z)\,\mathrm{sgn}\,z.$$

Performing a Fourier transformation of (8), we obtain the problem of conjugation of analytic functions on the contour $\mathrm{Im}\,k=0$[13]:

$$e^+(k)D(k)+e^+(-k)K_p(k)=h^-(k)+g(k),\qquad (9a)$$

or

$$D(k)[e^+(k)+e^-(k)]=\varphi^-(k)+g(k).\qquad (9b)$$

Here we have introduced the notations:

$$e^+(k)\equiv\frac{1}{2\pi}\int_{-\infty}^{+\infty}dz\,\mathscr{E}_+(z)e^{-ikz},$$
$$h^-(k)\equiv\frac{1}{2\pi}\int_{-\infty}^{+\infty}dz\,H_+(z)e^{-ikz},$$

$$g(k)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dz\,G_+(z)e^{-ikz},$$

$$D(k)=1-K_b(k)-K_p(k);\quad \varphi^-(k)=f^-(k)+e^+(-k)[K_b(k)-1],$$

($K_b(k)$ is analytic in the lower half-plane of k).

The solution of this boundary value problem depends on whether or not the condition for amplification of the slow charge-density wave ($\alpha=2$) is fulfilled. In the presence of amplification, the index κ (see [13]):

$$\varkappa=\frac{1}{2\pi}\arg\ln\frac{D(-k)}{D(k)}\Big|_{-\infty}^{+\infty},$$

of the equivalent (to (9)) boundary value problem for $\varphi^-(k)$:

$$\varphi^-(-k)=-\frac{D(-k)}{D(k)}\varphi^-(k)+g(k)\left[1-\frac{D(-k)}{D(k)}\right], \tag{10}$$

is equal to +2, and therefore its solution has the form

$$\varphi^-(k)=-\frac{1}{X_0^-(k)(k-i\rho)^2}\left\{Ak+\frac{1}{2\pi i}\int_C\frac{dx}{x-k}g(x)\right.$$
$$\left.\times[X_0^+(x)(x+i\rho)^2-X_0^-(x)(x-i\rho)^2]\right\}, \tag{10a}$$

where the contour C lies along the real axis, bypassing the point x = k from above, A is a constant, which will be determined below, $X_0^\pm(k)$ are the bounded (at infinity) solutions of the homogeneous conjugation problem with zero index:

$$X_0^+(k)=\frac{D(k)}{D(-k)}\left(\frac{k-i\rho}{k+i\rho}\right)^2X_0^-(k); \tag{10b}$$

and ρ is an arbitrary positive number.

Substituting (10a) and (10b) in (9b), we obtain the Fourier amplitude of the sought field. The constant A is then uniquely determined by the known value of the amplitude $\mathscr{E}_2$ of the amplified wave:

$$A=\frac{1}{k_2}\left\{X_0^-(k_2)(k_2-i\rho)^2\left[g(k_2)+\frac{\mathscr{E}_2}{2\pi i}D'(k_2)\right]\right.$$
$$\left.-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{dx}{x-k_2}g(x)[X_0^+(x)(x+i\rho)^2-X_0^-(x)(x-i\rho)^2]\right\}. \tag{10c}$$

Computing with the aid of the relations (9) and (10) the residues at the points $k=k_\alpha$, we obtain the final expressions for the tensor components $T_{4\alpha}^{(+)}$ in the presence of amplification ($\mathrm{Im}\,k_2<0$):

$$T_{42}^{(+)}=\frac{k_4}{k_2}\frac{X_0^-(k_2)}{X_0^-(k_4)}\left(\frac{k_2-i\rho}{k_4-i\rho}\right)^2\frac{D'(k_2)}{D'(k_4)},\quad D'\equiv\frac{dD}{dk}, \tag{11a}$$

$$T_{4\alpha}^{(+)}=-\frac{2k_4}{D'(k_4)}\left\{F_\alpha(k_4)-\frac{X_0^-(k_2)}{X_0^-(k_4)}\left(\frac{k_2-i\rho}{k_4-i\rho}\right)^2F_\alpha(k_2)\right.$$
$$\left.+\sum_{\beta=1,3}\frac{k_\beta}{k_4}\frac{F_\alpha(k_\beta)X_0^+(k_\beta)}{D'(k_\beta)X_0^-(k_4)}\left(\frac{k_\beta+i\rho}{k_4-i\rho}\right)^2[K_b(k_\beta)-K_b(-k_\beta)]\right\}, \tag{11b}$$

$$F_\alpha(k)\equiv\frac{K_p(k_\alpha)-K_p(k)}{k^2-k_\alpha^2},\quad \alpha=1,3.$$

In the absence of amplification ($\mathrm{Im}\,k_2>0$), the index of the problem (10) is equal to zero. Repeating the above-described computations for this case, we obtain the following expressions for $T_{4\alpha}^{(+)}$ ($\alpha=1,2,3$):

$$T_{4\alpha}^{(+)}=\frac{3k_4v_{T_p}^2}{\omega^2D'(k_4)}\left\{1+\sum_{\beta=1}^{3}\frac{k_\beta}{k_4}\frac{X_{00}^{(+)}(k_\beta)}{X_{00}^{(-)}(k_4)}\frac{K_b(k_\beta)-K_b(-k_\beta)}{(k_\beta-k_4)D'(k_\beta)}\right\}, \tag{12}$$

where $X_{00}^\pm$ are the bounded (at infinity) solutions of the homogeneous problem

$$X_{00}^+(k)=\frac{D(k)}{D(-k)}X_{00}^-(k). \tag{12a}$$

4. In the problem of transformation at the entrance boundary, an unmodulated electron beam with the same parameters (V_0, n_b) enters from a vacuum ($z<0$) into the plasma half-space ($z>0$). From the interior of the plasma, in the direction opposite to that of the beam's motion—toward the interface—propagates the counter plasma wave $\mathscr{E}_4\exp(ik_4z)$. Let us determine the tensor components $T_{\alpha4}^{(-)}$, which characterize the effectiveness of the transformation of this wave into the natural waves of the system that propagate along the direction of motion of the beam.

Let us seek the resultant field $E_-(z)$ in the plasma and the high-frequency corrections to the distribution functions in a form similar to (5):

$$E_-(z)=\mathscr{E}_-(z)+\sum_{\alpha=2,4}\mathscr{E}_\alpha\exp(ik_\alpha z), \tag{13a}$$

$$f_s(v,z)=F_s(v,z)+\frac{e}{m}\frac{df_{s0}}{dv}\sum_{\alpha=2,4}\frac{\mathscr{E}_\alpha\exp(ik_\alpha z)}{i(k_\alpha v-\omega)+\nu_s}. \tag{13b}$$

In the absence of amplification, we do not separate out the slow charge-density wave from the term $\mathscr{E}_-(z)$.

Solving the kinetic equation for the plasma and beam particles with the same boundary conditions as used in the preceding section, and substituting the corresponding expressions for f_s in (1), we obtain for the determination of $\mathscr{E}_-(z)$ the integral equation

$$\mathscr{E}_-(z)-\int_0^z dz'\mathscr{E}_-(z')K_b(z-z')-\int_0^\infty dz'\mathscr{E}_-(z')[K_p(|z-z'|)$$
$$-K_p(|z+z'|)]+g_-(z)=0,\quad z>0,$$
$$g_-(z)=\sum_{\alpha=2,4}\frac{4\pi e^2\mathscr{E}_\alpha}{i\omega m}\int_0^\infty v\,dv\left[\frac{df_p/dv}{i(k_\alpha v+\omega)-\nu_p}\exp\left(\frac{i\omega-\nu_p}{v}z\right)\right.$$
$$\left.-\sum_{s=p,b}\frac{df_s/dv}{i(k_\alpha v-\omega)+\nu_s}\exp\left(\frac{i\omega-\nu_s}{v}z\right)\right]. \tag{14}$$

To solve (14), we continue $\mathscr{E}_-(z)$ into the region $z<0$ by zero and introduce the auxiliary function $H_-(z)$:

$$H_-(z)=0,\quad z>0,$$
$$H_-(z)=-\int_0^{|z|}\mathscr{E}_-(z')dz'K_b(|z|-z') \tag{15}$$
$$-\int_0^\infty dz'\mathscr{E}_-(z')[K_p(|z-z'|)-K_p(|z+z'|)]+G_-(z),\quad z<0,$$
$$G_-(z)\equiv g_-(|z|)\,\mathrm{sgn}\,z.$$

The Fourier transformation of (15) leads to a boundary value problem for the determination of the Fourier component $e^-(k)$ of the field:

$$e^-(-k)=\frac{D(k)}{D(-k)}e^-(k)+\sum_{\alpha=2,4}\frac{\mathscr{E}_\alpha}{2\pi i}\left\{\frac{1}{k+k_\alpha}+\frac{D(k)}{(k-k_\alpha)D(-k)}\right\}. \tag{16}$$

At $\mathrm{Im}\,k_2<0$, the index κ of this conjugation problem is equal to −2, and the solution to (16) that decreases as we go to infinity takes, according to [13], the form

$$e^-(k)=\frac{1}{2\pi i}X_0^-(k)(k-i\rho)^2\sum_{\alpha=2,4}\mathscr{E}_\alpha\left\{\frac{2k_\alpha}{X_0^-(k_\alpha)}\right.$$
$$\left.\times\frac{1}{(k^2-k_\alpha^2)(k_\alpha-i\rho)^2}-\frac{1}{k-k_\alpha}\frac{1}{X_0^-(k)(k-i\rho)^2}\right\}, \tag{17}$$

if the condition

$$\int_{-\infty}^{+\infty}\frac{k\,dk}{X_0^+(k)(k+i\rho)^2}\sum_{\alpha=2,4}\mathscr{E}_\alpha\left[\frac{1}{k+k_\alpha}+\frac{X_0^+(k)(k+i\rho)^2}{X_0^-(k)(k-i\rho)^2}\frac{1}{k-k_\alpha}\right]=0 \tag{18}$$

is fulfilled. The last relation gives the equation for the amplitude $\mathscr{E}_2$ of the amplified wave in terms of the amplitude $\mathscr{E}_4$ of the incident wave. Their ratio is equal to the coefficient of transformation of the counter plasma wave into the amplified wave

$$T_{24}^{(-)}=\frac{\mathscr{E}_2}{\mathscr{E}_4}=-\frac{k_4}{k_2}\frac{X_0^-(k_2)}{X_0^-(k_4)}\left(\frac{k_2-i\rho}{k_4-i\rho}\right)^2. \tag{19}$$

Let us find the field in the region $z>0$ by performing the inverse Fourier transformation with the aid of (17):

$$\mathscr{E}_-(z)=\frac{\mathscr{E}_4}{2\pi i}\int_{-\infty}^{+\infty}dk\,e^{ikz}\frac{D(-k)}{D(k)}\frac{X_0^+(k)}{X_0^-(k_4)}\left(\frac{k+i\rho}{k_4-i\rho}\right)^2\left[\frac{2k_4}{k^2-k_4^2}-\frac{2k_4}{k^2-k_2^2}\right]. \tag{20}$$

The residues at the points $k=k_1$ and $k=k_3$ of the integrand in (20) yield the coefficients of transformation of the counter plasma wave into a fast charge-density wave and a plasma wave respectively:

$$T_{\alpha4}^{(-)}=\frac{2k_4D(-k_\alpha)}{D'(k_\alpha)}\frac{X_0^+(k_\alpha)}{X_0^-(k_4)}\left(\frac{k_\alpha+i\rho}{k_4-i\rho}\right)^2\left[\frac{1}{k_\alpha^2-k_4^2}-\frac{1}{k_\alpha^2-k_2^2}\right]. \tag{21}$$

In the absence of amplification ($\mathrm{Im}\,k_2>0$), the counter plasma wave $\mathscr{E}_4\exp(ik_4z)$ is transformed in the following manner

$$T_{\alpha4}^{(\pm)}=\frac{2k_4}{k_\alpha^2-k_4^2}\frac{X_{00}^+(k_\alpha)}{X_{00}^-(k_4)}\frac{[K_b(k_\alpha)-K_b(-k_\alpha)]}{D'(k_\alpha)},\quad \alpha=1,2,3, \tag{22}$$

in the bulk of the plasma into a wave propagating along the beam.

The general expressions obtained above for the conversion coefficients $T_{\alpha\beta}^{(\pm)}$ are valid for arbitrary beam densities. In the limiting case of low beam densities ($n_b\ll n_p$), these formulas can be substantially simplified. Introducing the dimensionless frequency

$$\Omega=\frac{\omega-\omega_0}{\mu\omega_0\delta_0}\quad\left(\omega_0=\frac{\omega_p}{\sqrt{1-\mu}},\ \delta_0=\left(\frac{n_b}{2n_p\mu}\right)^{1/2}\right)$$

and the dimensionless corrections to the wave numbers of the natural waves of the system

$$k_\alpha=\frac{\omega}{V_0}(1+\delta_0\Delta_\alpha),\quad \alpha=1,2,3,\qquad k_4=-\frac{\omega}{V_0}(1+\delta_0\Delta_4),$$

we obtain:

$$T_{41}^{(+)}=T_{43}^{(+)}=-\left\{1+\frac{\Delta_2^2}{(\Delta_2-\Delta_3)(\Delta_2-\Delta_1)}-\frac{1}{\Delta_2-\Delta_4}\sum_{\alpha=1,3}\frac{\Delta_\alpha}{\Delta_\alpha^3-2}\right\},$$

$$T_{42}^{(+)}=-\left(1-\frac{2}{\Delta_2^3}\right)\frac{\Delta_2^2}{(\Delta_2-\Delta_3)(\Delta_2-\Delta_1)} \tag{23}$$

$$T_{\alpha4}^{(-)}=\frac{-1}{1-2/\Delta_\alpha^3},\qquad T_{24}^{(-)}=-\frac{\Delta_2^2}{(\Delta_2-\Delta_3)(\Delta_2-\Delta_1)},\quad \alpha=1,3.$$

The dispersion equation (2) then assumes the form

$$\Delta_\alpha^3-\Omega\Delta_\alpha^2+1=0,\quad \alpha=1,2,3;\quad \Delta_4=\Omega. \tag{23a}$$

5. Substituting the analytic expressions obtained above for the components of the $T_{\alpha\beta}^{(\pm)}$ tensors in the determinant (4c), we obtain an equation for the spectrum of the system under consideration. In the general case this equation turns out to be complicated, so that it can apparently be solved only by numerical methods. We therefore consider below the limiting cases in which the analytic expressions for $T_{\alpha\beta}^{(\pm)}$ are so much simplified that the dispersion equation (4c) can be solved explicitly.

Let us first consider the frequency region sufficiently far away from the point ω_0 where the condition of equality of the beam velocity V_0 to the phase velocity of the plasma wave ($\alpha=3$) is fulfilled: $1\gg|\omega-\omega_0|/\omega_0\gg\mu\delta_0$. In this region the general expressions for $T_{\alpha\beta}^{(\pm)}$ assume the form:

$$T_{14}^{(-)}=\frac{1}{2\Omega^{3/2}}\left(1+\frac{2}{\Omega^{3/2}}\right),\qquad T_{24}^{(-)}=-\frac{1}{2\Omega^{3/2}}\left(1-\frac{2}{\Omega^{3/2}}\right),$$

$$T_{34}^{(-)}=-\left(1+\frac{2}{\Omega^3}\right),\quad T_{4\alpha}^{(+)}=-1,\quad \alpha=1,2,3.$$

Substituting these expressions in (4c), we obtain in the case of a relatively thin plasma layer ($\omega_pL\ll\delta_0V_0$) of interest to us the comparatively simple equation for the spectrum:

$$\exp[-2ik_3(\omega)L]=1+f(\theta)R^3,$$
$$k_3^2(\omega)=\frac{\omega^2-\omega_p^2}{\mu V_0^2},\quad \theta(\omega)=k_3L-\frac{\omega_0L}{V_0}=\frac{\omega_0L}{V_0}\frac{\omega-\omega_0}{\mu\omega_0}, \tag{24}$$
$$f(\theta)=\frac{2(1-\cos\theta)-\theta\sin\theta}{\theta^3},\qquad R=\delta_0\frac{\omega_pL}{V_0}<1.$$

In the zeroth approximation in R, we find from this equation the spectrum of the natural oscillations

$$\omega_n=\left[\omega_p^2+\frac{3}{2}\left(\frac{\pi n v_{Tp}}{L}\right)^2\right]^{1/2},\quad n=1,2,3;\quad \left(\frac{\pi n v_T}{\omega_pL}\right)^2\ll1, \tag{25}$$

and allowance for terms of order R^3 yields the increment

$$\omega_n''=\frac{\pi n}{4}\frac{\omega_b^2L}{V_0}f(\theta_n),$$
$$\theta_n=\pi n-\frac{\omega_pL}{V_0},\qquad |\theta_n|\gg\frac{\omega_pL}{V_0}\delta_0. \tag{26}$$

Thus, those oscillations of the plasma layer for which $f(\theta_n)>0$ grow in time with increments proportional to the beam density. The frequencies of the unstable oscillations then lie, according to the condition of applicability of the formulas given above ($|\theta_n|\gg R$), within the confines of the Cerenkov-amplification region in an infinite plasma: $\omega<\omega_b\equiv\omega_0(1+3\mu\delta_0/2^{2/3})$.

Physically, the quantity θ_n is the phase gain of the ($\alpha=3$)-wave field relative to a particle, positive θ_n corresponding to waves whose phase velocities $V_{ph}^{(n)}\equiv\omega_nL/\pi n$ are higher than the beam velocity V_0. As θ_n (the detunings $V_0-V_{ph}^{(n)}$) decrease in the region $|\theta_n|<\pi$, the increment (26) first increases and then decreases, becoming zero as $|\theta|\to0$. But for $|\theta_n|\lesssim R$, Eq. (24) and its solution (26) are not applicable, since the distortion of the field in the plasma by the charge-density waves cannot be neglected in this region of the parameters. Therefore, the resonance case ($V_0\sim V_{ph}^{(n)}$) requires special treatment.

As can be seen from (23), in the vicinity of the point $\omega=\omega_0$, where the phase velocity of the plasma wave is equal to the beam velocity, the effectiveness of the interconversion of the charge-density and plasma waves turns out to be high. Thus, in the zeroth approximation in the parameter Ω, we have:

$$T_{4\alpha}^{(+)}=-1,\qquad T_{\alpha4}^{(-)}=-1/3. \tag{27}$$

It follows from this that for $\Omega=0$ and $\theta_n=0$ Eq. (4c) becomes an identity up to quantities of the order of $R\ll1$. Therefore, for $|\theta_n|\ll1$, we can seek the function $\Omega(\theta)$ in the form of a series in powers of θ. We find up to terms of order θ

$$(\omega-\omega_0)''=\frac{27}{4}\mu\delta_0\omega_0\left(\pi n-\frac{\omega_0L}{V_0}\right),$$
$$(\omega-\omega_0)'=\sqrt{3}(\omega-\omega_0)''. \tag{28}$$

As can be seen from these formulas, in the resonance case the field buildup occurs at parameters for which the beam velocity is higher than the phase velocity of the wave; in the opposite case the beam absorbs the energy of the field, as a result of which the amplitude of the field attenuates exponentially.

Although formulas (28) were obtained under the assumption of small θ, they can nevertheless be used at the limit of the region of applicability ($|\theta|\sim R$ or $\Omega\sim1$)

to estimate the order of magnitude of the increment. By such means we can verify that the maximum of the increment in the system under consideration is, for $\theta_n \sim R$, proportional to the $\frac{2}{3}$ power of the beam density: $\delta_{max} \sim 6\pi n \delta_0 \delta_\infty$. For not too low beam currents, this increment is, in order of magnitude, comparable to the increment of the Cerenkov instability in an infinite plasma ($\delta_\infty \equiv \sqrt{3}\,\delta_0\omega_0\mu/2$).

Analysis of the foregoing results and the conditions for their applicability shows that the exponential growth of the amplitude of the field in the system under consideration is ensured as a result of the accumulation, in the plasma resonator, of the energy successively lost by the beam particles entering the resonator[2)]. For large differences between the beam velocity and the phase velocity of the plasma wave ($|\theta| \gg 1$), the dominant contribution to the positive part of the increment is made by the interconversion, accounted for by the last term on the right-hand side of Eq. (24), of the charge-density and plasma waves at the ends of the plasma resonator. Underlying such a transformation are the direct and inverse effects of the transition radiation of longitudinal waves by the beam particles: the amplification of the longitudinal wave ensures the transition radiation of a coherent succession of bunches (formed by the charge-density wave fields) at the exit end of the resonator, while the feedback, which is necessary for the exponential growth of the amplitude of the field, is an inverted transition-radiation effect (the modulation of the beam by the counter-wave field at the entrance to the plasma). The interference of the fields of the fast and slow charge-density waves in the process decreases the increment, so that it turns out, as a result, to be proportional to the beam density, although each of the products $T^{(-)}_{l4}T^{(+)}_{4l}$ ($l = 1, 2$) is proportional to ω_b. In the resonance region, the effectiveness of the interconversion of the plasma and beam waves is, as can be seen from (27), enhanced, while the interference effects are so weakened that the contributions of both longitudinal waves to the right-hand side of (4c) coincide and turn out to be equal to the contribution of the plasma waves. In consequence, the increment increases to a value comparable to the increment $\omega''_\infty \equiv \sqrt{3}\,\delta_0\omega_p\mu/2$ for the infinite plasma.

The results presented above are valid for sharp plasma-layer boundaries; in the case of a finite density gradient it is necessary to take into account the weakening of the intensity of the transition radiation [18], a weakening that leads to a decrease in the increment.

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1)The transformation of charge density waves in a nonequilibrium plasma was first investigated in [6]; plasma-wave transformation in an equilibrium plasma had been studied earlier (see, for example, the reviews [7-9]).

2)A quasilinear theory of beam relaxation in a plasma half-space with allowance for the effects of the energy storage has been developed by Vedenov [14] for the steady-state regime; the dynamics of the establishment of this regime has been investigated by Faĭnberg and Shapiro [15]. The growth of the amplitude of the normal oscillations in a resonator with a short transit time is considered in [16] in the specified spatial field pattern approximation; the corresponding experiment is described in [17].

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