# Theory of scattering of electromagnetic waves in a degenerate electron liquid

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The Rayleigh scattering of electromagnetic waves by density and current fluctuations in a degenerate electron liquid is investigated. Detailed formulas are obtained for the angular and frequency distributions of the intensity of the scattered radiation. Rough numerical estimates are presented.

It is well known that the interaction between the conduction electrons in a metal is not small compared with their kinetic energy. In a number of problems, this leads to the necessity of considering the conduction electrons not as an ideal gas of Fermi quasi-particles but as a degenerate electron liquid, describable by means of Landau's Fermi-liquid theory[1]. One of these problems is the Rayleigh scattering of electromagnetic waves by conduction electrons, inasmuch as the Fermi-liquid interaction substantially alters the spectrum of the characteristic oscillations of the system.

In the present article, a theoretical investigation is performed of the Rayleigh scattering of electromagnetic waves in an electron Fermi liquid, with the purpose of elucidating the angular and frequency distributions of the intensity of the scattered radiation.

The appearance of scattered waves is due to the interaction of the incident wave with fluctuation oscillations of the electron liquid, and the spectrum of the scattered radiation is determined by the fluctuation spectrum of the system. The calculation of the scattering coefficient can therefore be divided conveniently into two sections: 1) the determination of the dependence of the scattering coefficient on the correlators of the various physical quantities, which reduces to solving the problem of the interaction between a given incident wave and given fluctuations; 2) the calculation of the correlators of various physical quantities in an equilibrium electron liquid.

We shall be interested in the scattering of waves with high frequencies (ultraviolet and higher), i.e., in the case when $\hbar\omega$ is large compared with the binding energy of electrons in the atom and with their mutual interaction energy, or, in the language of the classical theory, when the frequency $\omega$ of the incident wave is large compared with the frequencies of the proper motion of the electrons in the system. Therefore, in treating the interaction between the incident wave and specified fluctuations, i.e., in Sec. 1, we can regard the electrons as a Fermi gas, neglecting both their interaction with the atomic nuclei and the Fermi-liquid interaction. This situation is completely equivalent to that which is found in the calculation of the high-frequency dielectric permittivity of a substance[2]. With regard to the Fermi-liquid interaction, it may be noted in addition that, since the Fermi-liquid corrections to the particle spectrum develop in a time $t \sim a/v_0$ ($a$ is the mean spacing between the atoms, and $v_0$ is the velocity of the electrons at the Fermi surface), for $\omega > v_0/a$ there is no point in taking into account the Fermi-liquid interaction in this part of the calculation.

As regards the second part of the calculation, i.e., the calculation of the spectral distributions of the fluctuations in an equilibrium electron liquid, it is now absolutely necessary to take complete and rigorous account of the interaction, by virtue of its determining role. Here it is necessary to retain all the terms associated with the Fermi-liquid interaction, inasmuch as they affect the spectrum of the collective oscillations of the system in an essential way and thereby influence the time behavior of the fluctuations, which, in its turn, determines the important, though relatively small, change of frequency in the Rayleigh scattering.

1. To find the dependence of the scattering coefficient on the correlators of the various physical quantities, we shall make use of a method usually used for a plasma[3]. In the propagation of an electromagnetic wave in a degenerate electron liquid, the total electric field $\mathbf{E}(\mathbf{r}, t)$ can be represented in the form

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}^0(\mathbf{r}, t) + \delta\mathbf{E}(\mathbf{r}, t) + \mathbf{E}'(\mathbf{r}, t),$$

where $\mathbf{E}^0(\mathbf{r}, t)$ is the field of the incident wave, $\mathbf{E}'(\mathbf{r}, t)$ is the field of the scattered wave, and $\delta\mathbf{E}(\mathbf{r}, t)$ is the fluctuation of the electric field in the degenerate electron liquid; correspondingly, the particle distribution function can be written in the form

$$n = n_0 + n_1,$$

where $n_0$ is the equilibrium Fermi function, and

$$n_1 = n_1^0 + \delta n + n_1',$$

where $n_1^0$ and $n_1'$ are the small corrections to the equilibrium function associated with the incident and scattered waves, and $\delta n$ is the fluctuation of the distribution function. Because of the smallness of the nonlinear interaction between different oscillations, we assume the incident wave to be given (here, the fluctuations are also assumed to be known—their calculation will be given below) and choose it in the form of a monochromatic plane wave:

$$\mathbf{E}^0(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{kr} - \omega t)}, \tag{1}$$

in this case, $n_1^0$ satisfies the linearized kinetic equation

$$\frac{\partial n_1^0}{\partial t} + \mathbf{v}\frac{\partial n_1^0}{\partial \mathbf{r}} + e\mathbf{E}^0\frac{\partial n_0}{\partial \mathbf{p}} = 0; \tag{2}$$

we consider the case when the collision integral can be neglected ($\omega\tau \gg 1$, where $\tau$ is the time between collisions).

As regards the scattered wave, since it appears as a result of the interaction of the incident wave with the fluctuations all the nonlinear terms associated with this interaction should be retained in the equation for $n_1'$. Thus, $n_1'$ satisfies the equation

$$\frac{\partial n_1'}{\partial t} + \mathbf{v}\frac{\partial n_1'}{\partial \mathbf{r}} + e\mathbf{E}'\frac{\partial n_0}{\partial \mathbf{p}} + e\left(\mathbf{E}^0 + \frac{1}{c}[\mathbf{v}\mathbf{H}^0]\right)\frac{\partial \delta n}{\partial \mathbf{p}} + e\left(\delta\mathbf{E} + \frac{1}{c}[\mathbf{v}\delta\mathbf{H}]\right)\frac{\partial n_1^0}{\partial \mathbf{p}} = 0, \tag{3*}$$

where $H^0$ is the magnetic field of the incident wave, and $\delta H$ is the fluctuation magnetic field. In accordance with what has been stated above, Eqs. (2) and (3) do not contain Fermi-liquid terms.

Using (1) and (2), we find from Eq. (3) an expression for the Fourier component $(n_1')_{\mathbf{k}',\omega'}$. Calculating next the current associated with $n_1'$, with the aid of Maxwell's equations we find the field $\mathbf{E}'$ of the scattered wave. For the case of scattering of a high-frequency transverse wave, taking into account that the calculation of the fluctuations is performed with allowance for the Fermi-liquid interaction, we obtain

$$\mathbf{E}'_{\mathbf{k}',\omega'}=\frac{4\pi i\omega'}{k'^2c^2-\varepsilon(\mathbf{k}',\omega')\omega'^2}\mathbf{J}_{\perp\mathbf{k}',\omega'},\tag{4}$$

where $\epsilon(\mathbf{k}',\omega')$ is the dielectric permittivity corresponding to the linearized kinetic equation of the type (2), and the quantity $\mathbf{J}$, which is due to the nonlinear terms in Eq. (3), is the current responsible for the appearance of the scattered waves:

$$\begin{aligned}\mathbf{J}_{\mathbf{k}',\omega'}=\frac{ie}{m\omega'}\Big\{&e\delta\rho_{\mathbf{q},\Delta\omega}\left[\mathbf{E}_0-\frac{\Omega^2c^2}{\omega\omega'}\frac{(\mathbf{k}'\mathbf{q})\mathbf{E}_0+\mathbf{q}(\mathbf{k}'\mathbf{E}_0)}{(\Delta\omega)^2-q^2c^2}\right]\\&+\frac{1}{\omega}(\mathbf{k}\delta\mathbf{i}_{\mathbf{q},\Delta\omega})\mathbf{E}_0-\frac{1}{\omega'}(\mathbf{k}'\delta\mathbf{i}_{\mathbf{q},\Delta\omega})\mathbf{E}_0-\frac{1}{\omega}(\mathbf{E}_0\delta\mathbf{i}_{\mathbf{q},\Delta\omega})\mathbf{k}-\frac{1}{\omega'}(\mathbf{k}'\mathbf{E}_0)\delta\mathbf{i}_{\mathbf{q},\Delta\omega}\\&+\frac{1}{\omega}\frac{\Omega^2}{(\Delta\omega)^2-q^2c^2}[(\mathbf{E}_0\delta\mathbf{j}_{\mathbf{q},\Delta\omega})\mathbf{k}'+(\mathbf{q}\delta\mathbf{j}_{\mathbf{q},\Delta\omega})\mathbf{E}_0]\Big\}.\end{aligned}\tag{5}$$

Here $m$ is the electron mass, $\Omega$ is the frequency of the Langmuir oscillations of the electron gas, $\delta\rho$ is the particle-density fluctuation,

$$\delta\mathbf{i}_{\mathbf{q},\Delta\omega}=\frac{\Omega^2}{(\Delta\omega)^2-q^2c^2}\delta\mathbf{j}_{\mathbf{q},\Delta\omega}-\frac{e}{m}\int\mathbf{p}(\delta n(\mathbf{p}))_{\mathbf{q},\Delta\omega}d\tau\tag{6}$$

and $\delta\mathbf{j}$ is the current-density fluctuation

$$\delta\mathbf{j}=e\int\frac{\partial\varepsilon_0}{\partial\mathbf{p}}\left\{\delta n(\mathbf{p},\mathbf{r},t)-\frac{\partial n_0}{\partial\varepsilon}\int f(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p}',\mathbf{r},t)d\tau'\right\}d\tau,\tag{7}$$

$\epsilon_0(\mathbf{p})$ is the equilibrium particle energy, and $f(\mathbf{p},\mathbf{p}')$ is the correlation function introduced in Landau's theory of the Fermi liquid[1]. The index $\perp$ denotes the projection of the vector on to a plane perpendicular to $\mathbf{k}'$, $\Delta\omega=\omega'-\omega$, $\mathbf{q}=\mathbf{k}'-\mathbf{k}$, and $d\tau=2d\mathbf{p}/(2\pi\hbar)^3$. Since we are considering scattering by the collective oscillations of the system, the change of wave-vector of the wave on scattering satisfies the condition $1/q\gg a$.

The total scattered intensity $I$ is determined by averaging the well-known expression for the increase of the energy of the electromagnetic field per unit time[4]:

$$I=-\frac{1}{2}\operatorname{Re}\int d\mathbf{r}\langle\mathbf{E}'(\mathbf{r},t)\mathbf{J}^*(\mathbf{r},t)\rangle.$$

Using (4), we obtain

$$I=\frac{1}{(2\pi)^4}\operatorname{Im}\int d\mathbf{k}'d\omega'd\omega_1'\exp\{-i(\omega'-\omega_1')t\}\times\omega'\langle\mathbf{J}_{\perp\mathbf{k}',\omega'}\mathbf{J}^*_{\perp\mathbf{k}',\omega_1'}\rangle/(k'^2c^2-\varepsilon'\omega'^2).$$

Here and below, $\epsilon'\equiv\epsilon(\mathbf{k}',\omega')$ and $\epsilon\equiv\epsilon(\mathbf{k},\omega)$.

Since, according to the general theory of fluctuations,

$$\langle\mathbf{J}_{\perp\mathbf{k}',\omega'}\mathbf{J}_{\perp\mathbf{k}',\omega_1'}\rangle\infty\delta(\omega'-\omega_1'),$$

the integrand is real and, consequently, the imaginary part of the integral can only be due to the poles of the integrand. We therefore replace $(k'^2c^2-\epsilon'\omega'^2)^{-1}$ by $i\pi\delta(k'^2c^2-\epsilon'\omega'^2)$ and, after integration over the modulus of the vector $\mathbf{k}'$, we obtain

$$I=\frac{1}{32\pi^3c^3}\int d\Omega'd\omega'd\omega_1'\sqrt{\varepsilon'}\,\omega'^2\langle\mathbf{J}_{\perp\mathbf{k}',\omega'}\mathbf{J}^*_{\perp\mathbf{k}',\omega_1'}\rangle;\tag{8}$$

here, the magnitude of the wave vector $\mathbf{k}'$ and the frequency $\omega'$ of the scattered wave are connected by the relation $k'=\sqrt{\epsilon'}\omega'/c$, and $d\Omega'$ is the element of solid angle in the direction of $\mathbf{k}'$.

We shall transform the correlator $\langle\mathbf{J}_{\perp\mathbf{k}',\omega'}\mathbf{J}^*_{\perp\mathbf{k}',\omega_1'}\rangle$ by means of formula (5). The calculation of the fluctuations given below can be carried through to the end for the case when the electron Fermi surface is isotropic. (In the general case, the problem is substantially more complicated and becomes equivalent to the determination of the dielectric permittivity tensor with the anisotropy, the electron-electron interaction and the spatial dispersion taken into account.) In this approximation (which can be assumed to be legitimate for the alkali metals, aluminum, lead, and also for liquid metals), the function $f(\mathbf{p},\mathbf{p}')$ depends only on the angle $\chi$ between the vectors $\mathbf{p}$ and $\mathbf{p}'$ at the Fermi surface and can be expanded in a series in Legendre polynomials:

$$F(\chi)=f(\chi)\left(\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}=\sum_k F_kP_k(\cos\chi);$$

The expression (7) for the current-density fluctuation then takes the form

$$\delta\mathbf{j}=\frac{e}{m^*}\left(1+\frac{F_1}{3}\right)\int\mathbf{p}\delta n(\mathbf{p},\mathbf{r},t)d\tau,\tag{9}$$

where $m^*$ is the total effective mass of a Fermi quasiparticle, and, according to formula (6),

$$\delta\mathbf{i}_{\mathbf{q},\Delta\omega}=\delta\mathbf{j}_{\mathbf{q},\Delta\omega}\left[\frac{\Omega^2}{(\Delta\omega)^2-q^2c^2}-\frac{m^*}{m(1+F_1/3)}\right].\tag{10}$$

Substituting (10) into (5), we can express $\langle\mathbf{J}_{\perp\mathbf{k}',\omega'}\mathbf{J}^*_{\perp\mathbf{k}',\omega_1'}\rangle$ in terms of the correlators of the charge and current densities

$$\langle\delta\rho_{\mathbf{q},\Delta\omega}\delta\rho^*_{\mathbf{q},\Delta\omega_1}\rangle,\ \langle\delta j_{i\mathbf{q},\Delta\omega}\delta j^*_{k\mathbf{q},\Delta\omega_1}\rangle,\ \langle\delta\rho_{\mathbf{q},\Delta\omega}\delta j^*_{i\mathbf{q},\Delta\omega_1}\rangle\quad(\Delta\omega_1=\omega_1'-\omega).$$

Since the scattering coefficient is additive with respect to the different types of scattering, it is convenient, in view of the cumbersome nature of the general formulas, to consider the scattering by the transverse and longitudinal oscillations separately.

We consider first the case when the fluctuation oscillations at which the scattering occurs are transverse:

$$\delta\rho_{\mathbf{q},\Delta\omega}=0,\quad\delta\mathbf{j}_{\mathbf{q},\Delta\omega}\perp\mathbf{q}.$$

Then

$$\begin{aligned}\mathbf{J}_{\perp\mathbf{k}',\omega'}=\frac{ie}{m\omega'^2}\left[\frac{\Omega^2}{(\Delta\omega)^2-q^2c^2}-\frac{m^*}{m(1+F_1/3)}\right]\Big\{&\frac{\Delta\omega}{\omega}(\mathbf{k}'\delta\mathbf{j}_{\mathbf{q},\Delta\omega})\mathbf{E}_0\\&-\frac{\omega'}{\omega}(\mathbf{E}_0\delta\mathbf{j}_{\mathbf{q},\Delta\omega})\mathbf{k}_\perp-(\mathbf{k}'\mathbf{E}_0)\delta\mathbf{j}_{\perp\mathbf{q},\Delta\omega}\Big\}\end{aligned}\tag{11}$$

and the correlator $\langle\mathbf{J}_{\perp\mathbf{k}',\omega'}\mathbf{J}^*_{\perp\mathbf{k}',\omega_1'}\rangle$ can be expressed in terms of the spectral distribution $\langle\delta j_t^2\rangle_{\mathbf{q},\Delta\omega}$ of the correlation function of the fluctuations of the transverse-current density in the following way:

$$\begin{aligned}\langle\mathbf{J}_{\perp\mathbf{k}',\omega'}\mathbf{J}^*_{\perp\mathbf{k}',\omega_1'}\rangle&=\pi V\delta(\omega'-\omega_1')\left(\frac{em^*}{m^2(1+F_1/3)\omega'^2}\right)^2\\&\times\left[1-\frac{\omega_0^2}{(\Delta\omega)^2-q^2c^2}\right]^2\Big\{\frac{(\Delta\omega)^2}{\omega^2}E_{0\perp}^2(k'^2-k_l'^2)+\frac{2\Delta\omega\omega'}{\omega^2}k_lE_{0l}(\mathbf{E}_0\mathbf{k}_\perp)\\&+\frac{2\Delta\omega}{\omega}k_l'E_{0\perp l}(\mathbf{k}'\mathbf{E}_0)+\frac{\omega'^2}{\omega^2}k_\perp^2(E_0^2-E_{0l}^2)+\frac{2\omega'}{\omega}[(\mathbf{E}_0\mathbf{k}_\perp)\\&-E_{0l}k_{\perp l}](\mathbf{k}'\mathbf{E}_0)+(\mathbf{k}'\mathbf{E}_0)^2(1+\cos^2(\widehat{\mathbf{k}'\mathbf{q}}))\Big\}\langle\delta j_t^2\rangle_{\mathbf{q},\Delta\omega}\equiv2\pi V\delta(\omega'-\omega_1')\langle J_\perp^2\rangle_{\mathbf{k}',\omega'},\end{aligned}\tag{12}$$

where

$$\omega_0^2=\Omega^2\frac{m(1+F_1/3)}{m^*},\tag{13}$$

$V$ is the total volume, and the index $l$ denotes the projection of a vector along the direction of $\mathbf{q}$.

Substituting (12) into (8) and performing the integration over $\omega_1'$, we find the scattering intensity in the frequency interval $d\omega'$ into the solid-angle element $d\Omega'$

$$dI=\frac{V}{16\pi^2c^3}\sqrt{\varepsilon'}\,\omega'^2\langle J_\perp^2\rangle_{\mathbf{k}',\omega'}d\omega'd\Omega'. \qquad (14)$$

In the case of scattering of an unpolarized wave, the expression (14) should be averaged over the different orientations of the vector $\mathbf{E}_0$. By calculating the different projections occurring in the formula (12), as a result of this averaging we obtain

$$dI=\frac{VE_0^2}{32\pi^2c^5}(\varepsilon')^{1/2}\left(\frac{em^*}{m^2(1+F_1/3)}\right)^2\left[1-\frac{\omega_0^2}{(\Delta\omega)^2-q^2c^2}\right]^2$$
$$\times\left\{\left(\frac{\Delta\omega}{\omega}\right)^2\sin(\widehat{\mathbf{k}'\mathbf{q}})\sin(\widehat{\mathbf{k}'\mathbf{q}}-\theta)\cos\theta+\frac{\Delta\omega}{\omega}\sqrt{\frac{\varepsilon}{\varepsilon'}}\sin^2\theta\cos\theta \qquad (15)\right.$$
$$\left.+\sin^2\theta\left(1+\frac{\varepsilon}{\varepsilon'}-\sqrt{\frac{\varepsilon}{\varepsilon'}}\cos\theta\right)\right\}\langle\delta j_t^2\rangle_{\mathbf{q},\Delta\omega}d\omega'd\Omega'$$

($\theta$ is the scattering angle). The formula (15) is valid for any frequency change $\Delta\omega$. According to the calculation performed below, $\langle\delta j_t^2\rangle_{\mathbf{q},\Delta\omega}$ is non-zero for $(\Delta\omega)^2=\omega_0^2+q^2c^2$ and for $\Delta\omega\lesssim qv_F$. In accordance with the requirements of the laws of energy and momentum conservation in the scattering, in the first case the expression in the square brackets in formula (15) vanishes, i.e., there is no scattering by high-frequency transverse oscillations, and in the second case scattering with only a small change of frequency is possible. Thus, $\Delta\omega/\omega\ll 1$. We shall perform a comparative estimate of the three terms in the curly brackets in formula (15). Using the relations

$$\sin(\widehat{\mathbf{k}'\mathbf{q}}-\theta)\cos\theta=\sin(\widehat{\mathbf{k}'\mathbf{q}})-\sin\theta\cos(\widehat{\mathbf{k}'\mathbf{q}}-\theta)$$

and

$$q\sin(\widehat{\mathbf{k}'\mathbf{q}})=k\sin\theta,$$

we write the first term in the form

$$\left(\frac{\Delta\omega}{\omega}\right)^2\frac{k^2}{q^2}\sin^2\theta-\left(\frac{\Delta\omega}{\omega}\right)^2\frac{k}{q}\sin^2\theta\cos(\widehat{\mathbf{k}'\mathbf{q}}-\theta).$$

Taking into account that

$$\frac{\Delta\omega}{\omega}\frac{k}{q}\lesssim\frac{kv_F}{\omega}\ll 1,$$

and

$$\left(1+\frac{\varepsilon}{\varepsilon'}-\sqrt{\frac{\varepsilon}{\varepsilon'}}\cos\theta\right)\sim 1,$$

we see that the first term is much less than the third at all scattering angles.

Furthermore, the ratio of the second term to the third is of the order of $\Delta\omega\cos\theta/\omega$. Thus, at all scattering angles, the first and second terms can be discarded. Replacing $\epsilon/\epsilon'$ by unity in the third term and dividing dI by the scattering volume V and the energy-flux density of the incident wave

$$S_0=\frac{c}{8\pi}\sqrt{\varepsilon}E_0^2,$$

we finally obtain the differential scattering coefficient for scattering by the transverse fluctuations:

$$dh_t=\frac{\varepsilon}{4\pi}\left(\frac{em^*}{m^2c^3(1+F_1/3)}\right)^2\left[1-\frac{\omega_0^2}{(\Delta\omega)^2-q^2c^2}\right]^2$$
$$\times\sin^2\theta\left(1+2\sin^2\frac{\theta}{2}\right)\langle\delta j_t^2\rangle_{\mathbf{q},\Delta\omega}d\omega'\,d\Omega'. \qquad (16)$$

In the case of scattering by the longitudinal fluctuations, when $\delta\rho_{\mathbf{q},\Delta\omega}\neq 0$ and the current-density fluctuations are expressed in terms of the charge-density fluctuations

$$\delta\mathbf{j}_{\mathbf{q},\Delta\omega}=e\frac{\Delta\omega}{q^2}\mathbf{q}\delta\rho_{\mathbf{q},\Delta\omega},$$

an analogous treatment gives for the differential scattering coefficient the well-known[2] formula:

$$dh_l=\frac{1}{4\pi}\left(\frac{e^2}{mc^2}\right)^2(1+\cos^2\theta)\langle\delta\rho^2\rangle_{\mathbf{q},\Delta\omega}d\omega'd\Omega', \qquad (17)$$

where $\langle\delta\rho^2\rangle_{\mathbf{q},\Delta\omega}$ is the spectral distribution of the correlation function of the particle-density fluctuations.

2. The calculation of the spectral distributions of the fluctuations is performed by the method of "random forces," developed by Rytov[5] and Landau and Lifshitz[6] for the calculation of fluctuations in electrodynamics and hydrodynamics, and applied by Abrikosov and Khalatnikov[7] to the kinetic equation.

As usual, the electron Fermi liquid is described by a closed self-consistent system of equations, consisting of the kinetic equation and Maxwell's equations[8]. We introduce an additional "random force" $y(\mathbf{p},\mathbf{r},t)$ into the right-hand side of the kinetic equation, so that the linearized kinetic equation for the fluctuation of the quasi-particle distribution function in an equilibrium electron liquid is written in the form

$$\frac{\partial\delta n}{\partial t}+\frac{\partial\delta n}{\partial\mathbf{r}}\frac{\partial\varepsilon_0}{\partial\mathbf{p}}-\frac{\partial n_0}{\partial\mathbf{p}}\int f(\mathbf{p},\mathbf{p}')\frac{\partial}{\partial\mathbf{r}}\delta n(\mathbf{p}',\mathbf{r},t)d\tau'$$
$$+e\delta\mathbf{E}\frac{\partial n_0}{\partial\mathbf{p}}=-\frac{\delta n}{\tau}+y(\mathbf{p},\mathbf{r},t), \qquad (18)$$

while the Maxwell equations remain unchanged. The term $-\delta n/\tau$ in Eq. (18) schematically describes the collisions of the particles. In the case $\Delta\omega\tau\gg 1$, which is the only one we shall be interested in, the collision integral plays only an auxiliary role in the calculations and its exact form is unimportant, since in the final results we shall eliminate it by taking the limit $\tau\to\infty$.

By means of Maxwell's equations, the fluctuation $\delta\mathbf{E}$ of the electric field is expressed in terms of the charge- and current-density fluctuations, which are determined in turn by the fluctuations of the distribution function. For the Fourier component of the electric-field fluctuation, we have

$$\delta\mathbf{E}_{\mathbf{q},\Delta\omega}=\delta\mathbf{E}_{l\mathbf{q},\Delta\omega}+\delta\mathbf{E}_{t\mathbf{q},\Delta\omega};$$
$$\delta\mathbf{E}_{l\mathbf{q},\Delta\omega}=-\frac{4\pi ie}{q^2}\mathbf{q}\delta\rho_{\mathbf{q},\Delta\omega}=-\frac{4\pi ie}{q^2}\mathbf{q}\int\delta n_{\mathbf{q},\Delta\omega}(\mathbf{p})d\tau,$$
$$\delta\mathbf{E}_{t\mathbf{q},\Delta\omega}=-\frac{4\pi i\Delta\omega}{(\Delta\omega)^2-q^2c^2}\delta\mathbf{j}_{t\mathbf{q},\Delta\omega} \qquad (19)$$
$$=-\frac{4\pi ie\Delta\omega}{(\Delta\omega)^2-q^2c^2}\left(1+\frac{F_1}{3}\right)\int\left(\frac{\partial\varepsilon_0}{\partial\mathbf{p}}\right)_t\delta n_{\mathbf{q},\Delta\omega}(\mathbf{p})d\tau.$$

$\delta\mathbf{E}_{l\mathbf{q},\Delta\omega}$ and $\delta\mathbf{E}_{t\mathbf{q},\Delta\omega}$ are respectively the longitudinal and transverse components of the field $\delta\mathbf{E}_{\mathbf{q},\Delta\omega}$; $(\partial\epsilon_0/\partial\mathbf{p})_t$ is the vector projection of the vector $\partial\epsilon_0/\partial\mathbf{p}$ on to a plane perpendicular to $\mathbf{q}$.

Fluctuations of the distribution function and of the "random force" occur only in the region of the Fermi surface and can be represented in the form

$$\delta n(\mathbf{p})=\nu(\vartheta,\varphi)\delta(\varepsilon_0-\mu),\quad y(\mathbf{p})=y^{\varepsilon_0}(\vartheta,\varphi)\delta(\varepsilon_0-\mu), \qquad (20)$$

where $\vartheta$ and $\varphi$ are the polar angles of the vector $\mathbf{p}$, and $\mu$ is the chemical potential.

According to the general theory of fluctuations, by following the same path as in [7] it is not difficult to convince oneself that the expression for the correlator of the "random forces" in a charged Fermi liquid has the same form as in a neutral Fermi liquid:

$$\langle y(\mathbf{p},\mathbf{r},t)\,y(\mathbf{p}',\mathbf{r}',t')\rangle=\frac{2T}{\tau}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\frac{d\varepsilon_0}{d\tau}\delta(\varepsilon_0-\mu)\delta(\varepsilon_0'-\mu)\times\left\{2\delta(\cos\chi-1)-\sum_{n=0}^{\infty}\frac{F_nP_n(\cos\chi)}{1+F_n/(2n+1)}\right\};\tag{21}$$

here and below, the temperature T is defined in energy units.

By means of the kinetic equation (18), we now express the fluctuations of the distribution function in terms of the fluctuations of the "random forces." Inasmuch as the case of an arbitrary function f is rather complicated, we confine ourselves to the case

$$F(\chi)=F_1P_1(\cos\chi)=F_1\cos\chi.$$

This is the simplest form of the function $F(\chi)$ for which the propagation of transverse zero sound is in principle possible.

Going over to Fourier components in Eq. (18), by using formulas (19) and (20) and choosing the polar axis along the direction of $\mathbf{q}$, we obtain

$$\int \nu_{\mathbf{q},\Delta\omega}(\vartheta,\varphi)\frac{d\Omega}{4\pi}=\left\{\int\frac{y^{\varepsilon_0=\mu}_{\mathbf{q},\Delta\omega}(\vartheta,\varphi)\,d\Omega/4\pi}{-i\Delta\omega+iqv_0\cos\vartheta+1/\tau}-\frac{F_1}{1+F_1/3}I_2(q,\Delta\omega)\int y^{\varepsilon_0=\mu}_{\mathbf{q},\Delta\omega}(\vartheta,\varphi)\frac{d\Omega}{4\pi}\right\}\left\{1+\frac{F_1}{1+F_1/3}\left(i\Delta\omega-\frac{1}{\tau}\right)I_2(q,\Delta\omega)+\frac{4\pi ie^2v_0}{q}\left(\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}I_1(q,\Delta\omega)\right\}^{-1},\tag{22}$$

$$\int \nu_{\mathbf{q},\Delta\omega}(\vartheta,\varphi)\sin\vartheta\cos\varphi\frac{d\Omega}{4\pi}=\int\frac{y^{\varepsilon_0=\mu}_{\mathbf{q},\Delta\omega}(\vartheta,\varphi)\sin\vartheta\cos\varphi}{-i\Delta\omega+iqv_0\cos\vartheta+1/\tau}\frac{d\Omega}{4\pi}\left\{1+iqv_0\frac{F_1}{2}(I_1(q,\Delta\omega)-I_3(q,\Delta\omega))+\frac{2\pi ie^2v_0^2\Delta\omega}{(\Delta\omega)^2-q^2c^2}\left(\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}\left(1+\frac{F_1}{3}\right)\times(I_0(q,\Delta\omega)-I_2(q,\Delta\omega))\right\}^{-1},\tag{23}$$

where $\mathbf{v}_0=(\partial\epsilon_0/\partial\mathbf{p})_{\epsilon_0=\mu}$, and

$$I_n(q,\Delta\omega)=\frac{1}{2}\int_{-1}^{1}\frac{\mu^n\,d\mu}{-i\Delta\omega+iqv_0\mu+1/\tau}.$$

These two integrals ((22) and (23)) give the possibility of constructing the correlators in which we are interested. The first integral is proportional to the density fluctuation, and the density correlator is

$$\langle\delta\rho_{\mathbf{q},\Delta\omega}\,\delta\rho^{\bullet}_{\mathbf{q}_1,\Delta\omega_1}\rangle=\left(\frac{d\tau}{d\varepsilon_0}\right)^2_{\varepsilon_0=\mu}\left\langle\int\nu_{\mathbf{q},\Delta\omega}(\vartheta,\varphi)\frac{d\Omega}{4\pi}\int\nu^{\bullet}_{\mathbf{q}_1,\Delta\omega_1}(\vartheta',\varphi')\frac{d\Omega'}{4\pi}\right\rangle.\tag{24}$$

The second integral, according to formula (9), is proportional to one of the components of the transverse-current fluctuation, and, by virtue of the isotropic distribution of the transverse-current fluctuations, the transverse-current correlator is

$$\langle\delta j_{t\mathbf{q},\Delta\omega}\,\delta j^{\bullet}_{t\mathbf{q}_1,\Delta\omega_1}\rangle=2e^2v_0^2\left(\frac{d\tau}{d\varepsilon_0}\right)^2_{\varepsilon_0=\mu}\left(1+\frac{F_1}{3}\right)^2\times\left\langle\int\nu_{\mathbf{q},\Delta\omega}(\vartheta,\varphi)\sin\vartheta\cos\varphi\frac{d\Omega}{4\pi}\int\nu^{\bullet}_{\mathbf{q}_1,\Delta\omega_1}(\vartheta',\varphi')\sin\vartheta'\cos\varphi'\frac{d\Omega'}{4\pi}\right\rangle.\tag{25}$$

Using formulas (23) and (21) and taking into account that

$$\delta(\cos\chi-1)=2\pi\delta(\varphi-\varphi')\delta(\cos\vartheta-\cos\vartheta'),$$
$$\cos\chi=\cos\vartheta\cos\vartheta'+\sin\vartheta\sin\vartheta'\cos(\varphi-\varphi'),$$

we find from formula (25)

$$\langle\delta j_{t\mathbf{q},\Delta\omega}\,\delta j^{\bullet}_{t\mathbf{q}_1,\Delta\omega_1}\rangle=2\pi e^2v_0^2\frac{T}{\tau}\left(\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}V\left(1+\frac{F_1}{3}\right)^2\delta(\Delta\omega-\Delta\omega_1)\times\delta_{\mathbf{q},\mathbf{q}_1}\left\{\int_{-1}^{1}\frac{(1-\mu^2)\,d\mu}{|-\Delta\omega+qv_0\mu+i/\tau|^2}-\frac{F_1}{1+F_1/3}|I_0-I_2|^2\right\}\times\left|1+iqv_0\frac{F_1}{2}(I_1-I_3)+\frac{2\pi ie^2v_0^2\Delta\omega}{(\Delta\omega)^2-q^2c^2}\left(\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}\left(1+\frac{F_1}{3}\right)(I_0-I_2)\right|^{-2}.\tag{26}$$

We must obtain the limiting value of this expression as $\tau\to\infty$.

The integrals $I_n$ are easily calculated. The zeros of the denominator of expression (26) as $\tau\to\infty$ coincide with the roots of the dispersion equation for the transverse oscillations of a degenerate electron liquid[9]. It is convenient to consider two cases:

1. $|\Delta\omega|<qv_0$.

In this case, the denominator does not vanish. As regards the numerator, the second term in the curly brackets is finite when $\tau\to\infty$, and after multiplication by $1/\tau$ gives zero; therefore, a contribution to the numerator is given only by the first term, which is easily calculated:

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_{-1}^{1}\frac{(1-\mu^2)\,d\mu}{(qv_0\mu-\Delta\omega)^2+\tau^{-2}}=\pi\int_{-1}^{1}(1-\mu^2)\delta(qv_0\mu-\Delta\omega)\,d\mu=\frac{\pi}{qv_0}\left[1-\left(\frac{\Delta\omega}{qv_0}\right)^2\right].$$

As a result, we obtain

$$\langle\delta j_{t\mathbf{q},\Delta\omega}\,\delta j^{\bullet}_{t\mathbf{q}_1,\Delta\omega_1}\rangle=2\pi^2e^2v_0^2T\left(\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}V\left(1+\frac{F_1}{3}\right)^2\delta(\Delta\omega-\Delta\omega_1)\times\delta_{\mathbf{q},\mathbf{q}_1}\frac{s}{\Delta\omega}(1-s^2)\left\{\left[1+\frac{F_1}{3}-\frac{s^2}{2}\left(F_1-\frac{3\omega_0^2}{q^2c^2}\right)\left(1+\frac{1-s^2}{2s}\ln\frac{1+s}{1-s}\right)\right]^2+\left[\frac{\pi s}{4}(1-s^2)\left(F_1-\frac{3\omega_0^2}{q^2c^2}\right)\right]^2\right\}^{-1},$$

where $s\equiv\Delta\omega/qv_0$.

2. $|\Delta\omega|>qv_0$.

In this case, all the integrals in the curly brackets in the numerator of the expression (26) are finite and the whole numerator tends to zero like $1/\tau$. Therefore, the correlator (26) can be nonzero only at those points at which the denominator also tends to zero, i.e., according to what has been said above, when

$$\Delta\omega=\pm s_0qv_0,$$

where $s_0$ satisfies the equation

$$(s_0^2-1)\left(\frac{s_0}{2}\ln\frac{s_0+1}{s_0-1}-1\right)=\frac{1}{3}-\frac{2}{F_1},\tag{27}$$

(transverse zero sound[9]), and when

$$\Delta\omega=\pm\sqrt{\omega_0^2+c^2q^2}$$

(a transverse high-frequency wave).

Considering the correlator (26) near these solutions of the dispersion equation and using the relation

$$\lim_{\tau\to\infty}\frac{\tau^{-1}}{(\omega-\omega')^2+\tau^{-2}}=\pi\delta(\omega-\omega'),$$

we obtain

$$\langle\delta j_{t\mathbf{q},\Delta\omega}\,\delta j^{\bullet}_{t\mathbf{q}_1,\Delta\omega_1}\rangle=4\pi^2e^2v_0^2\left(\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}\left(1+\frac{F_1}{3}\right)TV\cdot\delta(\Delta\omega-\Delta\omega_1)\delta_{\mathbf{q},\mathbf{q}_1}\left\{\frac{2}{F_1}\left(1+\frac{F_1}{3}\right)\frac{s_0^2-1}{1+F_1/3-3s_0^2}\times[\delta(\Delta\omega-s_0qv_0)+\delta(\Delta\omega+s_0qv_0)]+\frac{1}{3}\frac{\omega_0^2}{\omega_0^2+c^2q^2}\times[\delta(\Delta\omega-\sqrt{\omega_0^2+c^2q^2})+\delta(\Delta\omega+\sqrt{\omega_0^2+c^2q^2})]\right\}.$$

In a completely analogous way, starting from formulas (24), (22) and (21), we arrive at the following expressions for the density correlator:

$$\langle\delta\rho_{\mathbf{q},\Delta\omega}\,\delta\dot{\rho}_{\mathbf{q}_1,\Delta\omega_1}\rangle=2\pi^2T\left(\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}V\left(1+\frac{F_1}{3}\right)^2\delta(\Delta\omega-\Delta\omega_1)\,\delta_{\mathbf{q},\mathbf{q}_1}$$
$$\times\frac{s}{\Delta\omega}\left\{\left[1+\frac{F_1}{3}+s^2\left(F_1+\frac{3\omega_0^2}{(\Delta\omega)^2}\right)\left(1-\frac{s}{2}\ln\frac{1+s}{1-s}\right)\right]^2+\left[\frac{\pi s^3}{2}\left(F_1+\frac{3\omega_0^2}{(\Delta\omega)^2}\right)\right]^2\right\}^{-1}$$

in the case when $|\Delta\omega| < qv_0$, and

$$\langle\delta\rho_{\mathbf{q},\Delta\omega}\,\delta\dot{\rho}_{\mathbf{q}_1,\Delta\omega_1}\rangle=\pi^2T\left(\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}V\delta(\Delta\omega-\Delta\omega_1)\,\delta_{\mathbf{q},\mathbf{q}_1}[\delta(\Delta\omega-\omega_0)+\delta(\Delta\omega+\omega_0)]$$

when $|\Delta\omega| > qv_0$.

Transforming from the correlators to the spectral distributions, in accordance with the relations

$$\langle\delta\rho_{\mathbf{q},\Delta\omega}\,\delta\dot{\rho}_{\mathbf{q}_1,\Delta\omega_1}\rangle=2\pi V\delta(\Delta\omega-\Delta\omega_1)\,\delta_{\mathbf{q},\mathbf{q}_1}\langle\delta\rho^2\rangle_{\mathbf{q},\Delta\omega},$$

$$\langle\delta j_{t\mathbf{q},\Delta\omega}\,\delta\dot{j}_{t\mathbf{q}_1,\Delta\omega_1}\rangle=2\pi V\delta(\Delta\omega-\Delta\omega_1)\,\delta_{\mathbf{q},\mathbf{q}_1}\langle\delta j_t^2\rangle_{\mathbf{q},\Delta\omega},$$

substituting these into the formulas (16) and (17) and introducing the quantum correction factor

$$\frac{\hbar\Delta\omega}{T}(e^{\hbar\Delta\omega/T}-1)^{-1},$$

we finally obtain the distribution of the scattered radiation over the angles and frequencies: for the case of scattering by transverse fluctuations,

$$dh_t=\left(\frac{e^2}{mc^2}\right)^2\frac{v_0^2}{c^2}\varepsilon\left(\frac{m^*}{m}\right)^2\left(\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}\left[\frac{\omega_0^2}{(\Delta\omega)^2-q^2c^2}-1\right]^2\hbar\Delta\omega$$
$$\times(e^{\hbar\Delta\omega/T}-1)^{-1}\sin^2\theta\left(1+2\sin^2\frac{\theta}{2}\right)\left[\frac{s_0^2-1}{F_1(1+F_1/3-3s_0^2)}[\delta(\Delta\omega-s_0qv_0)\right.$$
$$+\delta(\Delta\omega+s_0qv_0)]+\frac{\theta(qv_0-|\Delta\omega|)s}{4\Delta\omega}(1-s^2)\left\{\left[1+\frac{F_1}{3}-\frac{s^2}{2}\left(F_1-\right.\right.\right.$$
$$\left.\left.\left.-\frac{3\omega_0^2}{q^2c^2}\right)\left(1+\frac{1-s^2}{2s}\ln\frac{1+s}{1-s}\right)\right]^2+\left[\frac{\pi s}{4}(1-s^2)\left(F_1-\frac{3\omega_0^2}{q^2c^2}\right)\right]^2\right\}^{-1}\Bigg]d\omega'\,d\Omega'$$

and for the case of scattering by longitudinal fluctuations

$$dh_l=\frac{1}{4}\left(\frac{e^2}{mc^2}\right)^2\left(\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}\frac{\hbar\Delta\omega}{e^{\hbar\Delta\omega/T}-1}(1+\cos^2\theta)$$
$$\times\left[\frac{\theta(qv_0-|\Delta\omega|)s}{\Delta\omega}\left(1+\frac{F_1}{3}\right)^2\left\{\left[1+\frac{F_1}{3}+s^2\left(F_1+\frac{3\omega_0^2}{(\Delta\omega)^2}\right)\left(1-\frac{s}{2}\ln\frac{1+s}{1-s}\right)\right]^2\right.\right.$$
$$\left.\left.+\left[\frac{\pi s^3}{2}\left(F_1+\frac{3\omega_0^2}{(\Delta\omega)^2}\right)\right]^2\right\}^{-1}+\frac{1}{2}[\delta(\Delta\omega-\omega_0)+\delta(\Delta\omega+\omega_0)]\right]d\omega'\,d\Omega',$$

where

$$\theta(y)=\begin{cases}1, & y>0\\ 0, & y<0\end{cases}.$$

The first formula differs essentially from the second by the factor $V_0^2/c^2$, since the scattering by the transverse oscillations is a relativistic effect.

Despite their cumbersome appearance, the formulas obtained have an intuitive physical meaning. In each case, the frequency spectrum consists of an average part $-qv_0 < \Delta\omega < qv_0$ and two sharp lines, at $\Delta\omega = \pm s_0qv_0$ in one case and at $\Delta\omega = \pm\omega_0$ in the other. The central plateau corresponds to the Doppler broadening of the main line, and the side lines are satellites in the Rayleight scattering, corresponding to transverse zero sound and the plasma oscillations. Since the frequencies of these oscillations in metals satisfy the condition $\hbar\Delta\omega \gg v_0\mu/c \sim \theta_D$, in the whole range of temperatures $T \lesssim \theta_D$ we have, for the satellites, the purely quantum case, and, consequently, only the Stokes satellite remains in each doublet. The existence of zero sound in metals depends on the magnitude of the Fermi-liquid constants, which, unfortunately, have been insufficiently fully studied. We remark, however, that the possibility of propagation of transverse zero sound in metals is problematical, since Eq. (27) has a solution only under the condition $F_1 > 6$, which is extremely restrictive.

Rough numerical estimates give for the intensity of the zero-sound satellite

$$dh_t\sim10^2\hbar\omega\sin\frac{\theta}{2}\sin^2\theta\,d\Omega'.$$

Propagation of transverse zero sound is possible only if q satisfies the conditions

$$\omega_0/c\ll q\ll\omega_0/v_0,$$

while, for Rayleigh scattering,

$$q=2\frac{\omega}{c}\sin\frac{\theta}{2}$$

Hence it follows, firstly, that a zero-sound satellite can be observed only in the scattering of waves with frequencies $\omega \gg \omega_0$ and, secondly, that with increasing frequency of the incident wave the maximum scattering angle at which the zero-sound satellite is still observed decreases and, since the product $\omega\sin(\theta/2)$ is not changed, the determining factor is $\sin^2\theta$. Therefore, the lowest possible incident-wave frequencies, i.e., $\omega \sim 10^{16}-10^{17}$ are optimal. At such frequencies, the region of angles at which the zero-sound satellite is observed is displaced toward larger angles, including $\theta \approx 90°$.

For frequencies $\omega \sim 10^{17}$ and angles $\theta \approx 90°$,

$$dh_t\sim10^{-8}d\Omega'\ \mathrm{cm}^{-1}.$$

For the intensity of the satellite corresponding to the plasma oscillations, we obtain

$$dh_l\sim10^{-3}-10^{-4}d\Omega'\ \mathrm{cm}^{-1}.$$

The region of angles at which this satellite should be observed is determined by the relation

$$2\omega\sin\frac{\theta}{2}\ll\omega_0\frac{c}{v_0}.$$

The total intensity of the central plateau (for $\omega \sim 10^{17}$ and $\theta \approx 90°$) is of the order of $10^{-8}-10^{-9}\ d\Omega'\ \mathrm{cm}^{-1}$.

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*$[\mathbf{v}\mathbf{H}_0] \equiv \mathbf{v}\times\mathbf{H}_0$.


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