Structure of the bound-state spectrum of two rotons in superfluid helium

L. P. Pitaevskiĭ and I. A. Fomin

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
Institute of Physical Problems, USSR Academy of Sciences
(Submitted July 25, 1973)
Zh. Eksp. Teor. Fiz. 65, 2516–2521 (December 1973)

The behavior of the branches of the superfluid helium spectrum corresponding to the bound states of two rotons is considered. An explicit expression is obtained for the dependence of the binding energy on the total momentum of the two rotons in the range of low momenta $|\mathbf{p}| \ll (\sqrt{\mu\epsilon})$. The asymptotic behavior of the branches in the high momentum range, $p_0 \gg |\mathbf{p}| \gg (\sqrt{\mu\epsilon})$, is found. An estimate is obtained for the width of the bound state due to the possibility of decay into two phonons.

As is well known, the infinitesimally weak attraction between rotons leads to the formation of a bound state with energy $E < 2\Delta$.[1-3] At $\mathbf{p}=0$ ($\mathbf{p}$ is the total momentum of the rotons), these states can be classified according to the value of the angular momentum $l=0, 2, 4, \ldots$ The experimental data on combination scattering of light in helium apparently indicate that the bound state with $l=0$ is lacking, but that there is a state with $l=2$. It is natural to suppose that there are bound states with larger l that have not yet been discovered experimentally. For $\mathbf{p}\neq 0$, the energy of the bound state depends on $|\mathbf{p}|\equiv p$ and this state can be regarded as a new branch of the spectrum in helium. The aim of the present note is a discussion of the path of this branch as a function of p.

We first note that at $p=0$, the state with $l=2$ is fivefold degenerate in its values of m (by m we understand the projection of the angular momentum in the $\mathbf{p}$ direction). The degeneracy in $|m|$ is removed at finite p; there remains only the degeneracy in the sign of m; therefore the state which has the angular momentum $l=2$ at $p=0$ is split at $p\neq 0$ into three states with $|m|=0, 1, 2$. These states possess a certain "helicity," but no longer have definite l, the contributions from the various l being intermingled.

The equation of the bound state is obtained by summation of diagrams with "dangerous" cross sections, i.e., cross sections in which there are two roton lines. The equation for the vertex part has the form [4]

$$\Gamma(Q,P-Q;\ P_3,P_4)=\tilde{\Gamma}(Q,P-Q;\ P_3,P_4)$$
$$+\frac{i}{2\pi}\int\tilde{\Gamma}(Q,P-Q;\ Q',\ P-Q')G(Q')G(P-Q')\Gamma(Q',\ P-Q',\ P_3,P_4)\frac{d^4Q'}{(2\pi\hbar)^3}. \tag{1}$$

The capital letters indicate "4-vectors;" $P=\{E, \mathbf{p}\}$, $Q=\{\omega, \mathbf{q}\}$ and so on. The energy of the bound state is determined by the pole of Γ. Near the pole one can neglect the free term with Γ, after which one obtains a homogeneous equation in which the eigenvalues of the total energy E of the rotons determine the energy of the bound states.

The momenta P_3 and P_4 are not affected by Eq. (1) and the dependence on them can be left out of account. We also note that one can neglect the dependence on the total momentum $\mathbf{p}$ in $\tilde{\Gamma}$. This is a well-based approximation. The fact is that, as we shall see below, the integral of the product of two G functions that appears in (1), depends on the momentum on the characteristic interval $[\mu(2\Delta-E)]^{1/2}$. One can expect the dependence of $\tilde{\Gamma}$ on p to be important only at $p\sim p_0$ (p_0 is, as usual, the location of the roton minimum in the spectrum of superfluid helium $\epsilon=\Delta+(p-p_0)^2/2\mu$). The experimental value $\epsilon\equiv 2\Delta-E\approx 0.37$ °K for $l=2$, i.e., it is very small in comparison with Δ and $p_0^2/2\mu$, which also validates the neglect in $\tilde{\Gamma}$ of the dependence on $\mathbf{p}$.

In the integral over d^4Q' the important values are $\omega'\approx\Delta$ and $|\mathbf{q}'|\equiv q'\approx p_0$; therefore we can assume ω' and q' in the arguments of $\tilde{\Gamma}$ and Γ to be equal to these values. There then remains in $\tilde{\Gamma}$ only the dependence on the angle between $\mathbf{q}$ and $\mathbf{q}'$, and in Γ the dependence on p, E and the angles between $\mathbf{q}$ and $\mathbf{p}$ and $\mathbf{q}'$ and $\mathbf{p}$. We can then integrate in (1) over dq' and $d\omega'$,[1)] as a result of which we obtain the equation

$$\Gamma=-\frac{\mu p_0^2}{\pi\hbar^3}\int\frac{do_{q'}}{4\pi}\frac{\tilde{\Gamma}\Gamma}{[4\mu\epsilon+p^2(x')^2]^{1/2}}, \tag{2}$$

where $\epsilon=2\Delta-E$ and x' is the cosine of the angle between $\mathbf{q}'$ and $\mathbf{p}'$.

The total momentum $\mathbf{p}$ of the rotons plays the role of a parameter in the above equation. We choose the direct ion of $\mathbf{p}$ as the polar axis, and locate the x axis in the plane of the vectors $\mathbf{p}_3\mathbf{p}_4$. We expand the quantities that enter into the equation in series in spherical harmonics:

$$\Gamma(Q,P-Q)=\sum_{lm}\Gamma_{lm}(\varepsilon,p;\ \omega,q)N_l^mP_l^m(x)e^{im\varphi_q}, \tag{3}$$

where $x=\cos\theta_q$, θ_q is the angle between $\mathbf{p}$ and $\mathbf{q}$,

$$N_l^m=\left[\frac{2l+1}{2}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2}$$

are the normalization factors for the Legendre polynomials $P_l^m(x)$.

$$\tilde{\Gamma}=\frac{\pi\hbar^3}{\mu p_0}\sum_l(2l+1)\gamma_lP_l(\cos\theta), \tag{4}$$

and θ is the angle between $\mathbf{q}$ and $\mathbf{q}'$.

Substituting these expansions in (2) and integrating over $do_{q'}$, we obtain the equation

$$\sum_\lambda(\gamma_lB_{lm}^{\lambda m}+\delta_l^\lambda)\Gamma_{\lambda m}=0, \tag{5}$$

where

$$B_{lm}^{\lambda m}=N_l^mN_\lambda^mp_0\int_{-1}^{+1}\frac{P_l^m(x)P_\lambda^m(x)}{[4\mu\epsilon+p^2x^2]^{1/2}}dx. \tag{6}$$

The first few functions $B_{lm}^{\lambda m}$ are given in Appendix A. The solvability condition for system (5)—the condition determining the energy levels of the bound states—reduces to vanishing of the determinant

$$|\delta_l^\lambda+\gamma_lB_{lm}^{\lambda m}|=0. \tag{7}$$

At p = 0

$$B^{\lambda m}_{lm} = \frac{p_0}{\sqrt{4\mu\varepsilon}}\delta_l^\lambda$$

so that the equations for the various l are decoupled and reduce to

$$\sqrt{4\mu\varepsilon_l} = -\gamma_l p_0 \tag{8}$$

and there is a bound state for $\gamma_l < 0$ with $\epsilon_l = \gamma_l^2 p_0^2/4\mu$, in agreement with the results of Iwamoto.[1]

For $p \neq 0$, as has been mentioned previously, ϵ begins to depend on p and $|m|$; we shall denote the corresponding levels as $\epsilon_l^m(p)$. We emphasize that since the equations with different l are intermingled at $p \neq 0$, the index l no longer denotes the angular momentum of the bound state literally, but only denotes the limiting value of the angular momentum at $p = 0$, and the quantity ϵ_l^m does not depend only on γ_l with the same l.

The first term of the expansion of ϵ_l^m in p has the form

$$\varepsilon_l{}^m = \varepsilon_l - p^2/2\mu_l{}^m \qquad (p^2 \ll \mu\varepsilon_l), \tag{9}$$

and the effective masses μ_l^m do not depend on the value of the interaction

$$\mu_l{}^m = \frac{2(2l-1)(2l+3)}{2l-1+2(l^2-m^2)}\mu, \tag{10}$$

and in particular

$$\mu_2{}^0 = \frac{42}{11}\mu, \quad \mu_2{}^1 = \frac{14}{3}\mu, \quad \mu_2{}^2 = 14\mu. \tag{11}$$

We now turn to the case of relatively large momenta $p_0 \gg p \gg \sqrt{\mu\epsilon}$. In this case, for even m, the important quantities in the integrals (5) are the small values of x and the equation could be simplified by calculating the coefficients $B_{lm}^{\lambda m}$ with logarithmic accuracy. However, it is simpler to turn directly to Eq. (2). Expanding $\tilde{\Gamma}$ and Γ in Fourier series in φ, we obtain a set of equations for various values of m (see [5]). In the integral (2) for even m, the small values of x are important, so that, with logarithmic accuracy, we can set $x = 0$ in $\tilde{\Gamma}$ and Γ. As a result, we have

$$1 = g_m \frac{p_0}{p} \ln \frac{4\mu\varepsilon}{p^2}, \tag{12}$$

where

$$g_m = \frac{\mu p_0}{\pi\hbar^3}\int \tilde{\Gamma}(\varphi, x=0)\frac{d\varphi}{2\pi}, \tag{13}$$

so that, with exponential accuracy,

$$\varepsilon^m \sim \frac{p^2}{\mu}\exp\left(\frac{p}{p_0 g_m}\right). \tag{14}$$

This solution naturally has meaning only if $g_m < 0$.

Bound states are lacking in this region for odd m. Substituting the expansion (4) for $\tilde{\Gamma}$ in (13), we express g_m in terms of γ_l:

$$g_m = \sum_{l \geq m} \gamma_l [N_l{}^m P_l{}^m(0)]^2. \tag{15}$$

It is seen from Eqs. (12), (13) that there is no more than one bound state in the considered region of "large" p for each even value of m, while as $p \to 0$ we had states with all possible m for those l for which $\gamma_l < 0$. This means that all the remaining $\epsilon_l^m(p)$ curves are "terminated," i.e., they reach the level $\epsilon_l^m = 0$ and go off into the continuous spectrum in the range of small p.

It is easy to see to which l (at $p = 0$) a single "surviving" branch of the spectrum corresponds at large p. For this purpose, we note that branches of the spectrum with the same value of m cannot intersect because of the theorem of the nonintersectibility of terms of the same symmetry.[2)] It is therefore clear that that branch of the spectrum which has a maximum value ϵ at $p = 0$ should be the last to terminate. In other words, the branch of the spectrum "surviving" at large p has a value of the energy as $p \to 0$ that is equal to the maximal $\epsilon_l^m(0)$ at the given $m \leq l$, which we shall denote by $\epsilon_{l\,\max}^m$. We also note that although the g_m are expressed in terms of γ_l with all $l \geq m$, it is natural to suppose that for the surviving branch $g_m \sim \gamma_{l\,\max}(m) \sim (\mu\epsilon_{l\,\max}^m/p_0^2)^{1/2}$. Inasmuch as it was assumed in the derivation of (12) and (13) that $g_m p_0/p \ll 1$, the condition of applicability of these formulas has the form $p^2 \gg \mu\epsilon_{l\,\max}^m$. The remaining branches for even m terminate at momenta $p^2 \sim \mu\epsilon_{l\,\max}^m$. The ending of these branches occurs in the manner described in [4], i.e.,

$$\varepsilon_l{}^m \sim \exp\left\{-\frac{\alpha_l{}^m}{p_{cl}{}^m - p}\right\}, \tag{16}$$

where p_{cl}^m is the momentum of the termination point.

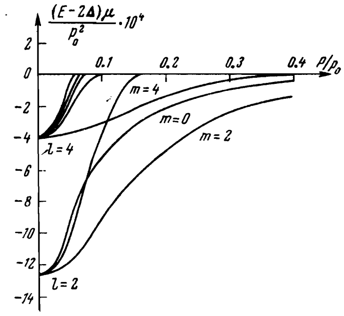
The branches ϵ_l^m with odd m terminate in the region $p^2 \sim \mu\epsilon_l^m(0)$, while the path of the curve near the termination point is determined by the formula

$$\varepsilon_l{}^m \ln\frac{\mu\varepsilon_l{}^m}{p^2} = \alpha_l{}^m(p - p_{cl}{}^m) \tag{17}$$

(see also Appendix B).

The figure shows schematically the character of the spectrum of the bound states of two rotons for the case in which only two constants $\gamma_2 \approx -0.07$ and $\gamma_4 \approx -0.04$ are different from zero and negative. The branch that survives at $p \gg \sqrt{\mu\epsilon_{l\,\max}^m}$ can terminate only at the point at which g_m changes sign, i.e., in the range of momenta in which $\tilde{\Gamma}$ depends significantly on **p**.

From the viewpoint of the possibility of experimental study of the picture of the spectrum just described, the problem of the damping of the bound states is of importance. As has already been shown by Landau,[6] excitations which have finite energy at $p = 0$ are unstable relative to decay into two phonons with oppositely directed momenta. In our case, the energy of each phonon should be $\Omega \approx \Delta$, and the momentum $k \approx \Delta/c \approx 0.48\ \text{Å}^{-1}$, where c is the sound velocity.[3)] The damping of the bound states due to the decay can be calculated if we take into account in $\tilde{\Gamma}$ the diagram with the intermediate state, in which there are two phonons:



$$\tilde{\Gamma}=\Gamma^{(1)}+\frac{i}{2\pi}\int\Gamma^{(1)}GG\tilde{\Gamma}\frac{d^4Q}{(2\pi\hbar)^3}. \quad (18)$$

As a result, an imaginary part appears in Eq. (5) for $\tilde{\Gamma}$. We set $\gamma_l \to \gamma_l - i\delta_l$. We then get for the energy at $p=0$

$$\varepsilon_l=(\gamma_l^2-2i\delta_l\gamma_l)p_0^2/4\mu. \quad (19)$$

A calculation by Eq. (18) gives for δ_l:

$$\delta_l=\frac{f_l^2}{4}\left(\frac{\Delta}{c}\right)^2\frac{1}{\mu c p_0}, \quad (20)$$

where f_l is in corresponding fashion the normalized irreducible amplitude of conversion of two rotons with angular momentum l into a pair of phonons. Direct estimation of f_l is difficult, since the character of the interaction of rotons with shortwave phonons is unknown. For the estimate, we assume that $f_l \approx \gamma_l$, i.e., the probability of two rotons being converted on collision into phonons ~1. It is probable that this gives the maximum estimate for f_l; then

$$\frac{\delta_l}{\gamma_l}\approx\frac{\gamma_l}{4}\left(\frac{\Delta}{c}\right)^2\frac{1}{\mu c p_0}\approx\frac{\gamma_l}{8}, \quad (21)$$

for example, $\delta_2/\gamma_2 \sim 0.01$, i.e., the damping is comparatively small.

We note that the existing experimental data on the lifetimes of rotons appear to indicate that γ_l cannot fall off too rapidly with increasing l.[7] This means that the entire range of energies can be filled with levels with different l.

The authors thank G. M. Éliashberg for useful comments.

APPENDIX A

$$B_{00}{}^{00}=\frac{p_0}{2p}L, \quad B_{20}{}^{00}=\frac{\sqrt{5}}{4}\frac{p_0}{p}\left[3\sqrt{1+\eta^2}-\left(\frac{3}{2}\eta^2+1\right)L\right],$$

$$B_{20}{}^{20}=\frac{5}{8}\frac{p_0}{p}\left[-\frac{3}{2}\sqrt{1+\eta^2}\left(1+\frac{9}{2}\eta^2\right)+\left(\frac{27}{8}\eta^4+3\eta^2+1\right)L\right],$$

$$B_{21}{}^{21}=\frac{15}{8}\frac{p_0}{p}\left[\sqrt{1+\eta^2}\left(1+\frac{3}{2}\eta^2\right)-\eta^2\left(1+\frac{3}{4}\eta^2\right)L\right],$$

$$B_{22}{}^{22}=\frac{15}{16}\frac{p_0}{p}\left[-\frac{3}{2}\sqrt{1+\eta^2}\left(1+\frac{\eta^2}{2}\right)+\left(1+\eta^2+\frac{3}{8}\eta^4\right)L\right];$$

$$\eta^2=\frac{4\mu\varepsilon}{p^2}, \quad L=\ln\frac{\sqrt{1+\eta^2}+1}{\sqrt{1+\eta^2}-1}$$

APPENDIX B

For odd m at small $\eta^2=4\mu\epsilon/p^2$

$$B_{lm}{}^{\lambda m}\approx\frac{p_0}{p}N_l^mN_\lambda^m\left\{\int_{-1}^{+1}\frac{P_l^m(x)P_\lambda^m(x)}{|x|}dx+\frac{1}{2}\frac{d}{d(x^2)}\right.$$

$$\left.\times[P_l^m(x)P_\lambda^m(x)]_{x=0}\eta^2\ln\eta^2\right\}. \quad (B.1)$$

Substituting this expression in Eq. (7) and expanding it in the small quantity $\eta^2 \ln \eta^2$, we get an equation of the form

$$D_0^{(m)}+D_1^{(m)}\eta^2\ln\eta^2=0, \quad (B.2)$$

where $D_0^{(m)}$ and $D_1^{(m)}$ are determinants

$$D_0^{(m)}=\left|\frac{p}{p_0}\delta_{l\lambda}+\gamma_l a_{l\lambda}^{(m)}\right|, \quad (B.3)$$

$$a_{l\lambda}^{(m)}=N_l^mN_\lambda^m\int_{-1}^{+1}\frac{P_l^{(m)}(x)P_\lambda^m(x)}{|x|}dx. \quad (B.4)$$

The explicit expression for $D_1^{(m)}$ is not needed.

For those p for which $D_0^{(m)}(p)=0$, Eq. (B.2) has the solution $\eta^2=0$, which corresponds to termination of one of the branches. Thus the terminal points of the branches p_{cl}^m which correspond to odd m are found from the equation

$$\left|\frac{p}{p_0}\delta_{l\lambda}+\gamma_l a_{l\lambda}^{(m)}\right|=0. \quad (B.5)$$

It can be shown that Eq. (B.5) has as many positive roots as it has negative γ_l with $l \gtrsim m$.

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1) In the integration over d^3q, we can make the substitution $q^2dq \approx p_0^2dq$; this corresponds to neglect of the next-higher-order terms in p/p_0 and $\sqrt{\mu\varepsilon}/p_0$.

2) Branches with different m have different symmetries and can therefore intersect.

3) As is well known, at $\Omega \sim \Delta$ the spectrum of the phonons still departs only slightly from linearity.

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Translated by R. T. Beyer
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