Bremsstrahlung from an electron passing by a nucleus situated in a constant external field

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A calculation based on the method of equivalent photons is made of the bremsstrahlung cross section for an electron passing by a nucleus situated in a constant crossed field. A generalization of the result to the case of an arbitrary field is given. It is shown that the greatest effect of a constant external field is manifested in the low frequency region, where the intensity of the synchroton radiation is maximal.

1. INTRODUCTION

In recent times it is becoming more and more obvious that a constant external field must exert an essential influence on quantum processes in which relativistic particles take part[1-4]. An exact calculation of processes in an external field is difficult and therefore one has to use approximate methods such as, for example, the crossed field approximation[5] or different asymptotic methods[6-8]. In this case, just as in the case when the external field is absent, one can utilize for processes involving a virtual photon the method of equivalent photons (cf., for example,[9,10]). It follows from the results of Nikishov[1] and Baĭer and Katkov[4] who considered bremsstahlung in the case of electron-electron (-positron) collisions that the external field affects most strongly the recoil electron (i.e., the equivalent photon spectrum). But in a collision with a massive particle (nucleus) one should evidently expect the greatest effect of the external field on the emission of a photon by the light particle (electron), while one can neglect the effect of the external field on the nucleus and therefore on the equivalent photon spectrum.

In the present paper we consider the bremsstrahlung from an electron passing by a nucleus situated in a constant crossed field ($\mathbf{E} \perp \mathbf{H}$, $|\mathbf{E}| = |\mathbf{H}|$). We note that the results of calculations in a crossed field are approximately applicable, as has been shown by Nikishov and Ritus[5] (cf., also[11]), to processes in a constant arbitrary field.

2. THE EQUIVALENT PHOTON APPROXIMATION

We specify by the four-potential[1)]

$$A^\mu = (nx)B^\mu, \quad n^2 = (nB) = 0, \tag{1}$$

(where B^μ is a constant four-vector) a constant and homogeneous crossed field

$$F^{\mu\nu} = n^\mu B^\nu - n^\nu B^\mu = \text{const}$$
$$(\mathbf{E} \perp \mathbf{H}, \quad |\mathbf{E}| = |\mathbf{H}|, \quad [\mathbf{E}\times\mathbf{H}] \parallel \mathbf{n}). \tag{2}$$

The Dirac equation for an electron moving in a given field can be solved exactly (cf., for example,[5,9]). The solution is characterized by four quantum numbers k^μ (quasimomentum) subject to the additional condition $k^2 = m^2$ (m is the electron mass) which go over into the components of the four-momentum of the free particle when the field is switched off. We note that the choice of the potential in the form (1) is evidently not the only one which leads to the given solution. In particular, a transformation (of the gradient type) is allowable

$$A_\mu \to A_\mu + \text{const}\ \partial_\mu (Bx),$$

which is equivalent to the parallel displacement of the coordinate system

$$(nx) \to (nx) + \text{const}.$$

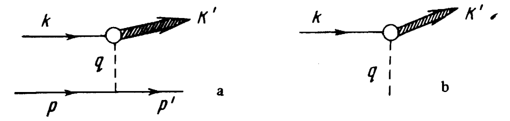

FIG. 1.

On the basis of the exact solutions indicated above we consider in the Furry representation the collision of an electron characterized by k^μ with a spinless nucleus characterized by p^μ ($p^2 = M^2 \gg m^2$) in the crossed field (2). In future we shall neglect the effect of the external field on the motion of the nucleus. As a result of the exchange of a virtual photon $q = p - p'$ ($q^2 < 0$) in the collision the nucleus acquires the momentum p' and a group of particles is produced with the total four-momentum K' (cf., Fig. 1a). We choose a special coordinate system in which the electron was at rest prior to the collision, i.e., $\mathbf{k} = 0$, and with regard to the crossed field we assume that $\mathbf{B} \perp \mathbf{p}$ (i.e., $\mathbf{n} \parallel \mathbf{p}$), and that $B^0 = 0$. In invariant form these conditions can be written in the following manner:

$$F^{\mu\nu} p_\mu k_\nu = 0. \tag{3}$$

We assume the nucleus to be relativistic, i.e., in the special coordinate system we assume $|\mathbf{p}| \gg M$. Then the process can be regarded as an interaction of electrons with the equivalent photons of the field of the nucleus being propagated in the direction $\mathbf{p} \parallel \mathbf{n}$. Neglecting the effect of the external crossed field on the nucleus we can assume that the equivalent photon spectrum is not altered compared to the free case, i.e.,

$$dN = \frac{2}{\pi} Z^2 e^2 \ln \frac{\mu}{\varkappa} \frac{d\varkappa}{\varkappa}, \tag{4}$$

where in place of the photon frequency q^0 we have introduced an invariant variable $\varkappa = 2|(kq)|/m^2$, while $\mu = (kp)/mM$. We shall calculate the cross section in this approximation by means of the Weizsäcker-Williams formula

$$d\sigma = \sigma^{\text{ph}} dN. \tag{5}$$

Here σ^{ph} is the cross section for the corresponding photoprocess in a crossed field (cf., Fig. 1b) involving the participation of a real photon q ($q^2 = 0$) propagated in the direction $\mathbf{n} \parallel [\mathbf{E}\times\mathbf{H}]$ (i.e., $qn = 0$). The conditions for the applicability of formula (5) are determined as is well known (cf., for example[10]) by the inequalities

$$-q^2/m^2 \ll 1, \quad -q^2/m^2 \ll \varkappa, \quad \mu \gg 1, \quad \mu \gg m\varkappa/M. \tag{6}$$

The cross section for the photoprocess σ^{ph} in an external crossed field depends on two invariant dimensionless parameters: $\varkappa$ and

$$\chi = \frac{e}{m^3}(nk)(-B^2)^{1/2} = \frac{e}{m^3}[-(F^{\mu\nu}k_\nu)^2]^{1/2} \tag{7}$$

where the parameters χ is expressed in terms of the tensor $F^{\mu\nu}$ of the field (2). As has been shown in[11] the cross section σ^{ph} calculated for the case of a crossed field approximately describes the process in an arbitrary constant external field $F^{\mu\nu}$ if the parameters

$$\chi=\frac{e}{m^3}[-(F^{\mu\nu}k_\nu)^2]^{1/2},\qquad \varkappa=\frac{2|(kq)|}{m^2},$$
$$f_1=\frac{e}{m^2}|F^{\mu\nu}F_{\mu\nu}|^{1/2},\qquad f_2=\frac{e}{m^2}|F_{\mu\nu}^{*}F^{\mu\nu}|^{1/2},$$
$$f_3=\frac{e}{m^4}|F^{\mu\nu}k_\mu q_\nu|,\qquad f_4=\frac{e}{m^4}|F_{\mu\nu}^{*}k^\mu q^\nu|, \tag{8}$$
$$f_5=\frac{e}{m^3}|(F^{\mu\nu}q_\nu)^2|^{1/2},\qquad f_6=\frac{e}{m^3}|F_{\mu\nu}F^{\nu\lambda}k^\mu q_\lambda|^{1/2}$$

($F^{*}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$) on which the cross section for the process depends satisfy the inequalities

$$f_i\ll 1,\quad f_i\ll\chi,\quad f_i\ll\varkappa\quad (i=1,\ldots,6) \tag{9}$$

(in the case of a crossed field all $f_i=0$). Thus, under the conditions (9), and also (3), formula (5) can be applied for the calculation of the cross section in the case of an external field of arbitrary form

3. THE CROSS SECTION FOR BREMSSTRAHLUNG

We consider the bremsstrahlung from an electron passing by a nucleus in the presence of a constant external field. The corresponding photoprocess-Compton scattering in the presence of an external field—has been investigated previously[11,12,8]. It was shown that in the process of scattering the electron not only can absorb incident photons, but can also emit photons identical with them. Moreover, at low energies of incident photons an essential contribution to the cross section is also given by terms associated with the synchrotron radiation of an electron in an external field.

Substituting the well-known[11,12,8] expression for the probability of the photoprocess

$$dw^{ph}=\frac{m^2\varkappa}{2q^0k^0}d\sigma^{ph}$$

into formula (5) and integrating over the equivalent photon spectrum κ, we obtain the differential cross section for bremsstrahlung

$$d\sigma=\frac{32Z^2e^6}{\sqrt{\pi}m^2}\frac{du}{(1+u)^2}\int_0^\infty\frac{d\varkappa}{\varkappa^2}\ln\frac{\mu}{\varkappa}\times[A(\varkappa,\chi,u)-A(-\varkappa,\chi,u)-2A_0(\varkappa^2,\chi,u)]. \tag{10}$$

Here the functions $A(\kappa,\chi,u)$ and $A(-\kappa,\chi,u)$ correspond to the emission and absorption of virtual photons q by the electron (cf., Figs. 2b, c), while the function $A_0(\kappa^2,\chi,u)$ corresponds to the product of the matrix elements for synchrotron radiation in the constant field $F^{\mu\nu}$ (Fig. 2a) and for bremsstrahlung with an exchange of two photons q one of which is absorbed and the other emitted by the electron (Fig. 2d). (Each diagram of Figs. 2b, c, d is to be taken as standing for the set of diagrams differing from each other by a permutation of the vertices.)

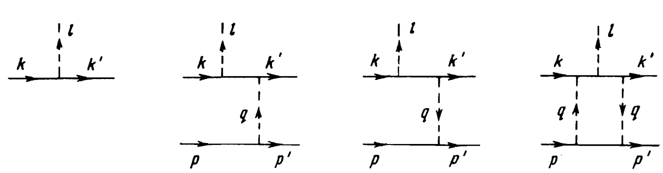

FIG 2.

The functions A and A_0 are of the following form:

$$A(\varkappa,\chi,u)=v_1\Phi_1(y_1)+v_2\Phi'(y_1)+v_3\Phi(y_1),$$
$$A_0(\varkappa^2,\chi,u)=v_4\Phi_1(y)+v_5\Phi'(y)+v_6\Phi(y) \tag{11}$$

where

$$v_1=\frac{1}{8}\left(2+\frac{u^2}{1+u}-\frac{4}{x}+\frac{4}{x^2}-\frac{16}{y^3x^4}\right),$$
$$v_2=-\frac{u^2}{2yx^2(1+u)}-\frac{4}{y^4x^4}\left(1+\frac{u^2}{2(1+u)}\right),$$
$$v_3=\frac{2}{y^2x^2}\left(1+\frac{u^2}{2(1+u)}\right)-\frac{2}{y^2x^3}\left(1+\frac{u^2}{1+u}\right),$$
$$v_4=v_1+\frac{1}{2x},\qquad v_5=\left(1+\frac{u^2}{2(1+u)}\right)\left(\frac{1}{yx^2}-\frac{4}{y^4x^4}\right),$$
$$v_6=\frac{y}{4}\left(1+\frac{u^2}{1+u}\right),$$

$\Phi(y)$ and $\Phi'(y)$ are the Airy function and its derivative,

$$\Phi_1(y)=\int_y^\infty\Phi(x)\,dx,\qquad y_1=y(1-x),\qquad y=\left(\frac{u}{\chi}\right)^{2/3},\qquad x=\frac{\varkappa}{u}.$$

The variable

$$u=\frac{\chi-\chi'}{\chi'},\qquad \chi'=\frac{e}{m^3}[-(F^{\mu\nu}k'_\nu)^2]^{1/2},$$

is related to the finite value of the quasimomentum of the electron k' after emission of radiation. It is important that, since the four-momentum is not conserved in the field, in formula (10) the integration over the equivalent photon spectrum κ should be carried out from zero and not from u, as in the case when the field is absent $F^{\mu\nu}=0$. In this case each term in the square brackets of (10) yields on integration over κ a result which diverges at the lower limit. However, integrating each term by parts in the logarithmic approximation an appropriate number of times one can separate out all the terms which diverge at the point $\kappa=0$ which cancel each other in the final sum. The result turns out to be finite:

$$d\sigma=\frac{8}{15}\frac{Z^2e^6}{\sqrt{\pi}m^2}\frac{du}{u(1+u)^2}\left(\frac{u}{\chi}\right)^{2/3}\left\{3\left(1+\frac{u^2}{2(1+u)}\right)I_1(y)+y^3\left(1+\frac{u^2}{1+u}\right)[I_1(y)-I_3(y)]-y\left(5+4\frac{u^2}{1+u}\right)I_2(y)\right\}. \tag{12}$$

The integrals I_1, I_2, I_3 can be expressed in the following form:

$$I_1=\int_0^\infty\frac{dx}{x}[\Phi(y_1)-\Phi(y_2)]\ln\frac{\mu}{ux},$$
$$I_2=\int_0^\infty\frac{dx}{x}[\Phi'(y_1)-\Phi'(y_2)]\ln\frac{\mu}{ux}, \tag{13}$$
$$I_3=\int_0^\infty dx[\Phi(y_1)+\Phi(y_2)]\ln\frac{\mu}{ux};\qquad y_2=y(1+x).$$

4. ASYMPTOTIC VALUES OF THE CROSS SECTION AND DISCUSSION OF THE RESULTS

The principal contribution to the formation of integrals (13) comes from the region $x\sim 1$ (i.e., $\kappa\sim u$). Because of this in place of the inequality $\kappa\gg f_i$ from (9), which is one of the conditions for the possibility of generalizing the result to the case of an arbitrary field, we obtain the following: $u\gg f_i$. For the sake of definiteness we assume that in the rest system of the nucleus ($p=0$) there is present a magnetic field H, and a relativistic electron of energy $\epsilon\gg m$ moves at right angles to it. Then condition (3) is satisfied:

$$F^{\mu\nu}k_\mu p_\nu = \mathbf{H}[\mathbf{kp}] = 0,$$

while for the parameter χ and the variable u we obtain

$$\chi = \frac{H}{H_0}\frac{\varepsilon}{m}, \quad u = \frac{\varepsilon - \varepsilon'}{\varepsilon'}, \qquad (14)$$

where $H_0 = m^2/e = 4.4 \times 10^{13}$ G. If at the same time

$$H \ll H_0 \quad u \gg H/H_0, \qquad (15)$$

then the inequalities (9), the validity of which makes possible the generalization of the result to the case of a magnetic field, will be satisfied.

Conditions (15) enable us to make another important deduction. As is well known (cf.,[13,4]), when a relativistic electron collides with a nucleus the formation of the principal part of the radiation at the frequency $\omega = \epsilon - \epsilon'$ occurs during a time $\tau_0 = 2\epsilon/m^2u$, while the time for the formation of radiation in the magnetic field is given by $\tau_H = H_0/Hm$. One would expect that the magnetic field would begin to have the greatest effect on the process starting with the region where these times become comparable:

$$\tau_H/\tau_0 = u/\chi \lesssim 1. \qquad (16)$$

For $H \ll H_0$ and $\epsilon \gg m$ the motion of an electron in a magnetic field is quasiclassical. The relative change in the momentum of the electron associated with the existence of acceleration in a magnetic field during a time τ,

$$\delta = \Delta|\mathbf{k}|/|\mathbf{k}| \sim \tau eH/\varepsilon,$$

turns out to be small both for $\tau = \tau_H$ ($\delta \sim m/\epsilon \ll 1$) and also for $\tau = \tau_0$ ($\delta \sim H/H_0u \ll 1$). Consequently, the magnetic field does not have time to have an appreciable dynamic effect on the motion of the electron during the characteristic time for the occurrence of the process. In other words, the interaction occurs over a region of space quite small compared to the radius of curvature of the trajectory, and this enables us to introduce the cross section in the usual manner independently of the instant at which the electron collides with the nucleus. However, in this case the magnetic field can affect the nature of the formation of radiation, particularly in the region determined by the inequality (16).

The cross section (12) depends on the value of the external field through the parameter χ which enters into the argument $y = (u/\chi)^{2/3}$ of the integrals (13). We consider two extreme cases, $y \gg 1$ and $y \ll 1$, for which we are able to obtain the following asymptotic representations for these integrals:

a) $y \gg 1$:

$$I_1 \approx \frac{\sqrt{\pi}}{y}\left(1 + \frac{2}{y^2}\right)\ln\frac{\mu}{u}, \quad I_2 \approx -\frac{\sqrt{\pi}}{y^2}\ln\frac{\mu}{u}, \quad I_3 \approx \frac{\sqrt{\pi}}{y}\ln\frac{\mu}{u}; \qquad (17)$$

b) $y \ll 1$:

$$I_1 \approx \frac{\sqrt{\pi}\Gamma(^1/_3)}{3^{2/3}\cdot 2}\ln\frac{\mu}{u}, \quad I_2 \approx \frac{\sqrt{\pi}\Gamma(^2/_3)}{3^{1/3}\cdot 2}\ln\frac{\mu}{u}, \quad I_3 \approx \frac{\sqrt{\pi}}{y}\ln\frac{\mu}{u}. \qquad (18)$$

Substituting (17) into formula (12) we find for $y \gg 1$

$$d\sigma = \frac{4Z^2e^6}{m^2}\frac{du}{u(1+u)^2}\left(\frac{4}{3} + \frac{u^2}{1+u}\right)\ln\frac{\mu}{u}, \quad \left(\frac{u}{\chi}\right)^{2/3} \gg 1. \qquad (19)$$

This result coincides exactly with the expression for the cross section for bremsstrahlung in the case of an electron colliding with a nucleus in the free case (cf.,[9], p. 455), if we assume

$$u = (\varepsilon - \varepsilon')/\varepsilon', \quad \mu = \varepsilon/m, \qquad (20)$$

where ϵ and ϵ' are the initial and the final values of the electron energy in the rest system of the nucleus in the case of a zero external field ($F^{\mu\nu} = 0$).

In the other limiting case $y \ll 1$, we obtain from formulas (18) and (12)

$$d\sigma = \frac{4\Gamma(^1/_3)}{3^{2/3}\cdot 5}\frac{Z^2e^6}{m^2\chi}\left(\frac{\chi}{u}\right)^{1/3}\left(1 + \frac{u^2}{2(1+u)}\right)\frac{du}{(1+u)^2}\ln\frac{\mu}{u} \qquad (21)$$
$$\left(\frac{u}{\chi}\right)^{2/3} \ll 1.$$

It can be seen that for $(u/\chi)^{2/3} \ll 1$ the cross section (21) grows with decreasing u as $(\chi/u)^{1/3}$, i.e., much more slowly than in the free case when it is proportional to 1/u (the infrared divergence).

Thus, the effect of a constant external field $F^{\mu\nu}$ on the bremsstrahlung of an electron in the field of a nucleus in accordance with (16) begins to manifest itself primarily when $(u/\chi)^{2/3} \lesssim 1$. In this domain the synchrotron radiation of an electron in the field $F^{\mu\nu}$ has its maximum intensity and therefore in addition to the diagrams of Figs. 2b, c an essential contribution to the cross section for this process is also given by the interference of the diagrams a and d of the same figure.

The example considered above involving a magnetic field corresponds to real possibilities experimentally. At the present time under laboratory conditions maximum field intensities have been obtained $H \lesssim 10^7$ G $\ll H_0$. In this case, however, values close to unity can be obtained for the parameter χ.

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1)We have adopted the metric (+ − − −), $x^\mu = \{x^0, \mathbf{x}\}$; the system of units is $c = \hbar = 1$.

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