# Non-Hamiltonian approach to conformal quantum field theory

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The completeness requirement for the set of operators appearing in field theory at short distances is formulated, and replaces the *S*-matrix unitarity condition in the usual theory. Explicit expressions are obtained for the contribution of an intermediate state with given symmetry in the Wightman function. Together with the "locality" condition, the completeness condition leads to a system of algebraic equations for the anomalous dimensions and coupling constants; these equations can be regarded as sum rules for these quantities. The approximate solutions found for these equations in a space of $4-\epsilon$ dimensions give results equivalent to those of the Hamiltonian approach.

## 1. INTRODUCTION

In recent years, the hypothesis of the conformal invariance of strong interactions at distances much shorter than $10^{-14}$ cm has been put forward and analyzed in detail (see the reviews[1,2]). It has been shown that the equations of quantum field theory are invariant under the conformal group, under the condition that anomalous values of the dimensions, which should be determined from the condition for solubility of the equations, are assigned to the different fields. All the observable consequences of the theory were expressed in terms of these dimensions and, in addition, in terms of a set of effective interaction constants at short distances.

At the same time, the equations for the determination of the above quantities (skeleton expansions for the vertex parts) were series with zero radius of convergence and therefore did not have well-defined mathematical meaning. The physical meaning of these equations was also highly obscure. The form of the equations depended in an essential way on the type of fundamental fields and on the form of their bare interaction, whereas the results of a theory with anomalous dimensions should not be sensitive to the choice of the initial Hamiltonian.

The purpose of the present article is to construct a more general formalism for the determination of the anomalous dimensions; this, on the one hand, would be "democratic" with respect to the different fields, and, on the other, would not contain meaningless series (these two properties turn out to be intimately related). Compared with the old approach, such a formalism plays the same role as the methods of S-matrix theory compared with Hamiltonian theory, and is a generalization of the S-matrix equations for the short-distance region.

## 2. THE OPERATOR-SET COMPLETENESS CONDITION AS DYNAMICAL EQUATIONS

In this section, we propose a self-consistency principle governing the interactions at short distances. This principle replaces the unitarity condition in ordinary field theory and, in brief, consists in the completeness of the set of operators entering in the massless theory, which, by assumption, is equivalent to the asymptotic theory of short distances.

The fundamental difficulty in the formulation of this principle lies in the classification of the appropriate operators. We recall that asymptotic "in" or "out" states do not exist in a massless theory. It is therefore necessary to find a replacement for this traditional complete set of operators. After this has been done, it may be hoped that, as in a theory with finite mass, combination of the completeness condition with the causality condition will give a dynamical system of equations, sufficient for the determination of the Wightman functions.

We shall consider a scalar field theory without internal degrees of freedom (later, it will be easy to generalize this treatment). We shall assume that there exists a set of local scalar operators $\{O_{0n}(x)\}$ with increasing dimensions $\{\Delta_{0n}\}$. This set is analogous to the set $:\varphi^n:$ of free field theory. We shall also assume that, at short distances, there is not only scale invariance but also conformal invariance with anomalous dimensions $\Delta_n$, and that under the action of a special conformal transformation the operators $\{O_{0n}(0)\}$ remain unchanged. Scalar operators cannot form a complete set, and we must therefore supplement them with the tensor operators $\{O^{(jn)}_{\alpha_1\ldots\alpha_j}(x)\}$, which transform according to an irreducible representation of the Lorentz group. The content of the latter assumption is that the set of operators $\{O^{(jn)}_{\alpha_1\ldots\alpha_j}\}$ and their derivatives with respect to the coordinates is complete. The number j will be called the "Lorentz spin," and the number n—the "principal quantum number"; operators with $n > 0$ will be called "satellites."

By virtue of the conformal symmetry, the operator basis introduced is orthogonal[2]:

$$\langle O^{(jn)}(0)O^{(j'n')}(x)\rangle \infty \delta_{jj'}\delta_{nn'}.$$

We shall show that the completeness condition leads to dynamical equations analogous to the unitarity equations in the ordinary theory. For this, we shall consider a product $\varphi(x)\varphi(0)$ of two scalar fields, and expand it in the proposed basis:

$$\varphi(x)\varphi(0)=\sum_{jns} C^{(jn)}_{\alpha_1\ldots\alpha_j|\mu_1\ldots\mu_s}(x)\,\partial_{\mu_1}\ldots\partial_{\mu_s} O^{(jn)}_{\alpha_1\ldots\alpha_j}(0). \tag{2.1}$$

Here, C is a c-number function of x, whose form is fixed, to within a few constants, by the conformal symmetry.

Equation (2.1) is not the asymptotic expansion that has been used in the papers[3-5], but should be understood in the following sense:

$$\Big\langle TA(x)A(0)\prod\varphi_{a_j}(z_j)\Big\rangle = \sum_{jns} C^{(jn)}_{\alpha_1\ldots\alpha_j|\mu_1\ldots\mu_j}(x)\,\partial_{\mu_1}\ldots\partial_{\mu_j}\Big\langle TO_{\alpha_1\ldots\alpha_j}(0)\prod\varphi_{a_j}(z_j)\Big\rangle \tag{2.2}$$

($\varphi_{aj}$ are arbitrary local fields).

By virtue of the analytic properties of the Wightman functions, the infinite series in the right-hand side of (2.2) converges for sufficiently small x and should be understood in the sense of an analytic continuation for other values of x. The dynamical equations for the C-function appear as the requirement that the quantity

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle$$

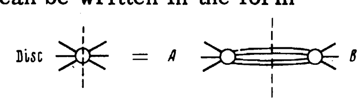
possess crossing symmetry after substitution in it of (2.2) for the different pairs of operators $\varphi(x_i)$.

We remark that knowledge of the function C is sufficient for the determination of the four-point, and even the n-point, Wightman functions, since successive use of the relation (2.2) reduces them to three-point functions, the explicit form of which is known[2]. The character of the functions C themselves is also uniquely fixed by the conformal invariance since, on substitution into the Wightman function $\langle\varphi(x)\varphi(0)O^{(jn)}(z)\rangle$, the only contribution arises from the term containing the operator $O^{(jn)}$.

Finally, our program consists in calculating all the functions C to within a few constants, substituting the operator expansion into the four-point function and finding the unknown constants from the crossing-symmetry requirement. On the basis of this program, a system of equations for the anomalous dimensions and the interaction constants at short distances will be obtained directly. Although it has not been possible to obtain a general solution of this system, particular solutions have been obtained in a number of cases (e.g., in a space of $4-\epsilon$ dimensions, where our methods lead to results equivalent to those in the approach of Wilson and Fisher[6]). In addition, general properties of the anomalous dimensions of operators with high spins have been derived.

## 3. UNITARITY CONDITIONS AND THE OPERATOR ALGEBRA

In a finite-mass theory, the unitarity condition arising from the completeness of the set of in- and out-operators can be written in the form[7]




(3.1)

This well-known equation can be interpreted in the following way. We shall consider the amplitude $F_{A\to B}$ in the space-like region for such configurations in which the relative distances between particles of group A ($r_{ij}^A$) and between particles of group B ($r_{ij}^B$) are much shorter than the distance $R_{AB}$ from A to B. In this case, the amplitude has the form

$$F_{AB}\propto\sum_n C_n{}^A(r_{ij}{}^A)C_n{}^B(r_{ij}{}^B)R_{AB}^{-\lambda_n}[\exp(-mR_{AB})]^n \qquad (3.2)$$

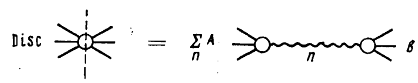
(m is the particle mass and $\lambda_n$ is a certain number). Formula (3.2) follows from the fact that the asymptotic form for $R_{AB}\to\infty$ is determined by singularities in the corresponding momentum variable, which, in turn, are given by (3.1).

Formula (3.2) has a simple physical meaning and implies that, by virtue of the small range of the interaction, correlation between the separated regions is effected by exchange of independent signals, each of which gives a contribution to the correlation proportional to $\exp(-mR_{AB})$.

What happens in the limit when the mass tends to zero? It is clear that the signals are no longer independent. Instead, signals of a new type arise, generated by the action of the operator $O_n(R)$ on the vacuum state. Such signals lead to correlation proportional to $R_{AB}^{-2\Delta_n}$ ($\Delta_n$ is the dimension of $O_n$). We can therefore expect that, in the configuration described above, $F_{AB}$ has the form

$$F_{AB}\propto\sum_n C_n{}^A(r_{ij}{}^A)C_n{}^B(r_{ij}{}^B)R_{AB}^{-2\Delta_n}. \qquad (3.3)$$

Formula (3.3) is not completely exact, since in it we have not taken account of the existence of the operators $\partial_{\mu_1}\ldots\partial_{\mu_s}O$, which lead to terms proportional to $R^{-2\Delta_n-p-q}$ (p and q are certain integers). To make (3.3) more precise, we shall consider the analog of formula (3.1) in the massless theory:




(3.4)

or

$$\mathrm{Disc}_q F_{AB}(\{p_A\},\{p_B\},q)=\sum_n F_{An}(\{p_A\},q)F_{Bn}(\{p_B\},q)\delta_q D_n(q).$$

Here, $F_{A\to n}$ is the transition amplitude, which will be determined later from the conformal invariance. It is regular at q = 0. The quantity $\delta_q D_n$ is the imaginary part of the Green function $\langle TO_n(0)O_n(x)\rangle$ in momentum space, or, which is the same thing, the Wightman function $\langle O_n(0)O_n(x)\rangle$ in this space.

We shall show that the relation written out is a consequence of the operator algebra (2.1). For simplicity, we shall consider the case when $F_{A\to B}$ is the four-point function of the operators $\varphi(x)$:

$$F(q,p,p')=\int d^4r\,d^4r'\,d^4R\exp(iqR+pr+p'r')\times\langle T\varphi(r)\varphi(0)\varphi(R)\varphi(R+r')\rangle.$$

It is easy to see that, for

$$p^2<0,\quad (q-p)^2<0,\quad p'^2<0,\quad (q-p')^2<0$$

the equality

$$\mathrm{Disc}_q F(q,p,p')=\mathrm{F.T.}\langle(T\varphi(r)\varphi(0))(\tilde{T}\varphi(R)\varphi(R+r'))\rangle \qquad (3.5)$$

holds ($\tilde{T}$ is an anti-chronological product, and F. T. symbolizes the Fourier transform).

Formula (3.5) can be verified in the finite-mass theory, by convincing oneself that the left creation amplitude in (3.1) is $\langle 0|T\varphi(z)\varphi(0)|n\rangle$, and the right is $\langle n|\tilde{T}\varphi(R)\varphi(R+r')\rangle$. Of course, (3.5) also remains valid in the zero-mass limit. We now substitute (2.1) into (3.5); we then obtain

$$\mathrm{Disc}\,F(q,p,p')=\sum_{jn}F^{jn}_{\alpha_1\ldots\alpha_j}(q,p)F^{jn}_{\beta_1\ldots\beta_j}\delta_q D^{(jn)}_{\alpha_1\ldots\alpha_j|\beta_1\ldots\beta_j}(q);$$

$$F^{jn}_{\alpha_1\ldots\alpha_j}(q,p)=\sum_s C^{(jn)}_{\alpha_1\ldots\alpha_j|\mu_1\ldots\mu_s}(p)q_{\mu_1}\ldots q_{\mu_s}. \qquad (3.6)$$

It follows from formula (3.6) that $F^{jn}(q,p)$ is analytic at q = 0. Equation (3.4) is thereby proved.

For what follows, it is important to note that the converse statement is incorrect. In order to see this and to understand the meaning of the unitarity condition, we again consider formula (3.5). If this formula were true for all values of the momenta, and not only for negative external masses, it would not be difficult to prove, by performing the Fourier transformation, that the unitarity leads to the operator algebra. However, the left-hand side of (3.5) is an analytic function of the external masses, whereas the right-hand side does not possess this property; such reasoning would therefore be incorrect.

To obtain a positive statement, we shall consider another relation:

$$\mathrm{Disc}_q F(q,\ p,\ p') = \mathrm{F.T.}\langle (R\varphi(z)\varphi(0)) A(\varphi(R)\varphi(R+r'))\rangle, \qquad (3.7)$$

where R and A denote the retarded and advanced commutators. The equivalence of (3.7) and (3.5) follows from standard arguments[7], based on the spectral condition. The usefulness of formula (3.7) is connected with the fact that, because of the retardation properties, the right-hand side is an analytic function of p in the absolute future and of p′ in the absolute past. Consequently, (3.7) remains valid for positive external masses, so long as we continue the left-hand side, putting $p \to p + i\eta$ and $p' \to p' - i\eta$ (where $\eta^2 > 0$, $\eta_0 > 0$). From the above discussion, we obtain the following theorem:

For a four-point function subject to the unitarity condition, the quantity

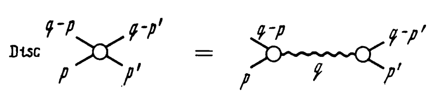
$$\langle R(\varphi(r)\varphi(0)) A(\varphi(R)\varphi(R+r'))\rangle,$$

which is the double discontinuity of the Wightman function in configuration space, satisfies the algebra.

## 4. UNITARY, CROSSING-SYMMETRIC AND CONFORMALLY INVARIANT GREEN FUNCTIONS

Because of the complexity of constructing amplitudes satisfying the operator algebra, we shall begin by constructing Green functions with the properties listed in the heading of this section. The requirements of the operator algebra will be imposed later.

We shall consider the four-point function $\langle T\varphi(x_1)\ \dots\ \varphi(x_4)\rangle$ and calculate the contribution to it of a scalar intermediate state generated by the action on the vacuum of a certain operator O with dimension d. According to (3.4), it has the form




$$= T(q,\ p)\,\mathrm{Im}\,D(q)\,T(q,\ p'). \qquad (4.1)$$

Here, because of the scale invariance,

$$\mathrm{Im}\,D(q) = \mathrm{const}\cdot(q^2)^{d-a/2},$$

holds, where a is the number of dimensions of space-time.

For the calculation of T, we note that the unitarity conditions for the vertex part $\langle TO(x)\varphi(y)\varphi(z)\rangle$ can be written in the form

$$\mathrm{Disc}_q \mathscr{T}(q,\ p) = T(q,\ p)\,\mathrm{Im}\,D(q), \qquad (4.2)$$

$$\mathscr{T}(q,p) = \int d^4R\, d^4r\, e^{iqR+ipr}\langle TO(R)\varphi(r)\varphi(0)\rangle. \qquad (4.3)$$

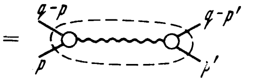
At the same time, the left-hand side of (4.2) is known from the conformal invariance. Comparing (4.3) and (4.2), we can determine T(q, p) explicitly, apart from the normalization. The corresponding expression is given in Appendix A. Substituting $T = T(q^2, p^2, (q-p)^2)$ into (4.1) and assuming that the function

$$F(q,\ p,\ p') = F(q^2,\ p^2,\ p'^2,\ (p-p')^2,\ (q-p)^2,\ (q-p')^2)$$

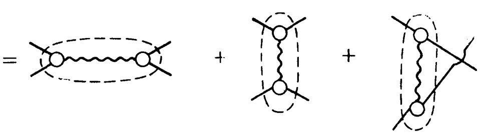
for negative external masses has only a right cut, we obtain

$$F^{(s)}(q,p,p') = \int_0^\infty ds\, \frac{s^{d-a/2}T(s,p^2,(q-p)^2)\,T(s\ p'^2,(q-p')^2)}{s-q^2+i0}. \qquad (4.4)$$

The amplitude (4.4) has no cuts in the t- and u-channels. Therefore, if we introduce the notation

$F^{(s)}(q,p,p') =$  (4.5)


the amplitude

$F(q,p,p') =$  (4.6)

will be a crossing-symmetric quantity, satisfying the unitarity condition. An analogous expression can be written for the contribution of an operator with arbitrary Lorentz spin j.

As can be seen, the unitarity conditions do not enable us to calculate the anomalous dimensions or to determine the amplitude completely. For this, we must impose the operator-algebra requirements on the expressions obtained. The solution of this problem encounters technical difficulties, associated with the necessity of calculating the amplitude (4.4) in coordinate space. Before proceeding to the treatment of general methods for overcoming these difficulties, we shall consider a simple particular case in which the difficulties are absent—the theory in a space of $4-\epsilon$ dimensions.

## 5. TREATMENT OF THE THEORY IN A SPACE OF $4-\epsilon$ DIMENSIONS, USING THE UNITARITY CONDITION

We shall consider an n-component field $\varphi_i$ in a $(4-\epsilon)$-space, and shall assume that we can achieve self-consistency by using only two operators in the expansion of $\varphi_i\varphi_j$. These two operators, $O^{(0)}$ and $O^{(2)}$, are assumed to have isotopic spins I = 0 and I = 2 and anomalous dimensions $d^{(0,2)} = 2 + \delta^{(0,2)}$, where δ is a certain small number (of order ε). The fields $\varphi_i$ are assumed to have almost canonical dimensions $\Delta = 1 - \epsilon/2 + O(\epsilon^2)$. The principal problem of this section consists in calculating the quantities $\delta^{(i)}$ from the operator algebra.

According to the assumptions made, the s-channel imaginary part of the scattering amplitude $A_{ikl m}(q, p, p')$ consists of two terms of the type (4.1):

$$\mathrm{Disc}_q A_{iklm}(q,p,p')\frac{1}{p^2(q-p)^2p'^2(q-p')^2}$$
$$= a_0\delta_{ik}\delta_{lm}T^{(0)}(q,p)T^{(0)}(q,p')\,\mathrm{Im}\,D^{(0)}(q) + a_2\left(\delta_{il}\delta_{km}+\delta_{im}\delta_{kl}-\frac{2}{n}\delta_{ik}\delta_{lm}\right)$$
$$\times T^{(2)}(qp)T^{(2)}(qp')\,\mathrm{Im}\,D^{(2)}(q) \qquad (5.1)$$

(the tensor structure of this formula corresponds to s-channel exchange of states with I = 0 and I = 2). The properties of $T^{(I)}$ for small ε and δ are extremely simple, since, if we disregard the external propagators, they are slowly-varying functions of their variables. This follows from the fact that, e.g., if the quantity $\delta = -\epsilon$, which corresponds to free fields, the vertex parts are constants.

For arbitrary but small δ, we obtain from Appendix A:

$$T_0(q,p)\equiv T(s,v_1,v_2)=\frac{1}{v_1v_2}\begin{cases}(\max(v_1,v_2))^{-\delta/2}, & s\ll v_{1,2}\\ s^{-\delta/2}, & s\gg v_{1,2}\end{cases}$$

$$\operatorname{Im} D(s)=s^{\delta+\varepsilon/2};$$

$$s=q^2,\quad v_1=p^2,\quad v_2=(q-p)^2.$$

If we represent $A(q, p, p')$ in the form

$$A_{iklm}(q,p,p')=c_0\delta_{ik}\delta_{lm}A^{(0)}(s,t,v_{1,2},w_{1,2})+c_2(\delta_{il}\delta_{km})$$
$$=(\delta_{im}\delta_{kl}-\frac{2}{n}\delta_{ik}\delta_{lm})A^{(2)}(s,t,v_{1,2},w_{1,2}),$$

we find, using the dispersion relations, that in the region $v_1\approx v_2\equiv v\gg w_1\approx w_2\equiv w$ (or $p\gg p'\gg q$)

$$A^{(I)}_{(s)}(s\ v,w)=\int_v^\infty\frac{ds'}{s'-s}\operatorname{Disc}_sA^{(I)}(s,v\ w)\approx\frac{1}{(vw)^{\delta/2}}\int_s^w\frac{ds'}{s'}(s')^{\delta+\varepsilon/2}$$
$$+\frac{1}{v^{\delta/2}}\int_w^v\frac{ds'}{s'}(s')^{(\delta+\varepsilon)/2}+\int_v^\infty\frac{ds'}{s'}(s')^{\varepsilon/2}=-\frac{1}{\delta+\varepsilon/2}\Big\{s^{\delta+\varepsilon/2}(vw)^{-\delta/2} \qquad (5.2)$$
$$+\frac{\delta}{\delta+\varepsilon}w^{(\delta+\varepsilon)/2}v^{-\delta/2}+\frac{\delta(\varepsilon+2\delta)}{\varepsilon(\varepsilon+\delta)}v^{\varepsilon/2}\Big\}$$

(in formula (5.2), higher powers of $\epsilon$ and $\delta$ are neglected).

In the region considered, the crossing variables t and u are approximately equal to v. Consequently, the contribution of the t- and u-channels is determined by the integral

$$\int_v^\infty\frac{ds'}{s'}(s')^{\varepsilon/2}=-v^{\varepsilon/2}\frac{2}{\varepsilon}. \qquad (5.3)$$

The full interaction amplitude has the form

$$A^{(I)}(q,p,p')=-\frac{a_I}{\delta_I+\varepsilon/2}\Big\{q^{2\delta_I+\varepsilon}(pp')^{-\delta_I}+\frac{\delta_I}{\delta_I+\varepsilon}(p')^{\delta_I+\varepsilon}p^{-\delta_I}\Big\}$$
$$-\frac{2}{\varepsilon}\Big\{a_I\frac{\delta_I}{\varepsilon+\delta_I}+\sum_{I'=0,2}c_{II'}a_{I'}\Big\}p^{-\varepsilon} \qquad (5.4)$$

Here $c_{II'}$ is a matrix whose elements give the admixture of isotopic spin I in the s-channel when isotopic spin I′ is exchanged in the t- and u-channels. This matrix is easily calculated and has the form

$$c_{00}=\frac{2}{n},\quad c_{02}=\frac{2n^2+2n-4}{n^2},$$
$$c_{20}=1,\quad c_{22}=\frac{n-2}{n}. \qquad (5.5)$$

To find the unknown quantities $a_I$ and $\delta_I$ we must calculate the Green functions in coordinate space and impose on them the operator-algebra condition, which reduces to the following.

We shall consider the Green function:

$$G_{iklm}(r,r',R)\equiv\langle T\varphi_i(r)\varphi_k(0)\varphi_l(R)\varphi_m(R+r')\rangle$$
$$=G^{(0)}(r,r',R)\delta_{ik}\delta_{lm}+G^{(2)}(r,r',R)\Big(\delta_{il}\delta_{km}+\delta_{im}\delta_{kl}-\frac{2}{n}\delta_{ik}\delta_{lm}\Big). \qquad (5.6)$$

Then the operator algebra

$$\varphi_i(r)\varphi_k(0)=r^{-2+\varepsilon}1\cdot\delta_{ik}+f_0r^{\varepsilon+\delta_0}\delta_{ik}O^{(0)}(0)+f_2r^{\varepsilon+\delta_2}O^{(2)}(0) \qquad (5.7)$$

(where 1 is the unit operator, and $O^{(0)}$ and $O^{(2)}$ are operators with isospin 0, 2 and anomalous dimensions $2+\delta_{0,2}$) gives the relations

$$G^{(0)}(r,r',R)\approx(rr')^{-2+\varepsilon}+f_0^2(rr')^{\varepsilon+\delta_0}R^{-4-2\delta_0}+\dots,$$
$$G^{(2)}(r,r',R)\approx f_2^2(rr')^{\varepsilon+\delta_2}R^{-4-2\delta_2}+\dots \qquad (5.8)$$

(here $r, r'\ll R$).

On the other hand, the transformation of the unitary amplitude to coordinate space is given by the formula

$$G_{iklm}(r,r',R)=\delta_{ik}\delta_{lm}(rr')^{-2+\varepsilon}+(\delta_{il}\delta_{km}+\delta_{im}\delta_{kl})R^{-4+2\varepsilon}$$
$$+\int\frac{d^{4-\varepsilon}p\,d^{4-\varepsilon}p'\,d^{4-\varepsilon}q}{p^2(q-p)^2p'^2(q-p')^2}e^{i(pr+p'r'+qR)}A_{iklm}(q,p,p'). \qquad (5.9)$$

The first term in (5.9) is the contribution of the unconnected diagrams. Formula (5.9) is equivalent to the following formula:

$$G^{(0)}(r,r',R)=(rr')^{-2+\varepsilon}+\frac{2}{n}R^{-4+2\varepsilon}\,\text{F.T.}\{A^{(0)}(q,p,p')\},$$
$$G^{(2)}(r,r',R)=R^{-4+2\varepsilon}+\text{F.T.}\ \{A^{(2)}(q,p,p')\}. \qquad (5.10)$$

The Fourier transforms are easily calculated, since in the region $r\sim r'\ll R$ only the region $q\ll p'\ll p\ll r^{-1}$ plays a role. Using this, it is not difficult to obtain the formula

$$\text{F.T.}\{A\}=R\frac{\partial}{\partial R}\int_{R^{-1}}^{r'^{-1}}\frac{dp'}{p'}(p')^{-\varepsilon}\Big(\int_{p'}^{1/r}+\int_{p'}^{1/r'}\Big)\frac{dp}{p}p^{-\varepsilon}A(q,p,p'),$$

from which follows the result

$$G^{(0)}(r,r',R)=(rr')^{-2+\varepsilon}+\frac{2}{n}R^{-4+2\varepsilon}$$
$$+\frac{a_0}{(\varepsilon+\delta_0)^2}\frac{(rr')^{\varepsilon+\delta_0}}{R^{4+2\delta_0}}-\frac{2a_0}{(\varepsilon+\delta_0)^2}R^{-4+2\varepsilon}$$
$$+\frac{2}{3}\ln\frac{R_2}{rr'}\Big\{a_0\frac{\delta_0}{\varepsilon+\delta_0}+\sum_{I'=0,2}c_{0I'}a_{I'}\Big\}, \qquad (5.11)$$

$$G^{(2)}(r,r',R)=R^{-4+2\varepsilon}+\frac{a_2}{(\varepsilon+\delta_2)^2}\frac{(rr')^{\varepsilon+\delta_2}}{R^{4+2\delta_2}}$$
$$-\frac{2}{(\varepsilon+\delta_2)^2}R^{-4+2\varepsilon}+\frac{2}{3}\ln\frac{R^2}{rr'}\Big(a_2\frac{\delta_2}{\varepsilon+\delta_2}+\sum_{I'}c_{2I'}a_{I'}\Big).$$

For arbitrary values of the parameters, the Green functions (5.11) contradict the algebra (5.7) and (5.8). However, if we select the parameters from the equations

$$a_I\frac{\delta_I}{\varepsilon+\delta_I}+\sum c_{II'}a_{I'}=0,$$
$$\frac{2}{n}=\frac{2a}{(\varepsilon+\delta_0)^2}\qquad\frac{2a_2}{(\varepsilon+\delta_2)^2}=1, \qquad (5.12)$$

we obtain an amplitude satisfying the algebra with

$$f_I^2=a_I/(\varepsilon+\delta_I)^2.$$

Solving the four equations for the four parameters (the two interaction constants $f_I$ and the two anomalous dimensions $\delta_I$), we obtain the result

$$\delta_0=-\frac{6}{n+8}\varepsilon,\quad\delta_2=-\frac{n+6}{n+8}\varepsilon;\quad f_0^2=\frac{1}{n},\quad f_2^2=\frac{1}{2}.$$

Consequently, there exists a unique set of dimensions and interaction constants for which a unitary crossing-symmetric amplitude satisfies the algebra. We note also that formula (5.11) confirms the theorem of Sec. 2—the double discontinuity of (5.11) with respect to $r^2$ and $r'^2$ coincides with the algebraic amplitude for an arbitrary choice of the parameters. Equation (5.12) was obtained from the condition for cancellation of the quasi-regular terms of the unitary amplitude, which made no contribution to the double discontinuity.

To obtain equations analogous to (5.12) in the general case, it is necessary to modify the primitive calculational methods of this section. For these purposes, an expansion in conformal partial waves turns out to be extremely useful, and we proceed to study this.

## 6. EXPANSION IN CONFORMAL PARTIAL WAVES

It is well known[8] that the scattering amplitude at zero momentum transfer can be expanded in the Lorentz partial waves corresponding to representations of the group O(4). This expansion is extremely convenient, since it diagonalizes the Bethe-Salpeter equation and other dynamical equations. However, in a finite-mass theory, this expansion cannot be generalized to the case of finite momentum transfers.

In conformal field theory, such a generalization turns out to be possible, and extremely useful for our purposes. This possibility is connected with the fact that a conformal transformation can change the momentum transfer and, in particular, transform the scattering amplitude at zero angle into an amplitude at arbitrary angle. This procedure is unique, since the conformal transformation depends on four parameters and by selecting these we can obtain an amplitude with any given momentum transfer.

To construct the required expansion, we select those conformally invariant expressions for the four-point function which, at zero momentum transfer, go over into the individual terms of the O(4) expansion. With this purpose, we shall consider the quantity $\mathcal{T}^{(\nu)}_{\alpha_1\ldots\alpha_j}$, which is defined as the three-point function

$$\langle TO^{(\nu)}_{\alpha_1\ldots\alpha_j}(z)\varphi(r_1)\varphi(r_2)\rangle,$$

where $O^{(\nu)}_{\alpha_1\ldots\alpha_j}$ is a fictitious operator of dimension $d \equiv \nu + 2$. The conformal invariance fixes $\mathcal{T}^{(\nu)}$ to within the normalization.

We note further that the amplitude

$$a^{(\nu j)}(r_{ik}) \equiv \int d^4z\, \mathcal{T}^{(\nu)}_{\alpha_1\ldots\alpha_j}(r_1 r_2 z)\mathcal{T}^{(-\nu)}_{\alpha_1\ldots\alpha_j}(r_3 r_4 z) \tag{6.1}$$

transforms like the four-point function

$$\langle T\varphi(r_1)\varphi(r_2)\varphi(r_3)\varphi(r_4)\rangle,$$

since the formal operator

$$\int d^4z\, O^{(\nu)}_{\alpha_1\ldots\alpha_j}(z)O^{(-\nu)}_{\alpha_1\ldots\alpha_j}(z)$$

is conformally invariant (a similar observation is made in[9]).

We shall show that an arbitrary four-point function can be represented in the form

$$A(r_{ik}) = \sum_{j=0}^{\infty}\int_{-i\infty}^{+i\infty} d\nu\, \psi_P(\nu)\, a_{\nu j}(r_{ik}), \tag{6.2}$$

and shall obtain the inversion formula for this integral transform. For this, we rewrite (6.2) in the momentum representation:

$$\begin{gathered} A(q,p,p') = \sum_{j=0}^{\infty}\int_{-i\infty}^{+i\infty} d\nu\, \psi_j(\nu)\mathcal{T}^{(\nu)}_{\alpha_1\ldots\alpha_j}(q,p)\mathcal{T}^{(-\nu)}_{\alpha_1\ldots\alpha_j}(q,p'), \\ A(q,p,p') = \int A(r,r',R)\, e^{i(pr+p'r'+qR)}\, dr\, dr'\, dR, \\ \mathcal{T}_{\alpha_1\ldots\alpha_j}(q,p) = \int \mathcal{T}_{\alpha_1\ldots\alpha_j}(r,R)\, e^{i(pr+qR)}\, dr\, dR \end{gathered} \tag{6.3}$$

(where $r_1 = 0$, $r_2 = r$, $r_3 = R$, and $r_4 = R + r'$).

For zero momentum transfer, we obtain

$$\begin{aligned} A(p,p',0) &= (pp')^{2\Delta-6}\sum_j U_j\left(\frac{\mathbf{pp}'}{pp'}\right) \\ &\times \int_{-i\infty}^{+i\infty} d\nu\, \psi_j(\nu)M_j(\nu)M_j(-\nu)\left(\frac{p}{p'}\right)^{\nu}. \end{aligned} \tag{6.4}$$

Here, $U_j$ is a Chebyshev polynomial, and $M_j(\nu)$ is defined by the formula (cf. Appendix B)

$$\mathcal{T}_{\alpha_1\ldots\alpha_j}(p,0) = \int d\mathbf{r}\, e^{i\mathbf{pr}}\int \frac{d\mathbf{R}}{R^{\nu+2}}\frac{[\nu_{\alpha_1}\ldots\nu_{\alpha_j}]}{|R-r|^{\nu+2}} = M_j(\nu)\, p^{2\Delta-6-\nu-j}[p_{\alpha_1}\ldots p_{\alpha_j}]. \tag{6.5}$$

The integral (6.5) must be understood as an analytic continuation from its region of convergence. In the derivation of (6.4), the relation

$$p^{-j}[p_{\alpha_1}\ldots p_{\alpha_j}](p')^{-j}[p_{\alpha_1}'\ldots p_{\alpha_j}'] = U_j\left(\frac{\mathbf{pp}'}{pp'}\right).$$

was used. Formula (6.4) is the O(4) expansion for A, the partial waves of which, in turn, are expanded in the dimensionless variable $p/p'$ in a Mellin integral.

Thus, formula (6.4) solves the problem posed, by giving a conformally invariant expansion that goes over into the O(4) expansion at zero momentum transfer.

The inversion formula for the transformation (6.2) is found using formula (6.4). Taking the inverse Mellin transform, we find

$$\begin{aligned} \psi_j(\nu) &= \frac{1}{M_j(\nu)M_j(-\nu)}\int \frac{d^4p'}{|p'|^4}U_j\left(\frac{\mathbf{pp}'}{pp'}\right) \\ &\times \left(\frac{p'}{p}\right)^{\nu}(pp')^{6-2\Delta}A(p,p'). \end{aligned} \tag{6.6}$$

Another equivalent inversion formula is obtained from consideration of the expression

$$A(\mathbf{r},\mathbf{r}',\mathbf{q}) = \sum_{j=0}^{\infty}\int d\nu\, \psi_j(\nu)\mathcal{T}^{(\nu)}_{\alpha_1\ldots\alpha_j}(r,q)\mathcal{T}^{(-\nu)}_{\alpha_1\ldots\alpha_j}(r',q). \tag{6.7}$$

Repeating the previous arguments, we find

$$\psi_j(\nu) = \frac{1}{N_j(\nu)N_j(-\nu)}\int \frac{d^4r_3\, d^4r_4}{r_{34}{}^8}\left(\frac{r_{12}}{r_{34}}\right)^{\nu+2}(r_{12}r_{34})^{2\Delta}A(r_{ik}), \tag{6.8}$$

where $N_j$ is defined by the formula

$$\int d^4R\, \mathcal{T}^{(\nu)}_{\alpha_1\ldots\alpha_j}(r,R) = N_j(\nu)\, r^{2-2\Delta-j-\nu}[r_{\alpha_1}\ldots r_{\alpha_j}].$$

The final results of this section consist of the expansion (6.2) in conformal partial waves, and the inversion formulas (6.6) and (6.8). Explicit expressions for $M_j(\nu)$ and $N_j(\nu)$ will be given in Appendix B.

## 7. CONSTRUCTION OF A UNITARY AMPLITUDE FOR ARBITRARY SPIN

In this section, the spectral function for a unitary amplitude will be calculated. The most direct way of doing this is to put $q = 0$ in (4.4) and use formula (6.5). Technically, however, this method is extremely complicated for non-zero spin values. We shall therefore use another method, based on the theorem of Sec. 3.

First we shall construct a conformally invariant operator expansion for $R(\varphi(x)\varphi(0))$ and $A(\varphi(x)\varphi(0))$ and calculate the quantity

$$\langle R(\varphi(r)\varphi(0))A(\varphi(R)\varphi(R+r'))\rangle.$$

We shall then select a function $\psi(\nu)$ such that the double discontinuity of (6.2) turns out to be equal to the expression obtained. If the resulting amplitude has the correct analytic properties in coordinate and momentum space, then, according to the theorem of Sec. 3, it will satisfy the unitarity condition and the dispersion relations.

The most general form of the contribution of the given operator to the retarded commutator is given by the formula

$$R(\varphi(x)\varphi(0)) = \sum_k a_{\alpha_1\ldots\alpha_j|\mu_1\ldots\mu_k}(x)\,\partial_{\mu_1}\ldots\partial_{\mu_k}O_{\alpha_1\ldots\alpha_j}, \tag{7.1}$$

where a is connected with the function C by the obvious relation

$$a(x) = \theta(x_0)[C(x-i\eta) - C(x+i\eta)]. \tag{7.2}$$

The function a, like C, is uniquely determined from the condition that the quantity

$$\langle R(\varphi(x)\varphi(0))O_{\alpha_1\ldots\alpha_j}(z)\rangle$$

should be equal to the vertex part. It is more convenient, however, to rewrite (7.1) in an explicitly conformally-covariant form.

This problem has already been considered by Ferrara, Grillo and Gatto[9], but the results of their work are inadequate for our purposes. In[9], it was asserted that the contribution of the operator O can be written in the form

$$\varphi(x)\varphi(0)\sim\int T(x,z)\tilde{O}(z)\,d^4z, \qquad (7.3)$$

where T(x, z) is the vertex part of the "shadow" operator $\Omega(z)$ with dimension $4-d$. This statement is correct only if we understand the integral (7.3) in a special sense. Namely, we must write

$$O(z)=O(0)+z_\mu\partial_\mu O(0)+{}^1/_2 z_\mu z_\nu\partial_\mu\partial_\nu O+\ldots,$$

substitute this expansion into the integral, and understand each of the integrals in the resulting sum in the sense of an analytic continuation from its region of convergence. Since these regions are different for different terms in the sum, the resulting Taylor series cannot be summed under the integral sign. In its original form, (7.3) violates causality, since the integration is taken over all space, while the left-hand side must necessarily commute with the operator O(y) if $y^2 < 0$ and $(x-y)^2 < 0$.

Thus, for an ordinary product, it is impossible to write convenient explicitly covariant formulas. For the retarded commutator, the situation is essentially different. We shall consider the expression

$$R(\varphi(x)\varphi(0))=\text{const}\times\int d^4z\,\theta(x-z)\theta(z)\langle\varphi(x)\varphi(0)\Omega_{\alpha_1\ldots\alpha_j}(z)\rangle O_{\alpha_1\ldots\alpha_j}(z), \qquad (7.4)$$

where $O_{\alpha_1\ldots\alpha_j}$ is an operator of dimension d; $\theta(x)\equiv\theta(x_0)\theta(x^2)$; $\Omega_{\alpha_1\ldots\alpha_j}$ is a fictitious operator of dimension $4-d$, so that the three-point function in (7.4) is given by (B.1). If (7.4) did not contain the $\theta$-functions, it would be explicitly covariant. It is not difficult to see that the $\theta$-functions do not violate the conformal invariance, since under the transformation

$$x_\mu'=x_\mu+\alpha_\mu x^2-2(\alpha x)x_\mu\equiv x_\mu+\delta x_\mu,$$
$$x'^2=(1-2\alpha x)x^2=x^2+\delta x^2$$

they transform like

$$\theta(x')=\theta(x)+\delta(x_0)\theta(x^2)\delta x_0+\delta x^2\theta(x_0)\delta(x^2)=\theta(x). \qquad (7.5)$$

The meaning of the result obtained is that the light cone is conformally invariant.

Comparing (7.1) and (7.4), we obtain

$$a_{\alpha_1\ldots\alpha_j|\mu_1\ldots\mu_k}(x)=\text{const}\cdot\frac{1}{k!}\int d^4z\,z_{\mu_1}\ldots z_{\mu_k}\theta(x-z)\theta(z)\langle\varphi(x)\varphi(0)\Omega_{\alpha_1\ldots\alpha_j}(z)\rangle. \qquad (7.6)$$

The integral (7.6) converges in the usual sense for all k, since the range of integration, defined by the $\theta$-functions, lies between the two light cones. It can also be verified that, because of the presence of the $\theta$-functions, Eq. (7.6) satisfies the causality condition.

We now establish the correspondence between the algebraic amplitude equal to

$$\langle R(\varphi(r)\varphi(0))A(\varphi(R)\varphi(R+r'))\rangle$$

$$=\int d^4z\,\theta(x-z)\theta(z)\langle\varphi(r)\varphi(0)\Omega_{\alpha_1\ldots\alpha_j}(z)\rangle\langle O_{\alpha_1\ldots\alpha_j}(z)A(\varphi(R)\varphi(R+r'))\rangle. \qquad (7.7)$$

and the usual "pole" term

$$\langle T\varphi(x)\varphi(0)\varphi(R)\varphi(R+y)\rangle=\int d^4z\langle T\varphi(x)\varphi(0)\Omega_{\alpha_1\ldots\alpha_j}(z)\rangle\times\langle TO_{\alpha_1\ldots\alpha_j}(z)\varphi(y)\varphi(R+y)\rangle\equiv(\mathcal{T}^{(4-d)}_{\alpha_1\ldots\alpha_j}\mathcal{T}^{(d)}_{\alpha_1\ldots\alpha_j})\equiv a^{(\nu,j)}. \qquad (7.8)$$

We shall show that there exists a relation

$$a^{(\nu,j)}(r_{ab})=N_j(-\nu)f_{\nu j}+N_j(\nu)f_{-\nu j}(r_{ab}), \qquad (7.9)$$

where $f_{\nu j}(r_{ab})$ is the T-product corresponding to the operator amplitude, and $N_j(\nu)$ are the functions introduced in Sec. 4 and calculated in Appendix B. For the proof, we shall calculate the discontinuity of the amplitude $a^{(\nu j)}$ with respect to $q^2$ in the momentum representation. In the calculation, it is necessary to use the two relations:

$$\delta_q\mathcal{T}^{(\nu)}(q,p)=T^{(\nu)}(q,p)\delta_q D^{(\nu)}(q)=\text{const}\cdot T^{(\nu)}(q,p)D^{(\nu)}(q), \qquad (7.10)$$

$$\mathcal{T}^{(+\nu)}(q,p)=\text{const}\cdot D^{(\nu)}(q)\mathcal{T}^{(-\nu)}(q,p). \qquad (7.11)$$

The relation (7.10) is the unitarity condition for the vertex, and the relation (7.11) follows from the fact, noted in Sec. 7, of the conformal invariance of

$$\int d^4z\,O^{(\nu)}(z)O^{(-\nu)}(z)$$

and from the uniqueness of the three-point function. If, using (7.10), we write

$$\mathcal{T}^{(\nu)}(q,p)=F^{(\nu)}(q,p)+T^{(\nu)}(q,p)D^{(\nu)}(q)$$

(where $F^{(\nu)}$ is regular at q = 0), the relation (7.11) leads to the equality

$$F^{(\nu)}(q,p)=T^{(-\nu)}(q,p),$$
$$\mathcal{T}^{(\nu)}(q,p)=T^{(-\nu)}(q,p)+T^{(\nu)}(q,p)D^{(\nu)}(q). \qquad (7.12)$$

The tensor indices, which we have omitted to write in deriving (7.12), do not change the situation. Formula (7.12) is verified by the calculation in Appendix A, where an explicit formula for $\mathcal{T}(q,p)$ is given in the spin-zero case.

Substituting (7.12) into (7.8), we find

$$a^{(\nu)}(q,p)=T^{(-\nu)}(q,p)T^{(\nu)}(q,p')+T^{(\nu)}(q,p)T^{(-\nu)}(q,p')$$
$$+T^{(\nu)}(q,p)D^{(\nu)}(q)T^{(\nu)}(q,p')$$
$$+T^{(-\nu)}(q,p)D^{(-\nu)}(q)T^{(-\nu)}(q,p') \qquad (7.13)$$

(here we have used the equality $D^{(\nu)}(q)D^{(-\nu)}(q)=1$). In formula (7.13), only the last two terms are singular at q = 0 (we recall that $T^{(\nu)}(q,p)$ is analytic at q = 0, and $D^{(\nu)}(q)\propto(q^2)^\nu$). Comparing with formula (3.6) for the algebraic amplitude $f_{\nu j}$ and using the dispersion relation, we convince ourselves that formula (7.9) is correct to within the normalization factors $N_j(\nu)$.

To find the normalization factors, we fix the normalization of $f_{\nu j}$ by the condition

$$f_{\nu j}(x,y,R)\underset{x,y\to 0}{\approx}x^{d-2\Delta}y^{d-2\Delta}\frac{1}{R^{2d}}U_j\left(\frac{x_\alpha g_{\alpha\beta}(R)y_\beta}{xy}\right)$$
$$g_{\alpha\beta}(R)=\delta_{\alpha\beta}-2R_\alpha R_\beta R^{-2}. \qquad (7.14)$$

Comparing with the asymptotic form of (7.14)

$$a^{(\nu j)}\underset{x,y\to 0}{\approx}y^{d-2\Delta-j}\frac{y_{\alpha_1}\cdots y_{\alpha_j}}{R^{2d}}g_{\alpha_1\beta_1}(R)\ldots g_{\alpha_j\beta_j}\cdot\int d^4z\,\langle T\varphi(x)\varphi(0)\Omega_{\alpha_1\alpha_j}(z)\rangle, \qquad (7.15)$$

where we have used the equality

$$n_\alpha=\frac{|R+y|R}{y}\left(\frac{R_\alpha}{R^2}-\frac{(R+y)_\alpha}{(R+y)^2}\right)\underset{y\to\infty}{\approx}\frac{y_\beta}{y}g_{\alpha\beta}(R),$$

we convince ourselves of the correctness of formula (7.9). This formula enables us to construct, without explicit calculations, a conformal expansion for the unitary amplitude and algebraic amplitude.

We shall begin by constructing the spectral function $\psi_j(\nu)$ for the algebraic amplitude, i.e., we shall seek the $\psi_j(\nu)$ ( such that

$$f_{\nu j}(r_{ab}) = \int_{-i\infty}^{+i\infty} \psi_j(\mu)\, a_{\nu j}(r_{ab})\, d\mu = 2\int_{-i\infty}^{+i\infty} \psi_{j\nu}(\mu) N_j(-\mu) f_{\mu j}(r_{ab})\, d\mu, \tag{7.16}$$

where, as shown in Appendix B,

$$N_j(\nu) = \frac{1}{\nu(\nu+j+1)} \beta\left(1+\frac{j-\nu}{2}\right) \Big/ \beta\left(1+\frac{j+\nu}{2}\right). \tag{7.17}$$

($\beta(x) \equiv \Gamma^2(x)/\Gamma(2x)$). The function $\psi_{j\nu}(\mu)$ should be an even function (this has already been used in (7.16)) and, in addition, $\psi_{j\nu}(\mu) N_j(-\mu)$ should have only one pole in the right half plane, at $\mu = \nu$, since it can be shown that $f_{\mu j}(r_{ab})$ is regular for Re $\mu > 0$; furthermore, $\psi(\mu)$ should fall off sufficiently rapidly as $\mu \to \infty$ for the integral to converge and to make it possible to close the contour to the right.

The only function satisfying all these requirements is

$$\psi_{j\nu}(\mu) = \text{const} + \frac{\chi_j(\mu)\chi_j(-\mu)}{\mu^2-\nu^2}, \qquad \chi_j(\mu) = \mu(j+1-\mu)\beta\left(1+\frac{j-\mu}{2}\right). \tag{7.18}$$

As shown in Appendix B, the algebraic amplitude possesses anomalous singularities in coordinate space, associated with the insufficiently rapid convergence of the integral (7.16).

We turn now to the construction of a spectral function for the unitary $U(r_{ab})$. This amplitude should not have anomalous singularities, and so $\psi(\mu)$ should fall off faster as $\mu \to \pm i\infty$ than in the case of the algebraic amplitude. Moreover, by virtue of the theorem of Sec. 3, the double discontinuities with respect to $x^2$ and $y^2$ should be the same for these amplitudes. Consequently, the new poles $\{\nu_n\}$ of $\psi(\mu)$, which ensure the rapid fall-off of the latter, should be positioned in such a way that the residues, proportional to $f(x, y, R)$, at these poles do not have double discontinuities with respect to $x^2$ and $y^2$. For small $x^2$ and $y^2$, the function $f_{\nu j}$ is given by formula (7.15), from which it is clear that the double discontinuities vanish if

$$d/2-\Delta-j/2 \equiv a/4+\mu/2-\Delta-j/2 = n \quad (n=0,1,2,\ldots). \tag{7.19}$$

Hence it follows that double poles at the points (7.19) are possible additional singularities in $\mu$.

It is easy to see that the spectral function with the most rapid fall-off along the imaginary axis and with the properties described is

$$\psi_{j\delta}{}^{U}(\mu) = \frac{g_j(\mu) g_j(-\mu)}{\mu^2-\delta^2}, \tag{7.20}$$

where

$$g_j(\mu) = \mu(j+1-\mu)\beta\left(1+\frac{j-\mu}{2}\right)\Gamma^2\left(\Delta-\frac{a}{4}-\frac{\mu}{2}+\frac{j}{2}\right).$$

The formula (7.20) solves the problem of finding the unitary amplitude. A more rigorous (but highly cumbersome) derivation of it for j = 0 is given in Appendix B. It is not difficult to see that it was precisely the double poles of the square of the Γ-function in (7.20) which were the source of the logarithmic terms in the formula (5.11).

## 8. CONSTRUCTION OF A CROSSING-SYMMETRIC ALGEBRAIC AMPLITUDE AND OF AN EQUATION FOR THE ANOMALOUS DIMENSIONS

The formulas of the preceding section enable us to construct a crossing-symmetric interaction amplitude, which will differ from the algebraic amplitude only by the logarithmic terms associated with the double poles of the spectral functions. The requirement that the residues at these poles vanish will give us a system of equations for the anomalous dimensions and internal interaction constants.

In carrying out this program, it is extremely important that the unitary amplitudes are constructed in such a way as to have the correct analytic properties in coordinate space, and therefore the expansion of the full amplitude in the unitary amplitudes should converge in the whole Euclidean region of the external variables. (A divergence of the series would lead to anomalous singularities). This property distinguishes the unitary from the algebraic amplitudes, making them more convenient to work with.

Thus, we shall consider the following expansion for the amplitude:

$$A = \sum_{j,n} C_{jn}[V_{jn}^{(s)}(r_{ik}) + V_{jn}^{(t)}(r_{ik}) + V_{jn}^{(u)}(r_{ik})], \tag{8.1}$$

where $V^{(s)}$ is an amplitude of the type (4.4). Here j is the operator spin, n is the "principal quantum number" (which distinguishes operators of a given spin but different dimensions; the dimensions are assumed to increase with the label n). As already stated, the series (8.1), if it represents the true amplitude, should converge for Euclidian $r_{ik}$ (of course, in the non-Euclidean region it diverges, generally speaking; this divergence leads to the appearance of Mandelstam spectral functions in the amplitude). This gives the possibility of calculating the asymptotic form of (8.1) for $x, y \ll R$. We have ($\mathbf{r}_1 = 0$, $\mathbf{r}_2 = \mathbf{x}$, $\mathbf{r}_3 = \mathbf{R}$, $\mathbf{r} = \mathbf{R} + \mathbf{y}$):

$$\begin{aligned} V_{jn}^{(s)}(r_{ik}) &= \int_{-i\infty}^{+i\infty} d\nu \frac{g_j(\nu) g_j(-\nu) N_j(-\nu)}{\nu^2-(d_{jn}-2)^2} f_{\nu j}(r_{ik}) \\ &= \int_{-i\infty}^{+i\infty} d\nu \frac{\nu(j+1+\nu)}{\nu^2-(d_{jn}-2)^2}\beta^2\left(1+\frac{j+\nu}{2}\right)\Gamma^2\left(\Delta-1+\frac{j+\nu}{2}\right) \\ &\times \Gamma^2\left(\Delta-1+\frac{j-\nu}{2}\right)\left(\frac{xy}{R^2}\right)^{\nu+2} U_j\left(\frac{x_\alpha g_{\alpha\beta}(R) y_\beta}{xy}\right) x^{-2\Delta} y^{-2\Delta} \\ &\approx \left\{ \alpha_{jn}\left(\frac{xy}{R^2}\right)^{d_{jn}} U_j\left(\frac{x_\alpha g_{\alpha\beta}(R) y_\beta}{xy}\right) \right. \\ &\left. + \left(\frac{xy}{R^2}\right)^{2\Delta+j} \beta_{jn} \ln\frac{R^2}{xy} + \gamma_{jn}\left(\frac{xy}{R^2}\right)^{2\Delta+j} \right\} x^{-2\Delta} y^{-2\Delta}, \end{aligned} \tag{8.2}$$

where

$$\alpha_{jn} = (d_{jn}+j-1)\beta^2\left(\frac{j+d_{jn}}{2}\right)\Gamma^2\left(\Delta+\frac{j-d_{jn}}{2}\right)\Gamma^2\left(\Delta+\frac{j+d_{jn}}{2}-2\right),$$

$$\beta_{jn} = \frac{\Gamma^2(2\Delta-2+j)\beta^2(j+\Delta)}{(2\Delta-2+j)^2-(d_{jn}-2)^2},$$

$$\gamma_{jn} = \frac{d}{d\nu}\left\{\frac{\nu(j+1+\nu)}{\nu^2-(d_{jn}-2)^2}\beta^2\left(1+\frac{j+\nu}{2}\right)\Gamma^2\left(\Delta-1+\frac{j+\nu}{2}\right) \cdot \Gamma^2\left(\Delta-1+\frac{j-\nu}{2}\right)\left(\Delta-1+\frac{j-\nu}{2}\right)\right\}\Bigg|_{\nu=2\Delta-2+j}.$$

The asymptotic form in the crossing channel is given by the expression

$$a_{\nu j}(r_{ik}) = \int dz \frac{U_j(n_{12}n_{34})}{|r_1-z|^d |r_2-z|^d |r_3-z|^{\tilde d} |r_4-z|^{\tilde d}} \cdot r_{12}^{d-2\Delta} r_{34}^{\tilde d-2\Delta} \underset{\substack{r_{12}\to 0\\ r_{34}\to 0}}{\approx}$$

$$\approx \frac{1}{R^{4\Delta}} \ln \frac{R^2}{xy}, \tag{8.3}$$

where

$$\mathbf{n}_{ik} = \frac{|\mathbf{r}_i - \mathbf{z}||\mathbf{r}_k - \mathbf{z}|}{r_{ik}} \left( \frac{\mathbf{r}_i - \mathbf{z}}{(\mathbf{r}_i - \mathbf{z})^2} - \frac{\mathbf{r}_k - \mathbf{z}}{(\mathbf{r}_k - \mathbf{z})^2} \right);$$

$$d = 4 - \tilde{d} = \nu + 2, \quad R \approx r_{12}, \quad x \approx r_{13}, \quad y \approx r_{24}.$$

Using the formulas (8.3) and (7.20), we find the contribution of the last two terms in (8.1):

$$V_{jn}^{(t)} + V_{jn}^{(u)} \approx \frac{2\rho_{jn}}{R^{4\Delta}} \ln \frac{R^2}{xy},$$

$$\rho_{jn} = \int_{-i\infty}^{+i\infty} \frac{d\nu\, g_j(\nu) g_j(-\nu)}{\nu^2 - (d_{jn} - 2)^2}.$$

It is not difficult to see that, on subsequent expansion of (8.3) in x and y, terms of the type

$$x^{-2\Delta} y^{-2\Delta} \left( \frac{xy}{R^2} \right)^{2\Delta + l + m} \ln \frac{R^2}{xy} U_l \left( \frac{x_\alpha g_{\alpha\beta}(R) y_\beta}{xy} \right)$$

will arise, where $l$ and m are positive integers.

Thus, the structure of the three terms in formula (8.1) for small x and y is as follows. The unitary amplitude in the direct channel (8.2) contains a contribution from the operator with spin $d_{jn}$ (the term with $d_{jn}$) and, in addition, has terms that are regular and logarithmic in x and y. The unitary amplitudes in the crossing channels contain terms logarithmic in x and y. The true amplitude should be subject to the algebra and should not contain regular and logarithmic terms. Consequently, the logarithmic terms in the direct channel should cancel with the corresponding terms in the crossing channels (we shall be concerned with the regular terms a little later).

We write the condition for cancellation of the logarithmic terms with zero spin:

$$\sum_n C_{0n} \beta_{0n} + 2 \sum_{j,n} \rho_{jn} C_{jn} = 0.$$

If in place of the $C_{jn}$ we introduce the interaction constants (or normalization constants of the three-point vertices) $f_{jn}$, we obtain the formula

$$\sum_{n=0}^{\infty} f_{0n}^2 \frac{\beta_{0n}}{\alpha_{0n}} + \sum_{j,n} f_{jn}^2 \frac{\rho_{jn}}{\alpha_{0n}} = 0. \tag{8.4}$$

It is not difficult to see that (8.4) is the exact variant of the first of the approximate equations (5.12). To obtain the analog of the second of these equations, we consider the condition for cancellation of the regular terms with zero spin. For this, we separate out from (8.1) the contribution of the unit operator or, which is the same thing, the contribution of the unconnected diagrams:

$$A = r_{12}^{-2\Delta} r_{34}^{-2\Delta} + r_{13}^{-2\Delta} r_{24}^{-2\Delta} + r_{14}^{-2\Delta} r_{34}^{-2\Delta} + \sum_{j,n} C_{jn} \{\ldots\}. \tag{8.5}$$

The second and third terms in formula (8.5) are proportional to $R^{-4\Delta}$ for x, y → 0. Consequently, the cancellation condition will have the form

$$2 + \sum \gamma_{0n} C_n = 0. \tag{8.6}$$

and this is the exact analog of the second Eq. (5.12).

Up to now, we have considered the condition for cancellation of the anomalous terms with zero spin and zero principal quantum number. As can be seen from formula (8.2), other anomalies, associated with the next poles of the square of the Γ-function, are also present in the amplitude. They give a contribution of the form

$$U_j \left( \frac{x_\alpha g_{\alpha\beta}(R) y_\beta}{xy} \right) \left( \frac{xy}{R^2} \right)^{2\Delta + j + 2m} \left[ \beta_{jn}^{(m)} \ln \frac{R^2}{xy} + \gamma_{jn}^{(m)} \right] \tag{8.7}$$

(m = 0, 1, ...). However, a contribution of precisely such a structure arises from the crossing-channel terms. By requiring that the anomalies cancel, we obtain

$$\sum_n \beta_{jn}^{(m)} C_{jn} + 2 \sum_{l,n} \rho_{ln}^{(jm)} C_{ln} = 0 \tag{8.8}$$

and, analogously to (8.6),

$$\sum \gamma_{jn}^{(m)} C_n + \sum \eta_{ln}^{(jm)} C_{ln} + 2\xi_{jm} = 0 \tag{8.9}$$

(here, $\xi_{jm}$ is the component corresponding to spin j and dimensions $\Delta_m = 2\Delta + j + 2m$ from the unconnected diagrams).

In the case of different incoming and outgoing particles, i.e., for the reaction AB → CD, it is not difficult to write the generalization of Eq. (8.8):

$$\sum_n \mathscr{K}_{AB|CD}^{(jnm)} f_{AB}^{(jn)} f_{CD}^{(jn)} + \sum_{l,n} \mathscr{L}_{AB|CD}^{(jm|lm)} (f_{AC}^{(ln)} f_{BD}^{(ln)} + f_{AD}^{(ln)} f_{BC}^{(ln)}) = 0. \tag{8.10}$$

We have been able to calculate the functions $\mathscr{K}$ and $\mathscr{L}$, which are very simply connected with the generalization of β and ρ to the case of different incoming particles, only for scalar external particles and for m = 0. Therefore, we can write out explicitly only a part of the equations for the dimensions and coupling constants.

## 9. DISCUSSION OF THE EQUATIONS, AND ADDITIONAL CONDITIONS ON THE SOLUTIONS

In the preceding section, it was shown that the requirements of the operator algebra lead to a system of equations for the interaction constants and the anomalous dimensions. Unfortunately, it has been possible to write down explicitly only some of the equations arising, which are found to be extremely complicated. We should not be surprised about this, since the equations we are studying replace the summation of all Reynman diagrams.

The relations we have written out can be regarded as the exact sum rules that are satisfied by the anomalous dimensions and coupling constants. It has been shown that if these are fulfilled, the Green functions satisfy the algebra. The question of the validity of the converse statement arises. We have not been able to prove it, but an examination of a number of models in which our sum rules are fulfilled makes this statement plausible.

We note, finally, that is is necessary to supplement the equations with an additional condition concerning the existence of conserved operators with normal dimensions. For example, in the channel with j = 2, an energy-momentum tensor with d = 4, whose coupling constants with all other fields are universal, must necessarily be present. This requirement is supplementary to the Wightman axioms, which guarantee the existence only of the total momentum, but not of its density.

A general study of the solutions of Eqs. (8.4) and (8.6) and a discussion of the possibility of their experimental verification go beyond the scope of the present article.

## APPENDIX A

## THE FOURIER TRANSFORMATION OF THE VERTEX PART

For a scalar operator O, the vertex ⟨TO(R)φ(r)φ(0)⟩ has the following Fourier transform:

$$\mathscr{T}(q,p) = \int d^4 r\, e^{ipr} r^{d-2\Delta} \int \frac{d^4 R e^{iqR}}{R^d |R-r|^d}. \tag{A.1}$$

To calculate this integral, we use a Gaussian parametrization of the propagators, which gives

$$\mathcal{T}(q,p)=\text{const}\cdot\int_0^\infty dx\,dy\,dz\,\frac{(xy)^{d/2-1}z^{\Delta-d/2-1}}{(x+y+z)^2}\times\exp-\left\{\frac{xyq^2+yzp^2+xz(q+p)^2}{x+y+z}\right\}. \tag{A.2}$$

Making the replacement $z = v\rho$, $x = \rho\lambda$, $y = \rho(1-\lambda)$, calculating the integrals over v and $\rho$ and using properties of the hypergeometric function, we obtain

$$\begin{gathered}\mathcal{T}(q,p)=N(d)T_{4-d}(q,p)+L(d)T_d(q,p)q^{2d-4};\\ N(d)=\frac{\Gamma(d-2)}{\Gamma(-d)}\frac{\Gamma^2(-d/2)}{\Gamma^2(d/2+2)},\quad L(d)=\frac{\Gamma(2-d)}{\Gamma(d)},\\ T_d(q,p)=\frac{1}{B(d/2,d/2)}\int_0^1\frac{d\lambda[\lambda(1-\lambda)]^{d/2-1}}{[\lambda p^2+(1-\lambda)(p+q)^2]^{2+d/2-\Delta}}\\ \times{}_2F_1\left(2+\frac{d}{2}-\Delta,\Delta+\frac{d}{2}-2,\frac{\lambda(1-\lambda)q^2}{\lambda p^2+(1-\lambda)(p+q)^2}\right).\end{gathered} \tag{A.3}$$

## APPENDIX B

## CALCULATION OF $N_j(\nu)$ AND $M_j(\nu)$

To calculate $N_j$, we write an explicit expression for the three-point function:

$$\begin{gathered}\langle TO_{\alpha_1\ldots\alpha_j}(0)\varphi(y)\varphi(z)\rangle=|y|^{-d}|z|^{-d}|y-z|^{d-2\Delta}[\nu_{\alpha_1}\ldots\nu_{\alpha_j}],\\ \nu_\alpha=\frac{|y||z|}{|y-z|}\left(\frac{y_\alpha}{y^2}-\frac{z_\alpha}{z^2}\right).\end{gathered} \tag{B.1}$$

we multiply (B.1) by the light-like vector $\xi$, and put $r^2 = (y-z)^2 = 1$; then, by virtue of the definition of $N_j$, we obtain

$$\int\frac{d^4R}{(R^2-i0)^{(\nu+2-j)/2}((R-r)^2-i0)^{(\nu+2-j)/2}}\left(\frac{\xi R}{R^2}-\frac{\xi(R-r)}{(R-r)^2}\right)^j=N_j(\nu)(\xi r)^j. \tag{B.2}$$

From (B.2), we find

$$N_j(\nu)=\sum_{m_1+m_2=j}\frac{j!}{m_1!\,m_2!}J^j_{m_1m_2}, \tag{B.3}$$

where

$$J^j_{m_1m_2}=\int d^4R\,\frac{(\xi R)^{m_1}(\xi(R-r))^{m_2}}{(R^2-i0)^{(\nu+2-j+2m_1)/2}((R-r)^2-i0)^{(\nu+2-j+2m_2)/2}}. \tag{B.4}$$

The integral (B.4) is calculated by means of Gaussian parametrization with allowance for the fact that, by virtue of the condition $\xi^2 = 0$, we can assume that $R = R_0$ (where $R_0$ is the stationary point of the Gaussian integral) in the numerator of (B.4). Simple but tedious calculations give

$$J^j_{m_1m_2}=\Gamma(\nu)\beta\left(1+\frac{j-\nu}{2}\right)\Gamma^{-1}\left(\frac{2-\nu+m_1-m_2}{2}\right)\Gamma^{-1}\left(\frac{2-\nu+m_2-m_1}{2}\right). \tag{B.5}$$

Substitution of (B.5) into (B.3) and use of integral representations for $1/\Gamma(z)$ give the result (7.17).

Knowing $N_j$, we can find $M_j$ by means of the formula ($\kappa^2 = 1$)

$$\int d^4r[r_{\alpha_1}\ldots r_{\alpha_j}]e^{i\kappa r}r^{-2\delta}=i^j4^{-\delta}\frac{\Gamma(2-\delta+j)}{\Gamma(\delta)}[\kappa_{\alpha_1}\ldots\kappa_{\alpha_j}]. \tag{B.6}$$

From (B.6), we obtain

$$\begin{gathered}M_j(\nu)=K_j(\nu)N_j(\nu),\\ K_j(\nu)=\frac{i^j}{2^{j+\nu}}\Gamma\left(3-\Delta+\frac{j-\nu}{2}\right)\Gamma^{-1}\left(\Delta-1+\frac{j+\nu}{2}\right).\end{gathered}$$

We also give the formula for $N^j_{AB}$ in the case of fields with different dimensions $\Delta_A$ and $\Delta_B$. In this case,

$$\begin{gathered}N^j_{AB}(\nu)=\frac{1}{\nu(\nu+j+1)}\beta_{AB}\left(1+\frac{j-\nu}{2}\right)\beta^{-1}_{AB}\left(1+\frac{j+\nu}{2}\right),\\ \beta_{AB}(x)=B(x+\Delta_A-\Delta_B,\,x+\Delta_B-\Delta_A).\end{gathered}$$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