The kinetics and integral characteristics of synchrotron radiation at high energies

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(Submitted June 13, 1973)
Zh. Eksp. Teor. Fiz. **66**, 81–95 (January 1974)

The energy distribution function $\rho(\epsilon, t)$ of particles radiating in a magnetic field during the time interval t is discussed with allowance for quantum corrections for the case when $\chi < 1$ (χ is a parameter characterizing the quantum effects). The investigation is based on the use of the energy distribution moments at the time t, moments which prove to be analytically calculable. From these moments the function $\rho(\epsilon, t)$ is determined with the aid of an interpolation procedure. Also found with allowance for the quantum corrections are the integral spectral and angular distributions of the radiation for the entire period of time during which the particle is in the field.

1. INTRODUCTION

We have of late been witnessing a new upsurge in interest in the problem of the synchrotron radiation of ultrarelativistic particles. This is connected, on the one hand, with the experiments planned for the most powerful accelerators, in which attempts will be made to extract the hard photons produced when high-energy electrons (up to hundreds of GeV) pass through a strong ($\sim 10^6$ Oe) magnetic field (magnetic converters[1]) and, on the other, with the fact that the available indications of the existence of very strong fields (up to 10^{12} Oe) in astronomical objects[2] make it important that we know the characteristics of the radiation of relativistic particles in such fields. The instantaneous characteristics of the radiation (the spectrum, angular distribution, polarization, etc.) of particles with the energies in question have been thorougly investiaged in both the classical and quantum regions[3-5]. However, if we consider the problem of synchrotron radiation of a particle which is located in a magnetic field for a sufficiently long period of time and which does not receive energy during this time, then of primary interest are the integral characteristics of the radiation of a particle that has traversed a definite path in the field[1)] (the integral spectrum and angular distribution of the radiation, the energy spread of the particles, etc.). To analyze this problem, we must solve the corresponding kinetic problem. The analysis in the classical region turns out to be quite simple. Under the above-indicated conditions, however, the parameter

$$\chi(\varepsilon) = (H/H_0)(\varepsilon/m)$$

(ϵ is the particle energy, H is the magnetic field intensity, and $H_0 = m^2c^3/e\hbar = 4.41 \times 10^{13}$ Oe is the critical field) characterizing the quantum-mechanical effects is not negligibly small, and the quantum-mechanical approach must be used, which complicates to a great extent the solution of the kinetic problem. In such a formulation, the problem is also of theoretical interest. The present paper is devoted to the above-indicated range of problems.

In Sec. 2 we formulate the computational procedure and determine the energy distribution moments for the particles at the time t. In Sec. 3 we obtain the integral characteristics of the radiation—the spectral and angular distributions of the radiation for the entire period of time during which the particle is in the field—with allowance for the quantum effects. In Sec. 4 we discuss the form of the particle distribution function $\rho(\epsilon, t)$.

2. THE DISTRIBUTION MOMENT

The phenomena that occur during the motion of a high-energy charged particle in a magnetic field essentially depend on the value of the parameter χ. For $\chi \gg 1$ the emitted photons can (with a probability of the same order as the emission probability) produce particle pairs, i.e., the problem of the electron-photon shower should be considered. For $\chi \lesssim 1$ the probability of pair production is exponentially suppressed, and it is sufficient to solve the problem of the radiation of the charged particle. It is precisely to this region that the presently known applications pertain, and we shall restrict ourselves to it in the present paper.

The kinetic equation for the energy distribution function $\rho(\epsilon, t)$ of an electron located in a magnetic field for a time interval t has the form

$$\frac{\partial\rho(\varepsilon,t)}{\partial t} = -W(\varepsilon)\rho(\varepsilon,t) + \int_{\varepsilon}^{\infty} W(\varepsilon,\varepsilon')\rho(\varepsilon',t)\,d\varepsilon'. \tag{2.1}$$

Here $W(\epsilon, \epsilon')$ is the probability density for the transition of an electron from the state with energy ϵ' to the state with energy ϵ with the emission of a photon of frequency $\omega = (\epsilon' - \epsilon)t\hbar$:

$$W(\varepsilon,\varepsilon') = \frac{\alpha m^2}{\sqrt{3}\pi\hbar}\frac{1}{\varepsilon'^2}\left\{\frac{(\varepsilon'-\varepsilon)^2}{\varepsilon\varepsilon'}K_{2/3}(v) + \int_v^{\infty} K_{5/3}(y)\,dy\right\}, \tag{2.2}$$

where $v = 2(\epsilon' - \epsilon)/3\epsilon\chi(\epsilon')$, and

$$W(\varepsilon) = \int_0^{\varepsilon} W(\varepsilon',\varepsilon)\,d\varepsilon'. \tag{2.3}$$

We shall consider magnetic fields which satisfy the inequality $H \ll H_0$, and in which the motion of a high-energy particle is always quasi-classical[2)]; this implies, in particular, that the energy spectrum of the particle is quasi-continuous. In view of this, it is sufficient to use the kinetic equation in the quasi-classical form (2.1).

The expression (2.1) is the balance equation for the particle number in the energy representation. This type of equation is encountered in, for example, the problem of the electron-photon cascade in a medium, but its solution in a magnetic field meets with great difficulties owing to the complex nature of the kernel of (2.2).

Let an electron beam in which the energy distribution is described by the function $\Phi(\epsilon)$ enter a magnetic field H perpendicular to the beam at time t = 0, i.e., $\rho(\epsilon, 0) = \Phi(\epsilon)$. Let us define the function

$$F_{\Phi}(t) = \int_0^{\infty} \rho(\varepsilon,t)g(\varepsilon)\,d\varepsilon, \tag{2.4}$$

where, evidently,

$$F_\Phi(0)=\int_0^\infty \Phi(\varepsilon)g(\varepsilon)d\varepsilon, \tag{2.5}$$

$g(\epsilon)$ being an arbitrary function of the energy. If we set $g(\epsilon) = \epsilon^m$ in (2.4), then we obtain the distribution moments, which, if known, provide significant physical information. In the case when all the moments are known, we can, in principle, determine the distribution function.

Let us differentiate the expression (2.4) n times with respect to time and express the time derivative entering into the right-hand side with the aid of the formula (2.1):

$$\frac{\partial^n F_\Phi(t)}{\partial t^n}=\int_0^\infty \frac{\partial^{n-1}\rho(\varepsilon',t)}{\partial t^{n-1}}[Gg(\varepsilon')]d\varepsilon', \tag{2.6}$$

where G is the linear integral operator

$$[Gg(\varepsilon')]=\int_0^{\varepsilon'} W(\varepsilon,\varepsilon')[g(\varepsilon)-g(\varepsilon')]d\varepsilon. \tag{2.7}$$

Repeating this procedure, we obtain

$$\frac{\partial^n F_\Phi(t)}{\partial t^n}=\int_0^\infty \rho(\varepsilon,t)[G^n g(\varepsilon)]d\varepsilon, \tag{2.8}$$

where

$$\begin{aligned}[G^n g(\varepsilon')]&=\int_0^{\varepsilon'} W(\varepsilon,\varepsilon')[G^{n-1}g(\varepsilon)]d\varepsilon-W(\varepsilon')[G^{n-1}g(\varepsilon')]\\ &=\int_0^{\varepsilon'} W(\varepsilon,\varepsilon')\{[G^{n-1}g(\varepsilon)]-[G^{n-1}g(\varepsilon')]\}d\varepsilon.\end{aligned} \tag{2.9}$$

Expanding the function $F_\Phi(t)$ in a Taylor series, and using (2.8) and (2.5), we find

$$F_\Phi(t)=\sum_{n=0}^\infty \frac{t^n}{n!}\int_0^\infty \Phi(\varepsilon)[G^n g(\varepsilon)]d\varepsilon=\int_0^\infty d\varepsilon\,\Phi(\varepsilon)[e^{tG}g(\varepsilon)]. \tag{2.10}$$

If the initial beam is monochromatic (i.e., $\Phi(\epsilon) = \delta(\epsilon-\epsilon_0)$), then

$$F_{\varepsilon_0}(t)=\sum_{n=0}^\infty \frac{t^n}{n!}G^n g(\varepsilon_0)=\sum_{n=0}^\infty \frac{(-1)^n t^n}{n!}\varphi_n(\varepsilon_0), \tag{2.11}$$

where $\varphi_n(\epsilon_0)$ satisfies, in accordance with (2.9), the relation

$$\varphi_n(\varepsilon_0)=\int_0^{\varepsilon_0} d\varepsilon\, W(\varepsilon,\varepsilon_0)[\varphi_{n-1}(\varepsilon_0)-\varphi_{n-1}(\varepsilon)],\qquad \varphi_0=g(\varepsilon_0). \tag{2.12}$$

The above results are valid for an arbitrary equation of the type (2.1). Let us now proceed to consider the equation with the kernel (2.2), writing it in the form

$$\begin{aligned}W(u,\varepsilon_0)=&\frac{\alpha\sqrt{3}m}{\pi\hbar}\frac{H}{H_0}\frac{1}{8\pi i}\int_{-C-i\infty}^{-C+i\infty} ds\,\Gamma\left(-\frac{s}{2}-\frac{1}{3}\right)\\ &\times\Gamma\left(-\frac{s}{2}+\frac{1}{3}\right)\frac{u^s}{(1+u)^3}(3\chi_0)^{-(s+1)}\left[\left(1-\frac{2}{3s}\right)(1+u)+u^2\right],\end{aligned} \tag{2.13}$$

where $C > 2/3$, $\chi_0 = \chi(\epsilon_0)$; here we have made the change of variable $u = (\epsilon_0-\epsilon)/\epsilon$, which is often done in the synchrotron-radiation problem, and have used the integral representation of the K functions (see[11], p. 658).

Let us find the energy distribution moments when $g(\epsilon) = \epsilon^m$. Then the quantity $\varphi_n(\epsilon_0)$, which can be computed according to the formulas (2.12) and (2.13), can be represented in the form of an n-tuple contour integral, there arising at each step of the iterative procedure in (2.12) an integral of the following form:

$$\int_0^\infty du\frac{u^{s_n}}{(1+u)^3}\left[(1+u)\left(1-\frac{2}{3s_n}\right)+u^2\right][1-(1+u)^{p_n}]=J_{(p_n,s_n)}, \tag{2.14}$$

where s_n is the variable of the n-th integration when the representation (2.13) is used and $(-p_n)$ is the power (of the energy, which enters into the expression through the quantity χ) arising in the preceding step in the iteration of the recurrence relation. Evaluating the integral (2.14), we find

$$\begin{aligned}J_{(p_n,s_n)}=-\frac{\Gamma(s_n)}{6}\Big\{&\Gamma(1-s_n)(3s_n^2+3s_n+10)+6\frac{\Gamma(-s_n-p_n)}{\Gamma(3-p_n)}\Big[s_n^3\\ &+s_n^2(p_n+1)+\frac{s_n}{3}(3p_n^2-8p_n+10)-\frac{2}{3}p_n(p_n-2)\Big]\Big\}.\end{aligned} \tag{2.15}$$

Using (2.15), we obtain the following expression for $\varphi_n(\epsilon_0)$, (2.12), in the case $g(\epsilon) = \epsilon^m$ under consideration:

$$\begin{aligned}\varphi_n(\varepsilon_0)=&\varepsilon_0^m\left(-\frac{\alpha m\sqrt{3}}{24\pi\hbar}\frac{H}{H_0}\right)^n\frac{1}{(2\pi i)^n}\int_{-C-i\infty}^{-C+i\infty}ds_n\ldots\int_{-C-i\infty}^{-C+i\infty}ds_1 D(s_1)\ldots\\ &\ldots D(s_n)\psi(s_1,s_2,\ldots,s_n)\psi(s_1,s_2,\ldots,s_{n-2},-1,s_{n-1})\psi(s_1,s_2,\ldots\\ &\ldots,s_{n-3},-1,-1,s_{n-2})\ldots\psi(s_1,-1,\ldots,-1,s_2)\psi(-1,-1,\ldots,-1,s_1),\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}D(s)&=\Gamma\left(-\frac{s}{2}+\frac{1}{3}\right)\Gamma\left(-\frac{s}{2}-\frac{1}{3}\right)\Gamma(s)(3\chi_0)^{-(s+1)},\\ \psi(s_1,\ldots,s_n)&=-\frac{6}{\Gamma(s_n)}J_{(p_n,s_n)}\equiv\tau(p_n,-s_n),\\ p_n&=n-(m+1)+s_1+s_2+\ldots+s_{n-1}.\end{aligned} \tag{2.17}$$

The integration in the formula (2.16) should be performed in the order $s_1, s_2, \ldots$. The form of p_n ensues from the following argument: at each step in the iteration, starting from the second, there appears (see (2.13)) an additional factor $\chi^{-(s_i+1)}$, which in the aggregate makes to p_n the contribution $n-2+s_2+\ldots+s_{n-1}$, while the power of the quantity χ in the first iteration is (s_1+1-m).

For $\chi \ll 1$ we can, by successively closing the contours on the left and evaluating the integrals in the formula (2.16), obtain an expression for $\varphi_n(\epsilon_0)$ in the form of an asymptotic series:

$$\varphi_n(\varepsilon_0)=\sum_k a_{n,k}(3\chi_0)^k. \tag{2.18}$$

Substituting the values thus computed into (2.11), and carrying out the summation over n, we find an expression for the moments. In the case when $\chi \sim 1$, however, the integration contours should be closed on the right, and a converging series for $\varphi_n(\epsilon_0)$ is then obtainable. Let us discuss the case $\chi \ll 1$ in greater detail. It is convenient here to write the recurrence relations (2.12) in the form

$$\varphi_n(\varepsilon_0)=Q_n(\varepsilon_0)\varphi_{n-1}(\varepsilon_0), \tag{2.19}$$

where

$$\begin{aligned}Q_n&=-\frac{\alpha\sqrt{3}m}{24\pi\hbar}\frac{H}{H_0}\frac{1}{2\pi i}\int_{-C-i\infty}^{-C+i\infty}ds\,D(s)\tau(p_n,-s)\\ &=\frac{3\sqrt{3}}{16\pi}\frac{I_c(\varepsilon_0)}{\varepsilon_0}\sum_{l=2}^\infty b_l\tau(p_n,l)(3\chi_0)^{l-2}.\end{aligned} \tag{2.20}$$

Here

$$b_l=\frac{(-1)^{l+1}}{l!}\Gamma\left(\frac{l}{2}+\frac{1}{3}\right)\Gamma\left(\frac{l}{2}-\frac{1}{3}\right), \tag{2.21}$$

and $I_c(\epsilon_0) = 2\alpha m^2\chi_0/3\hbar$ is the classical intensity of the synchrotron radiation of particles with energy ϵ_0. The values of p_n naturally depend on the points at which the residues were calculated in the preceding integrals. Substituting into (2.19) the functions $\varphi_n(\epsilon_0)$ and $\varphi_{n-1}(\epsilon_0)$ in the form of the expansions (2.18), and equating the coefficients of the same powers of χ, we obtain the following recurrence relation:

$$a_{n,k}=\frac{3\sqrt{3}}{16\pi}\frac{I_c(\varepsilon_0)}{\varepsilon_0}\sum_{r=0}^k b_{k+2-r}a_{n-1,r}\tau(-(m+n+r-1),k+2-r). \tag{2.22}$$

Notice here that to describe the classical limit, we need to calculate all the residues at the point $l = 2$, the quantum corrections being the contributions of the poles at $l > 2$. Substituting $\varphi_n(\epsilon_0)$ from (2.18) into (2.11), we have

$$\frac{\langle\varepsilon^m\rangle}{\varepsilon_0^m}=\sum_{n=0}^\infty\frac{(-t)^n}{n!}\sum_{k=0}(3\chi_0)^k a_{n,k}=\sum_{k=0}(3\chi_0)^k f_k(t). \tag{2.23}$$

The formulas found allow us to compute the distribution moments $\langle\epsilon^m\rangle$ in the form of a power series in χ_0. In view of the well-known unwieldiness, we shall write out $\langle\epsilon^m\rangle$ here up to terms $\sim\chi_0^2$ and $\langle\epsilon\rangle$ up to the χ_0^3 terms.

The following are the values of the quantities b_i and $\tau(-q, i)$ $(i = 2, 3, 4, 5)$ which will be needed below:

$$b_2=-\frac{2\pi}{6\sqrt{3}},\quad b_3=\frac{10\pi}{6^3},\quad b_4=-\frac{2\pi}{\sqrt{3}\cdot 3^4},\quad b_5=\frac{77\cdot 2\pi}{4!6^4};$$
$$\tau(-q,2)=-16q,\quad \tau(-q,3)=-22q(5+q),\qquad (2.24)$$
$$\tau(-q,4)=-4q(218+63q+7q^2),$$
$$\tau(-q,5)=-2q[17q^2(q+14)+1387q+3878].$$

Let us now compute the functions $f_k(t)$ for $k = 0, 1, 2$. For the chosen initial condition $\rho(\epsilon, 0) = \delta(\epsilon - \epsilon_0)$, we have from (2.23): $f_0(0) = 1$, $f_{k\geq 1}(0) = 0$. For this reason, $a_{0,0} = 1$, $a_{0,k\geq 1} = 0$. Then

$$f_0(t)=\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}a_{n,0}=1+\sum_{n=1}\frac{(-t)^n}{n!}\frac{I_c(\varepsilon_0)}{\varepsilon_0}(n+m-1)a_{n-1,0}$$
$$=1-\frac{I_c(\varepsilon_0)}{\varepsilon_0}\Big[tf_0(t)+(m-1)\int_0^t dx\ f_0(x)\Big].\qquad (2.25)$$

Differentiating (2.25) with respect to t, we find for $f_0(t)$ the equation:

$$\frac{df_0(t)}{dt}\left(1+t\frac{I_c(\varepsilon_0)}{\varepsilon_0}\right)+m\frac{I_c(\varepsilon_0)}{\varepsilon_0}f_0(t)=0,\qquad (2.26)$$
$$f_0(0)=1,$$

whose solution is

$$f_0(t)=\frac{1}{(1+z)^m},\qquad z=\frac{I_c(\varepsilon_0)}{\varepsilon_0}t.\qquad (2.27)$$

Using (2.22) and (2.24) and the procedure employed in (2.25), we obtain for $f_1(t)$ the following equation:

$$\frac{df_1(t)}{dt}(1+z)+(m+1)\frac{z}{t}f_1(t)=-\frac{55\sqrt{3}}{6\cdot 48}\left[(m+5)\frac{df_0(t)}{dt}+t\frac{d^2f_0(t)}{dt^2}\right],\qquad (2.28)$$
$$f_1(0)=0,$$

whose solution is

$$f_1(t)=\frac{m}{(1+z)^{m+1}}\cdot\frac{55\sqrt{3}}{3\cdot 48}\left[2\ln(1+z)+\frac{z}{1+z}\left(\frac{m+1}{2}\right)\right].\qquad (2.29)$$

A successive calculation yields for any k the equation

$$\frac{df_k}{dt}(1+z)+(m+k)\frac{z}{t}f_k=\eta_k(t),\qquad f_k(0)=\delta_{k,0},\qquad (2.30)$$

where $\eta_k(t)$ is expressible in terms of the derivatives of the functions f_l $(l = 0, \ldots, k-1)$ up to the $(k + 1 - l)$-th derivative.

Let us give the mean quantities $\langle\epsilon^m\rangle$ up to terms $\sim\chi_0^2$:

$$\frac{\langle\varepsilon^m\rangle}{\varepsilon_0^m}=\frac{1}{(1+z)^m}\Big\{1+\frac{55\sqrt{3}\chi_0}{48}\frac{m}{1+z}\left[2\ln(1+z)+\frac{z}{1+z}\left(\frac{m+1}{2}\right)\right]$$
$$-\frac{\chi_0^2 m}{(1+z)^2}\left[27z+\frac{7z(m+1)(m+2)}{6(1+z)}+7(m+1)\ln(1+z)\right]$$
$$+\left(\frac{55\sqrt{3}}{48}\right)^2\frac{\chi_0^2 m}{(1+z)^2}\Big[2(m+1)\ln^2(1+z)+\frac{\ln(1+z)}{1+z}[m(m+5)z+2(m-1)]\qquad (2.31)$$
$$+\frac{z}{(1+z)^2}\left[5z^2+\frac{z}{8}(62-5m+6m^2+m^3)-2(m-1)\right]\Big]\Big\}.$$

ε_0, GeV	$\chi\cdot 10^2$	$\frac{\varepsilon_c}{\varepsilon_0}$	$\frac{\delta_1\langle\varepsilon\rangle}{\varepsilon_0}$	$\frac{\delta_2\langle\varepsilon\rangle}{\varepsilon_0}$	$\frac{\delta_3\langle\varepsilon\rangle}{\varepsilon_0}$	$\frac{\langle\varepsilon(t)\rangle}{\varepsilon_0}$
100	0.887	0.94299	0.00276	−0.00017	0.00001	0.94489
600	5.320	0.731	0.051	−0.011	0.003	0.773
1200	10.64	0.576	0.107	−0.026	0.006	0.663
1800	15.96	0.476	0.144	−0.031	0.006	0.594
2400	21.28	0.405	0.166	−0.029	0.005	0.548

Here the first term in the curly brackets gives the classical result, the remaining terms being quantum corrections of order χ_0 and χ_0^2.

In order to give an idea of the nature of the expressions that arise when the subsequent corrections in χ_0 are computed, and to determine the value of $\langle\epsilon\rangle$ more accurately, let us write out the correction $\delta_3\langle\epsilon\rangle$ proportional to χ_0^3:

$$\frac{\delta_3\langle\varepsilon\rangle}{\varepsilon_0}=\chi_0^3\Big\{\frac{77\sqrt{3}}{128}\left[68\frac{\ln(1+z)}{(1+z)^4}+\frac{z}{(1+z)^5}(113z^2+471z+392)\right]$$
$$+\left(\frac{55\sqrt{3}}{48}\right)^3\Big[\frac{8\ln^3(1+z)}{(1+z)^4}+\frac{4\ln^2(1+z)}{(1+z)^5}(7z+1)-\frac{2\ln(1+z)}{(1+z)^6}(9+22z$$
$$-12z^2-10z^3)+\frac{z}{2(1+z)^7}(15z^4+133z^3+210z^2+98z+36)\Big]\qquad (2.32)$$
$$-\frac{55\sqrt{3}}{48}\Big[\frac{84\ln^2(1+z)}{(1+z)^4}+\frac{\ln(1+z)}{(1+z)^5}(108z^2+180z-68)$$
$$+\frac{z}{6(1+z)^6}(461z^3+2960z^2+3327z+408)\Big]\Big\}.$$

It can be seen from the formulas (2.31) and (2.32) that the quantum corrections to $\langle\epsilon^m\rangle$ decrease with time (with distance traversed in the field) more rapidly than the classical term, so that at very large times the quantity $\langle\epsilon^m\rangle$ is determined by the classical term. This circumstance will be discussed below.

In the table we give values of $\langle\epsilon(t)\rangle$ computed from the formulas (2.31) and (2.32) for a magnetic field $H = 2\times 10^6$ Oe, a depth ct = 1.21 cm, and different initial energies ϵ_0; ϵ_c is the classical value of $\langle\epsilon\rangle$ at the given depth and $\delta_k\langle\epsilon\rangle$ are the corrections of order χ_0^k. As is well known, the expansion of the magnetic-bremsstrahlung intensity in an asymptotic power series in χ yields satisfactory numerical results only for $\chi < 0.1$. This, however, applies to the formulas (2.31) and (2.32) for small z. For $z > \chi_0$ the region of applicability of these formulas broadens, which is illustrated by the table. For $z \gg 1$ the leading terms of the expansion are, in comparison with the classical term $1/z$, of order χ_0^k/z. The values of $\langle\epsilon\rangle$ for the values of H and t used by us have also been computed by White[8] by a direct numerical solution of the basic integro-differential equation (2.1). The results obtained by him are of low accuracy (to within 4%), despite the use of high-performance electronic computers, and, within the limits of this accuracy, they agree with the results given in the table.

Let us draw attention to the following important circumstance. Nowhere in the derivation of (2.16)–(2.31) did we assume that m was an integer. In fact, the results obtained are valid for any, including complex, m. The procedure developed allows us, in principle, to obtain the form of any integral transform of the function $\rho(\epsilon, t)$ with respect to ϵ, respectively choosing the function $g(\epsilon)$ in (2.4)–(2.10). The inversion of this transform solves the problem of finding $\rho(\epsilon, t)$. In this sense, the formulas (2.11) and (2.16) are Mellin transforms of the function $\rho(\epsilon, t)$. Notice that to find integral characteristics of the type $\langle A(t)\rangle = \int A(\epsilon)\rho(\epsilon, t)d\epsilon$, it is sufficient to know the form of the transform of $\rho(\epsilon, t)$ in some integral transformation. Indeed, let

$$A(\varepsilon)=\int \bar{A}(s)\xi(s,\varepsilon)\,ds;$$

then

$$\langle A(t)\rangle=\int \bar{A}(s)\,ds\int d\varepsilon\,\xi(s,\varepsilon)\rho(\varepsilon,t)=\int ds\,\bar{A}(s)\,\bar{\rho}(s,t).$$

3. THE INTEGRAL CHARACTERISTICS OF THE RADIATION: ANGULAR DISTRIBUTION AND SPECTRUM

Besides the energy averages at a given t, which were considered in the preceding section, of great interest are the integrals of the characteristics of the radiation over some interval of time (over the radiation observation time, over the entire period of time during which the particle is in the field, etc.). In this case it is convenient to first compute

$$\int_{t_1}^{t_2}\rho(\varepsilon,t)\,dt.$$

We shall consider this problem using as an example the following characteristics of the radiation: the total spectral distribution of the radiation for the entire period and the total angular distribution (over the angle of emission of the radiation relative to the plane of motion of the particle). With that end in view, let us take the time integral of the right-hand side of (2.31) and invert it. As a result, we obtain (up to terms $\sim\chi^2$)

$$\begin{aligned}\int_0^\infty \rho(\varepsilon,t)\,dt=\frac{1}{I_c(\varepsilon)}\Big\{&1+c(\varepsilon)\left[2+\frac{\varepsilon_0}{2}\delta(\varepsilon-\varepsilon_0)\right]\\ &-\chi^2(\varepsilon)\left[20+7\varepsilon_0\delta(\varepsilon-\varepsilon_0)-\frac{7}{6}\varepsilon_0^2\frac{d}{d\varepsilon}\delta(\varepsilon-\varepsilon_0)\right]\\ &+c^2(\varepsilon)\left[3+2\varepsilon_0\delta(\varepsilon-\varepsilon_0)-\frac{1}{4}\varepsilon_0^2\frac{d}{d\varepsilon}\delta(\varepsilon-\varepsilon_0)\right]\Big\},\end{aligned} \qquad (3.1)$$

$$c(\varepsilon)=\frac{55\sqrt{3}}{48}\chi(\varepsilon).$$

Using (3.1), we can find any total characteristics of the radiation up to terms $\sim\chi^2$. For this purpose, we must integrate the corresponding instantaneous characteristics with (3.1). For the spectral distribution of the radiation for the entire period of the motion of an electron in the field (until the particle ceases to be ultrarelativistic[3]), we have up to terms $\sim\chi$

$$\frac{1}{\varepsilon_0}\frac{d\mathscr{E}}{d\varkappa}=\frac{9\sqrt{3}}{8\pi}\varkappa^2\int_1^\infty K_{5/3}(\varkappa y)\left[\frac{y^{1/3}-1}{3}+c(\varepsilon_0)\left(y-\frac{1}{2}\right)-\frac{3}{4}\chi(\varepsilon_0)\varkappa y^2\right]dy, \qquad (3.2)$$

where $\kappa = 2\hbar\omega/3\chi(\epsilon_0)\epsilon_0$ and the variable $y = (\epsilon_0/\epsilon)^2$.

The first term in (3.2) is the classical result, which can be obtained from the instantaneous classical value of the intensity as a function of the frequency by integrating it over the time with allowance for the time dependence of the energy of the emitting particle[3] (this problem is considered in[10]). The remaining terms are the first quantum correction ($\sim\chi$). The forms of the classical spectrum c and the quantum corrections for $\chi_0 = 0.177$ (the curve 1) and $\chi_0 = 0.089$ (curve 2) are shown in Figs. 1a and 2a. We show at the same time (Figs. 1b and 2b) the fraction ξ of the classical energy loss that is accounted for by the radiation in the interval $0-\kappa$, to wit

$$\xi=\frac{1}{\varepsilon_0}\int_0^\varkappa\frac{d\mathscr{E}}{d\varkappa}d\varkappa.$$

It can be seen that the dominant contribution to the energy loss falls in the region where $\kappa \lesssim 1$. For $\kappa \ll 1$ the spectrum behaves like $1/\sqrt{\kappa}$, but the contribution of this

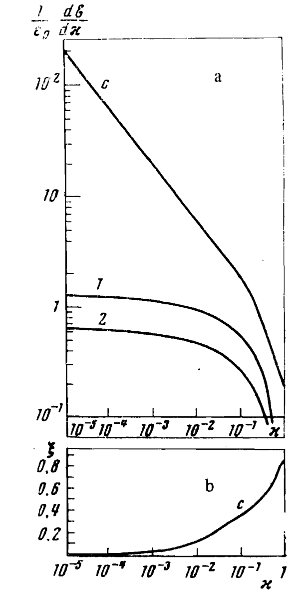

FIG. 1. The plots – the curves c – of the integral spectrum (a) and the classical energy loss (b) as functions of $\kappa = 2\hbar\omega/3\epsilon_0\chi(\epsilon_0)$; the curves 1 and 2 represent the first quantum corrections to the spectrum for $\chi_0 = 0.177$ and $\chi_0 = 0.089$ respectively.

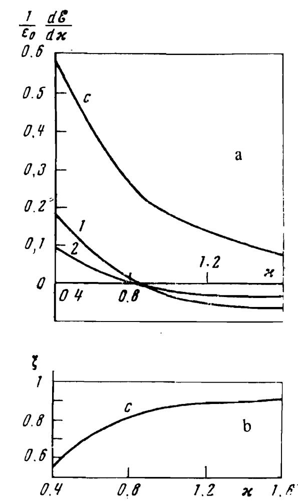

FIG. 2. Same curves as in Fig. 1 for the region $\kappa \sim 1$.

region to the energy loss is, as can be seen from Fig. 1, quite small (it is clear that the losses in this region are proportional to $\sqrt{\kappa}$). For $\kappa > 1$ the spectral distribution falls off exponentially. It is evident that the relative magnitude of the quantum corrections (as compared to the classical result) increases in the hard section of the spectrum. This is also evident from Figs. 1a and 2a. Notice that the integrals of the quantum corrections of each order in χ over the entire region of variation of κ vanish. This circumstance is a consequence of the law of conservation of energy, since the total energy lost during the radiation emission is ϵ_0.

For the angular distribution of the energy losses, we obtain the following expression (up to terms $\sim\chi$):

$$\begin{aligned}\frac{1}{\varepsilon_0}\frac{d\mathscr{E}}{dy}=\frac{3}{32}\frac{1}{y^2}\Big\{&3-4R^3+R^5+\frac{55\sqrt{3}}{72}\chi(\varepsilon_0)y^2\left[7R^3+3y^2R^5+\frac{3}{4}(12y^2+7)R^7\right]\\ &-\frac{2\chi(\varepsilon_0)}{3\sqrt{3}\pi}\left[\frac{39\,\mathrm{arctg}\,y}{y}+[39(y^6-1)+143y^4+17y^2]R^8\right]\Big\}, \end{aligned}\qquad (3.3)$$

where $y = \epsilon_0\vartheta/m$, $R = (1 + y^2)^{-1/2}$, and ϑ is the angle of emission of the radiation relative to the plane of motion of the particle. In Fig. 3 the curve c is the plot of the classical part of (3.3). The curves 1 and 2 represent the entire function (3.3) (with allowance for the first quantum correction) respectively for $\chi = 0.266$ and $\chi = 0.133$. The

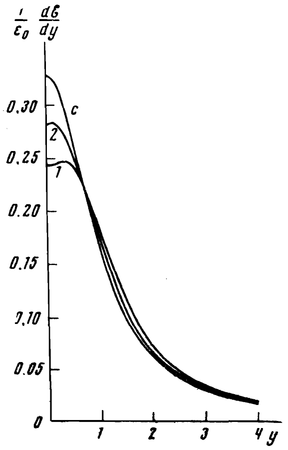

FIG. 3. The integral angular distribution as a function of $y = \epsilon_0 \vartheta/m$ (ϑ is the angle of emission of the radiation). The curve c is the classical distribution, while the curves 1 and 2 are the distributions with allowance for the first quantum correction for χ_0 respectively equal to 0.266 and 0.133.

points at which all the three curves intersect correspond to the value $y = y_0$ at which the quantum corrections vanish, these corrections being negative when $y < y_0$ and positive when $y > y_0$. Thus, the quantum corrections decrease the function $\epsilon_0^{-1} d\mathscr{E}/dy$ in comparison with its classical part when $y < y_0$ and increase it when $y > y_0$. This is a reflection of the well-known fact[5] that the instantaneous angular distribution broadens with increasing χ. With allowance for the quantum corrections, the maximum of the distribution lies at $\vartheta \neq 0$. Besides the quantum corrections, the decrease of the energy $\langle\epsilon\rangle$ in time also leads to the broadening of the angular distribution (in comparison with the instantaneous distribution).

4. THE DISTRIBUTION FUNCTION

The process of the successive emission of photons by a particle in quantum electrodynamics is discrete in nature, the time intervals between the moments of emission being random. Therefore, it might be expected that the distribution function for a long interval of time (after a large number of emission events) would be of the Gaussian type. As the time increased further, the particles would, owing to the decrease with energy of the radiation intensity, accumulate in the low-energy region. This circumstance and the quantum retardation—the nonvanishing probability of finding a particle in an energy region from which it should escape according to the classical theory—lead to an asymmetry in the distribution curve about the point $\epsilon = \epsilon_{max}$ where the distribution has its maximum; moreover, $\epsilon_{max} < \langle\epsilon\rangle$ and there is a "tail" on the higher-energy side. If at $t = 0$ we have $\rho(\epsilon, 0) = \delta(\epsilon - \epsilon_0)$, then for small times (when only a few emission events have occurred) the particles are concentrated in a narrow (of width $\sim\chi_0\epsilon_0$) energy region contiguous to ϵ_0, there is a "tail" on the low-energy side, the quantum fluctuations in the emission process are important, and $\epsilon_{max} > \langle\epsilon\rangle$. Thus, the physical pictures of the process are different in the large- and small-time regions, and we shall therefore consider them separately[4)]. Notice that after many emission events the mean particle-energy loss is well described by a continuous (classical) emission process (see (2.31)).

Let us now proceed to find the distribution $\rho(\epsilon, t)$. The inversion of the classical part of $\langle\epsilon^m\rangle$, given by (2.31), yields directly the classical distribution function

$$\rho_c(\varepsilon, t) = \delta\left(\varepsilon - \frac{\varepsilon_0}{1+z}\right), \tag{4.1}$$

where $z = I_c(\epsilon_0)t/\epsilon_0$ (see (2.27)), i.e., the distribution preserves the δ-function form, while its center is displaced in the time t to the point $\epsilon = \epsilon_0/(1 + z)$. The result (4.1) can also be easily obtained from the basic equation (2.1). Let us write it in the form

$$\frac{\partial\rho(\varepsilon, t)}{\partial t} = -\int_0^\infty W(u, \varepsilon)\rho(\varepsilon, t)\,du + \int_0^\infty W(u, \varepsilon(1+u))\rho(\varepsilon(1+u), t)\,du, \tag{4.2}$$

where the probability $W(u, \epsilon)$ is defined in the same way as in (2.13), and let us take into account the fact that for $\chi \ll 1$ the dominant contribution to the integral over u is made by small $u \sim\chi$ (see, for example,[5]). Then the function of $\epsilon(1 + u)$ can be expanded in powers of u. We obtain as a result

$$\frac{\partial\rho(\varepsilon, t)}{\partial t} = \sum_{n=1}^{\infty} \frac{1}{n!}\frac{\partial^{(n)}}{\partial\varepsilon^n}[I_n(\varepsilon)\rho(\varepsilon, t)], \tag{4.3}$$

where

$$I_n(\varepsilon) = \int_0^\infty W(u, \varepsilon)(\varepsilon u)^n\,du.$$

If we allow $h \to 0$ on the right-hand side of the formula (4.3), which corresponds to a transition to the classical limit, then only the first term of the series will remain, in which I_1 should be replaced by the classical value of the intensity, i.e., we have

$$\frac{\partial\rho_c(\varepsilon, t)}{\partial t} = \frac{\partial}{\partial\varepsilon}[I_c(\varepsilon)\rho_c(\varepsilon, t)]. \tag{4.4}$$

The solution of this equation for the initial condition $\rho_c(\epsilon, 0) = \Phi(\epsilon)$ will be the function

$$\rho_c(\varepsilon, t) = \Phi\left(\frac{\varepsilon}{1 - I_c(\varepsilon)t/\varepsilon}\right)\frac{1}{(1 - I_c(\varepsilon)t/\varepsilon)^2} \tag{4.5}$$

which clearly agrees with (4.1). To allow for the quantum corrections on the right-hand side of Eq. (4.3), we must retain the terms of the appropriate order in χ. We must however, bear in mind that for the initial condition $\rho(\epsilon, 0) = \delta(\epsilon - \epsilon_0)$ the expansion in powers of u in the second term of Eq. (4.2) makes sense only when $(\epsilon_0 - \epsilon)/\epsilon \gg \chi$, since the upper limit of the u integral is practically equal to $u_{max} = (\epsilon_0 - \epsilon)/\epsilon$ (when this inequality is fulfilled, the upper limit of the u integral can be replaced by ∞). The cited inequality is violated when $\epsilon - \epsilon_0 \sim \epsilon\chi$, i.e., in the region contiguous to the initial energy. For large t (when many emission events have taken place), the majority of the particles escape from this section, while for small t they are, on the contrary, concentrated in it. For this reason, Eq. (4.3) with the quantum corrections is valid only for sufficiently large t, which leads to certain difficulties in the imposition of the initial (at t = 0) condtion.

Besides the solution to Eq. (4.3), the distribution function $\rho(\epsilon, t)$ can also be obtained by inverting the above-found distribution moments (2.31), which were computed with the singular initial condition $\rho(\epsilon, 0) = \delta(\epsilon - \epsilon_0)$. To us, that is naturally a more direct approach. As a result of the inversion of (2.31) (see, for example,[12]), we obtain the function $\rho(\epsilon, t)$ expressed in terms of $\rho_c(\epsilon, t)$ and its derivatives. The singular nature of such a distribution is due to the allowance for the finite number of terms in the expansion of $\langle\epsilon^m\rangle$ in powers of χ. Although the indicated form of the distribution function is suitable for computing averages with, it does not impart the true character of the distribution, which, naturally, is a smooth function of ϵ for $t > 0$. Therefore, of certain interest is the interpolation of the function $\rho(\epsilon, t)$ by a smooth curve, such that it yields in the given order in χ the correct values of the distribution moments.

On the basis of the above-expounded physical considerations, we can, in the first order in χ, interpolate the function $\rho(\epsilon, t)$ by the Gaussian distribution:

$$\rho_1(\varepsilon, t) = C_1 \exp[-b(\varepsilon - \varepsilon_{max})^2].$$

Computing with this function the first two moments of the distribution up to terms $\sim\chi$, and taking into account the normalization condition for $\rho(\epsilon, t)$, we find

$$\rho_1(\varepsilon, t) = \frac{1}{\sqrt{2\pi\Delta^2}} \exp\left\{-\frac{(\varepsilon - \langle\varepsilon\rangle)^2}{2\Delta^2}\right\}, \tag{4.6}$$

where

$$\Delta^2 \equiv \langle\varepsilon^2\rangle - \langle\varepsilon\rangle^2 = \frac{55\sqrt{3}}{48} \frac{\varepsilon_0^2 \chi_0 z}{(1+z)^4} + O(\chi_0^2)$$

is the variance of the distribution; in $\langle\epsilon\rangle$, the first quantum correction should be retained. The distribution (4.6) gives (up to terms $\sim\chi$) the correct values of all the moments $\langle\epsilon^m\rangle$; as $t \to 0$, the function $\rho_1(\epsilon, t) \to \delta(\epsilon - \epsilon_0)$, while for large t ($z \gg 1$) it goes over into the classical distribution $\rho_c(\epsilon, t)$, (4.1). This is connected with the above-noted fact that when $z \gg 1$ the quantity $\langle\epsilon^m\rangle$ is determined by the classical term. The width of the distribution curve first increases, attains its maximum at $z \approx 1/3$, and then decreases again.

To take the subsequent quantum corrections into account, we seek, in accordance with the foregoing, the distribution function in the form

$$\rho_2(\varepsilon, t) = C_2 \begin{cases} \exp\{-\lambda^2 b(\varepsilon - \varepsilon_{max})^2\}, & \varepsilon \geqslant \varepsilon_{max} \\ \exp\{-b(\varepsilon - \varepsilon_{max})^2\}, & \varepsilon \leqslant \varepsilon_{max} \end{cases}. \tag{4.7}$$

Let us compute with this function the first three distribution moments up to terms $\sim\chi^2$, and solve the resulting system of equations for C_2, b, λ, and ϵ_{max}. This requires the solution of the following cubic equation.

$$y^3(\pi - 3) - \Delta^2 y + B = 0, \tag{4.8}$$

where

$$B = 3\langle\varepsilon\rangle\langle\varepsilon^2\rangle - \langle\varepsilon^3\rangle - 2\langle\varepsilon\rangle^3 = \frac{\varepsilon_0^3 \chi_0^2 z}{(1+z)^6}\left[7 - 6\left(\frac{55\sqrt{3}}{48}\right)^2 \frac{z}{1+z}\right].$$

Notice that for $z \sim 1$ (more precisely, for $z \gg \chi_0$), the ration B/Δ^2 is a small parameter ($\sim\chi_0$), which significantly facilitates the solution of Eq. (4.8). In this case

$$y = \frac{B}{\Delta^2}\left[1 + \frac{B^2}{\Delta^6}(\pi - 3)\right], \quad \varepsilon_{max} = \langle\varepsilon\rangle + y,$$

$$\frac{1}{\sqrt{b}} = \left[2\Delta^2 - y^2\left(\frac{3\pi}{8} - 1\right)\right]^{1/2} + \frac{y\sqrt{\pi}}{2}, \tag{4.9}$$

$$\lambda^2 = \frac{1}{(1 - y\sqrt{\pi b})^2}, \quad C_2 = \sqrt{\frac{b}{\pi}} \frac{1}{1 - y\sqrt{\pi b}/2}.$$

This solution shows that: 1) the quantity $y = \epsilon_{max} - \langle\epsilon\rangle$ is at first positive, passes through zero at $z \sim 0.4$, and subseuqently remains negative; 2) the width of the distribution curve first increases, and then decreases, ρ_2 going over into $\rho_c(\epsilon, t)$ at $z \gg 1$; 3) the height of the distribution curve first decreases, and then increases. Everything said here is well illustrated by the plots of the function[5] $\epsilon_0\rho_2(\epsilon, t)$ for $\chi = 0.089$ and $\chi = 0.177$ shown in Fig. 4. Notice again that the function $\rho_2(\epsilon, t)$ in the form (4.7) gives the correct values of all the moments $\langle\epsilon^m\rangle$, (2.31), up terms $\sim \chi^2$.

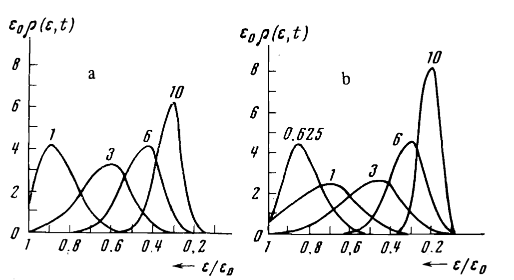

FIG. 4. The distribution function for a) $\chi_0 = 0.089$ and b) $\chi_0 = 0.177$ ($H = 5 \times 10^6$ Oe). The numbers on the curves are the values of the effective range lH (l is in cm, H in units of 10^6 Oe).

Let us now consider the region of small times t. For the initial condition $\rho(\epsilon, 0) = \delta(\epsilon - \epsilon_0)$ chosen by us, of interest is the probability of finding a particle in some interval $\epsilon_0 - \epsilon_0 x$ ($x < 1$), a probability which is given by the quantity

$$A(\varepsilon_0, x, t) = \int_{\varepsilon_0 x}^{\varepsilon_0} \rho(\varepsilon, t)\, d\varepsilon, \tag{4.10}$$

with the proviso that $A(\epsilon_0, x, 0) = 1$. Using the basic equation (2.1), we can obtain the expansion of $A(\epsilon_0, x, t)$ for small t. Indeed,

$$\frac{\partial^n A}{\partial t^n} = -\int_{\beta}^{\varepsilon_0} d\varepsilon \frac{\partial^{n-1}\rho(\varepsilon, t)}{\partial t^{n-1}} \sigma(\varepsilon, \beta), \tag{4.11}$$

where $\beta = \epsilon_0 x$ and

$$\sigma(\varepsilon, \beta) = \int_0^{\beta} W(\varepsilon', \varepsilon)\, d\varepsilon'. \tag{4.12}$$

Then, clearly,

$$\left.\frac{\partial A}{\partial t}\right|_{t=0} = -\sigma(\varepsilon_0, \beta), \quad \left.\frac{\partial^2 A}{\partial t^2}\right|_{t=0} = \sigma^2(\varepsilon_0, \beta) - \int_{\beta}^{\varepsilon_0} W(\varepsilon, \varepsilon_0)[\sigma(\varepsilon, \beta) - \sigma(\varepsilon_0, \beta)]\, d\varepsilon. \tag{4.13}$$

To calculate the highest derivatives $\partial^n A/\partial t^n$, we must substitute into (4.11) the functions $\partial^{n-1}\rho(\epsilon, t)/\partial t^{n-1}$, obtained by differenting the basic equation (2.1), expressing the derivative on the righthand side in terms of $\rho(\epsilon, t)$ with the aid of (2.1), and repeating the process $n - 2$ times. We have as a result

$$A(\varepsilon_0, x, t) = 1 - \sigma(\varepsilon_0, \beta)t + \frac{t^2}{2!}\left[\sigma^2(\varepsilon_0, \beta) - \int_{\beta}^{\varepsilon_0} W(\varepsilon, \varepsilon_0)[\sigma(\varepsilon, \beta) - \sigma(\varepsilon_0, \beta)]\, d\varepsilon\right] + \ldots \tag{4.14}$$

It follows directly from the formula (4.10) that

$$\rho(\varepsilon, t) = -\partial A(\varepsilon_0, \varepsilon/\varepsilon_0, t)/\partial\varepsilon. \tag{4.15}$$

If $(1-x)/x \gg \chi$, then the quantity $\sigma(\epsilon_0, \beta)$ is exponentially small, which is due to the nature of the instantaneous spectrum of the radiation: a particle radiating energy in small portions $\sim\epsilon_0\chi$ cannot escape from the indicated region after a small number of emission events. Of primary interest at small t, however, is the energy region where $(1-x)/x \sim \chi$ and in which the integrals entering into (4.14) can be evaluated only numerically.

The authors are very grateful to D. V. Pestrikov for his help in the numerical computations.

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[1]The individual problems for such a formulation have been considered in a number of papers[6-10].

[2]The principal quantum number $n = H_0(\gamma^2 - 1)/2H$ is, under the assumed conditions, clearly much larger than one.

[3]Let us recall that when the energy of an electron becomes of the order of its mass, all the formulas used here (including the instantaneous characteristics of the radiation) cease to be valid. However, for $\epsilon_0 \gg m$, this region makes a negligibly small contribution to the total energy of the radiation.

[4]Nowhere in the preceding sections did we, proceeding directly from Eq. (2.1), use the explicit form of the distribution function; the obtained results are therefore valid for any t.

[5]The function $\rho(\epsilon, t)$ was computed by a direct numberical solution of the basic integro-differential equation (2.1) in[7, 8], and is given in the form of graphs for some values of the parameters. Our results essentially agree in this region with those obtained in[7, 8].

[1]T. Erber, Concepts in Hadron Physics (Acta Phys. Austriaca, Suppl., Vol. VIII), Springer, Wien, 1971.

[2]F. Dyson and D. Ter Haar, Neutron Stars and Pulsars (Russ. Transl.), Mir, 1973.

[3]L. D. Landau and E. M. Lifshitz, Teoriya polya (The Classical Theory of Fields), Nauka, 1967 (Eng. Transl., Pergamon Press, New York, 1971).

[4]A. A. Sokolov and I. M. Ternov, Sinkhrotronnoe izluchenie (Synchrotron Radiation), Nauka, 1966 (Eng. Transl., Pergamon Press, New York, 1968).

[5]V. N. Baĭer, V. M. Katkov, and V. S. Fadin, Izluchenie řelyativistskikh elektronov (Radiation by Relativistic Electrons), Atomizdat, 1973.

[6]C. S. Shen, Phys. Rev. Lett. 24, 410 (1970).

[7]C. S. Shen and D. White, Phys. Rev. Lett. 28, 455 (1972).

[8]D. White, Phys. Rev. D5, 1930 (1972).

[9]C. S. Shen, Phys. Rev. D6, 2736 (1972).

[10]D. V. Gal'tsov and V. A. Losev, Yad. Fiz. **17**, 109 (1973).

[11]I. S. Gradshtein and I. M. Ryzhik, Tablitsi integralov, summ, ryadov i proizvedenii (Tables of Integrals, Series and Products), Fizmatgiz, 1962 (Eng. Transl., Academic Press, 1966).

[12]E. Titchmarsh, Introduction to the Theory of Fourier Integrals, 2nd ed., Oxford University Press, 1948 (Russ. Transl., Gostekhizdat, 1948).

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