Invariants of the nonlinear oscillation equation

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Problems associated with the phenomenon of self-focusing in nonlinear media are considered. Invariants of the nonlinear oscillation equation are determined. The sufficient conditions for self-focusing to occur are elucidated on their basis, and the possibility of estimating the parameters of the beam thus formed on the basis of known boundary conditions is studied.

In the present work, we consider the problem of establishing oscillations in a semi-infinite medium described by the equation

$$\Delta \mathbf{E} - \frac{\partial^2 \mathbf{E}}{\partial^2 t}(\varepsilon_0 + \Phi(|\mathbf{E}|^2)) = 0,$$

where the positive function $\Phi(\xi) = 0(\xi)$ as $\xi \to 0$. As is well known, the propagation of waves in such a medium is accompanied by the phenomenon of self-focusing ([1,2]). If harmonic oscillations are excited on the boundary of the medium at $z = 0$, then $\mathbf{E}$ is representable in the form $\mathbf{E} = e^{i(\omega t - kz)}\mathbf{A}$, where $k^2 = \omega^2\epsilon_0$, and the function $\mathbf{A}$ satisfies the equation

$$2ik\partial\mathbf{A}/\partial z = \Delta\mathbf{A} + F(|\mathbf{A}|^2)\mathbf{A}, \tag{1}$$

in which $F(\xi) = \omega^2\Phi(\xi)$. In the present paper, we study several problems on the behavior of the solution of Eq. (1) at infinity.

Frequently, in the study of the phenomenon of self-focusing, in place of Eq. (1), the studies are limited to the consideration of the so-called parabolic approximation, i.e., the approximation in which k is assumed to be large and the quantity $\partial^2 A/\partial z^2$ is neglected in comparison with $k\partial A/\partial z$. Then, in place of (1), we have

$$2ik\partial\mathbf{A}/\partial z = \Delta_\perp\mathbf{A} + F(|\mathbf{A}|^2)\mathbf{A}, \tag{2}$$

where $\Delta_\perp$ is an operator in the coordinates of the plane S orthogonal to the z axis. The behavior of the solution of Eq. (2) at large z was considered in [3]. In that work, it was established that Eq. (2) has two invariants relative to z, i.e., the quantities I_1 and I_2 such that $\partial I_1/\partial z = \partial I_2/\partial z = 0$. Furthermore, using these invariants, the sufficient conditions are established under which the phenomenon of self-focusing will be observed for the solution of Eq. (2) and some other problems.

However, it remains unclear at the present time in what measure the parabolic approximation is applicable to the solution of the entire equation (1) at large values of z. Therefore, it is of interest to obtain the invariants of the entire equation and analyze the solution on their basis. In Sec. 1 of the present work, the invariants of Eq. (1) are indicated and the sufficient conditions established under which the phenomenon of self-focusing at large z will be observed. In Sec. 2, the possibility is studied of estimating the parameters of the formed self-focused beam from the given boundary conditions.

1. We establish the fact that Eq. (1) has four invariants relative to z. To be precise, we represent A in the form

$$\mathbf{A} = \sum_{\alpha=1}^{3} \mathbf{n}_\alpha u_\alpha e^{i\eta_\alpha},$$

where u_α and η_α are real functions, and $\mathbf{n}_\alpha$ are unit orthogonal vectors. Then, after cancelling the factor $\exp(i\eta_\alpha)$ and equating the real and imaginary parts in Eq. (1), we obtain

$$2k\frac{\partial u_\alpha}{\partial z} = u_\alpha\Delta\eta_\alpha + 2\nabla\eta_\alpha\cdot\nabla u_\alpha$$

or

$$k\frac{\partial u_\alpha^2}{\partial z} = \nabla_\perp(u_\alpha^2\nabla_\perp\eta_\alpha) + \frac{\partial}{\partial z}\left(u_\alpha^2\frac{\partial\eta_\alpha}{\partial z}\right).$$

Thence

$$\frac{\partial}{\partial z}\int_S u_\alpha^2\left(k - \frac{\partial\eta_\alpha}{\partial z}\right)d\sigma = \int_S \nabla_\perp(u_\alpha\nabla_\perp\eta_\alpha)\,d\sigma = 0.$$

Consequently, the quantities

$$P_1^{(\alpha)} = 2k\int_S u_\alpha^2\left(k - \frac{\partial\eta}{\partial z}\right)d\sigma \tag{3}$$

do not depend on the coordinate z.

Furthermore, we consider the expression

$$P_2 = \int_S\left[\int_0^{|\mathbf{A}|^2} F(\xi)\,d\xi + \sum_{\alpha=1}^{3}\left(2\left|\frac{\partial A_\alpha}{\partial z}\right|^2 - |\nabla A_\alpha|^2\right)\right]d\sigma. \tag{4}$$

Designating by $\mathbf{A}^*$ the vector that is the complex conjugate of $\mathbf{A}$, we get from (1)

$$-2ik\frac{\partial\mathbf{A}^*}{\partial z} = \Delta\mathbf{A}^* + F(|\mathbf{A}|^2)\mathbf{A}^*.$$

Taking this into account, we obtain

$$\begin{aligned}\frac{\partial P_2}{\partial z} &= \int_S\Big[\left(\frac{\partial\mathbf{A}}{\partial z}\mathbf{A}^* + \frac{\partial\mathbf{A}^*}{\partial z}\mathbf{A}\right)F(|\mathbf{A}|^2) + \sum_{\alpha=1}^{3}\Big(2\frac{\partial A_\alpha}{\partial z}\frac{\partial^2 A_\alpha^*}{\partial z^2} \\ &\quad + 2\frac{\partial A_\alpha^*}{\partial z}\frac{\partial^2 A_\alpha}{\partial z^2} - \nabla A_\alpha\cdot\nabla\frac{\partial A_\alpha^*}{\partial z} - \nabla A_\alpha^*\cdot\nabla\frac{\partial A_\alpha}{\partial z}\Big)\Big]d\sigma \\ &= \int_S\left[\frac{\partial\mathbf{A}}{\partial z}(F\mathbf{A}^* + \Delta\mathbf{A}^*) + \frac{\partial\mathbf{A}^*}{\partial z}(F\mathbf{A} + \Delta\mathbf{A})\right]d\sigma \\ &= \int_S\left[\frac{\partial\mathbf{A}}{\partial z}\left(-2ik\frac{\partial\mathbf{A}^*}{\partial z}\right) + \frac{\partial\mathbf{A}^*}{\partial z}2ik\frac{\partial\mathbf{A}}{\partial z}\right]d\sigma = 0.\end{aligned}$$

Then, in the case in which $\mathbf{E}$ and $\mathbf{A}$ are vectors, the invariants (relative to z) for the solution of Eq. (1) are the quantities $P_1^{(\alpha)}$ ($\alpha = 1, 2, 3$) and P_2. If E and A are scalars, then Eq. (1) has two invariants, P_1 and P_2.

We note the following circumstance. The invariants of the parabolic approximation (see [3]) differ from the invariants P_1 and P_2 of the total equation by the fact that they do not contain the terms $\partial\eta/\partial z$ and $|\partial\mathbf{A}/\partial z|^2$. The contribution of these terms to the values of the invariants P cannot be assumed to be small. Actually, in addition to the self-focused wave, which propagates in the z direction, the boundary oscillations and the oscillations in the process of beam formation produce waves that diverge in the radial direction. Inasmuch as $\partial\eta/\partial z \sim k$ and $|\partial A/\partial z|^2 \sim |\nabla_\perp A|^2$ in the latter, and the quantity $\int u^2 d\sigma$ can also not be small in comparison with the corresponding quantity in the self-focused beam, we can assume in the general case that

$$\left|\int_S u^2\frac{\partial\eta}{\partial z}d\sigma\right| \ll k\int_S u^2\,d\sigma, \quad \int_S\left|\frac{\partial A}{\partial z}\right|^2 d\sigma \ll \int_S|\nabla_\perp A|^2\,d\sigma.$$

We now consider the solution of Eq. (1) with boundary conditions (at z = 0) that differ from zero in a certain limited region G. We introduce a spherical system of coordinates $\{R, \theta, \varphi\}$ such that its center lies in G and the angle θ is the angle between the direction to the point and the z axis. Then the following statement is valid: if the boundary conditions are such that $P_2 > 0$, then $\max|\mathbf{A}| \geq q > 0$ as $R \to \infty$, and this means that a self-focused beam is formed. Actually, let us assume the opposite. Then, as $R \to \infty$, the amplitude of the oscillations $|\mathbf{A}|$ tends to zero and the waves are propagated as if the medium were linear. In this case, the following asymptotic relations hold:

$$u_\alpha = \frac{Y_\alpha(\theta,\varphi)}{R} + O\left(\frac{1}{R^2}\right), \quad \frac{\partial\eta_\alpha}{\partial z} = k(1-\cos\theta) + O\left(\frac{1}{R^2}\right), \quad (5)$$

where $Y_\alpha(\theta, \varphi)$ are certain functions determined by the boundary conditions:

The expression (4) for the invariant P_2 can be transformed [by using Eq. (1)] to a different form that is useful for what follows:

$$P_2 = \int_S \Big[\int_0^{|\mathbf{A}|^2} F(\xi)\,d\xi - F(|\mathbf{A}|^2)|\mathbf{A}|^2 + \sum_{\alpha=1}^{3} \left(2u_\alpha^2 \left(\frac{\partial\eta_\alpha}{\partial z}\right)^2 + \left(\frac{\partial u_\alpha}{\partial z}\right)^2 - u_\alpha \frac{\partial^2 u_\alpha}{\partial z^2} - 2k\frac{\partial\eta_\alpha}{\partial z}u_\alpha^2 \right) \Big] d\sigma. \quad (6)$$

As $R \to \infty$, taking the asymptotic form (5) into account, we have

$$\left| \int_S \Big[\int_0^{|\mathbf{A}|^2} F(\xi)\,d\xi - F(|\mathbf{A}|^2)|\mathbf{A}|^2 + \sum_{\alpha=1}^{3} \left(\left(\frac{\partial u_\alpha}{\partial z}\right)^2 - u_\alpha \frac{\partial^2 u_\alpha}{\partial z^2} \right) \Big] d\sigma \right| \ll \int_S 2\sum_{\alpha=1}^{3} u_\alpha^2 \frac{\partial\eta_\alpha}{\partial z}\left(k - \frac{\partial\eta_\alpha}{\partial z}\right) d\sigma.$$

Then, keeping the principal terms of the asymptotic expression, we get, as $z \to \infty$:

$$P_2 = -\int_S 2\sum_{\alpha=1}^{3} \frac{Y_\alpha^2(\theta,\varphi)}{R^2} k^2 (1-\cos\theta)\cos\theta\, d\theta = -2k^2 \int_0^{2\pi} d\varphi \int_0^{\pi/2} \sum_{\alpha=1}^{3} Y^2(\theta,\varphi)(1-\cos\theta)\sin\theta\, d\theta < 0,$$

which contradicts the assumption that $P_2 > 0$. Thus the condition $P_2 > 0$ is the sufficient condition that $\max|\mathbf{A}|^2 \geq q > 0$ will be accomplished as $R \to \infty$. The quantity $|\partial\mathbf{A}/\partial z|^2$ enters into the expression for the invariant P_2. If not the normal derivative of the function $\mathbf{A}$ but only the function itself is known on the boundary, then the condition $P_2 > 0$ can be replaced by the equation

$$I = \int_S \Big[\int_0^{|\mathbf{A}|^2} F(\xi)\,d\xi - \sum_{\alpha=1}^{3} |\nabla_\perp A_\alpha|^2 \Big] d\sigma > 0,$$

since $P_2 > I$ always. (We note that the condition $I > 0$ is also a sufficient condition for self-focusing of the solution of Eq. (2), see[3].)

2. As is well known ([2,4]), Eq. (1), unlike the corresponding equation for a linear medium, admits of a solution in the form of a wave of finite energy, which has constant shape and amplitude relative to the coordinate z. Such a wave has a plane wave front. For simplicity, we consider the scalar case, when $A = ue^{i\eta}$. Then the functions u and η in the plane wave mentioned above satisfy the relations

$$\Delta_\perp u + F(u^2)u = k^2(C^2+2C)u = Bu, \quad (7a)$$

$$\partial\eta/\partial z = -kC \quad (7b)$$

and the condition for (7a) at infinity: $u \to 0$, $\rho \to \infty$. Inasmuch as $F(u^2)u \ll Bu$ as $\rho \to \infty$, we have $u = O(\exp(-B^{1/2}\rho)/\rho^{1/2})$ for large ρ. We also note that Eq. (7a) admits of solutions of different types, i.e., solutions which have different numbers of maxima (see[5]).

Let oscillations be excited on the boundary, at z = 0, in a certain restricted region, such that $P_2 > 0$. Here, as was established in Sec. 1, a self-focusing beam is formed. (For simplicity, we shall assume that both the boundary conditions and the beam are axially symmetric.) If $\nabla_\perp\eta \neq 0$, a change takes place in the shape of the beam, which is generally accompanied by radiation of radial waves. Inasmuch as the quantity max u does not tend toward zero as $z \to \infty$, it is natural to assume that $\nabla_\perp\eta \to 0$ as $z \to \infty$ and the phase front of the self-focused beam becomes plane.

Taking into account the leading considerations set forth above, we consider the case in which a self-focused plane wave of any type satisfying Eq. (7a), and certain waves diverging in the radial direction, are formed in the medium under the action of boundary oscillations. In this case, let us assess the possibility determining the parameters of the self-focused wave as $z \to \infty$ from the boundary conditions with the help of the invariants.

In accord with the assumption, the phase front tends as $R \to \infty$ to the shown "limiting" form, where the angle θ_0 is determined from the condition of the identity of the phases of the plane and diverging waves at the junction point (see Fig. 1). At large z, the plane wave 1 moves with the velocity $\omega/k(1 + C)$ in the z direction, and the diverging wave 2 has a small amplitude as $R \to \infty$ and consequently its phase front is propagated in the radial direction—just as if the medium were linear—with the velocity ω/k. Thence, at the junction point, as $R \to \infty$, the ratio z/R tends toward the value $1/(1 + C)$. Therefore, in the "limit" we have

$$\theta_0 = \arccos\frac{1}{1+C}.$$

Further, in the plane wave, in accord with (7), as $R \to \infty$

$$u(R,\theta) = O\left(\frac{\exp(-B^{1/2}R\sin\theta)}{(R\sin\theta)^{1/2}}\right).$$

Consequently, $u(R, \theta_0) = O(1/R^2)$. Therefore, matching (as $R \to \infty$) the asymptotic form of the function u in the plane and the diverging waves on the ray θ_0, we have in the diverging wave, as $R \to \infty$,

$$u(R,\theta) = \frac{Y(\theta)}{R} + O\left(\frac{1}{R^2}\right), \quad \frac{\partial\eta}{\partial z} = k(1-\cos\theta) + O\left(\frac{1}{R^2}\right),$$

where $Y(\theta)$ is some function determined by the initial conditions and by the processes which take place upon formation of the wave, while $Y(\theta_0) = Y(\pi/2) = 0$.

We represent P_1 as the sum $P_1 = P_1^f + \Delta P_1$, where P_1^f is the integral (3) over the part of S corresponding to the self-focused wave, i.e., in that part of S which is seen from the point O at the angle $\theta \leq \theta_0$, and ΔP_1 is the integral (3) over the part of S corresponding to the diverging waves and seen from the origin at the angle $\theta \geq \theta_0$. Similarly, we represent P_2 in the form $P_2 = P_2^f + \Delta P_2$.

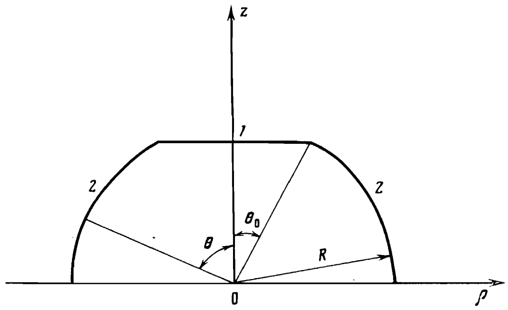

FIG. 1

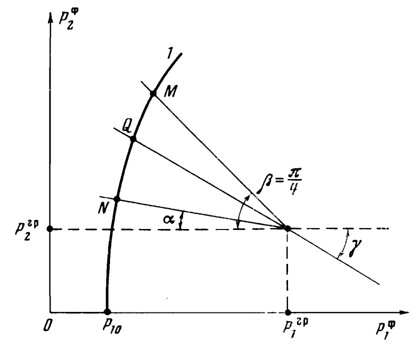

FIG. 2

As $z \to \infty$, keeping the principal terms of the asymptotic expression, we have

$$\Delta P_1 = 2k \int_{z\,\mathrm{tg}\,\theta_0}^{\infty} u^2 \left(k - \frac{\partial\eta}{\partial z}\right) \rho\, d\rho \to 2k^2 \int_{\theta_0}^{\pi/2} Y^2(\theta) \sin\theta\, d\theta. \qquad (8)$$

Similarly, we get from (6) as $z \to \infty$,

$$\Delta P_2 = \int_{z\,\mathrm{tg}\,\theta_0}^{\infty} \left[\int_0^{u^2} F(\xi)\, d\xi - F(u^2) u + 2u^2 \left(\frac{\partial\eta}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 - u\frac{\partial^2 u}{\partial z^2} - 2k\frac{\partial\eta}{\partial z} u^2 \right] \rho\, d\rho \to 2 \int_{z\,\mathrm{tg}\,\theta_0}^{\infty} \frac{\partial\eta}{\partial z}\left(\frac{\partial\eta}{\partial z} - k\right) u^2 \rho\, d\rho \qquad (9)$$

$$\to -2k^2 \int_{\theta_0}^{\pi/2} Y^2(\theta) \sin\theta (1 - \cos\theta)\, d\theta.$$

Then

$$\Delta \bar{P}_1 \geqslant -\Delta \bar{P}_2 \geqslant \Delta \bar{P}_1 (1 - \cos\theta_0) = \Delta \bar{P}_1 \frac{1}{1+C}, \qquad \Delta \bar{P}_{1,2} = \lim_{z\to\infty} \Delta P_{1,2}. \qquad (10)$$

We now ascertain how the quantities P_1^f and P_2^f are connected as $z \to \infty$. As mentioned above, we consider the case in which the function $u \to \bar{u}$ in the plane wave as $z \to \infty$, where $\bar{u}$ is the solution of Eq. (7a). From (7a), we have the relation

$$\int_0^\infty \rho\, d\rho \int_0^{\bar{u}^2} F(\xi)\, d\xi = B \int_0^\infty \bar{u}^2 \rho\, d\rho.$$

Taking it into account, we have from (3), (4), and (7), as $u \to \bar{u}$:

$$P_1{}^f \underset{z\to\infty}{\longrightarrow} \bar{P}_1{}^f = 2k^2 (1+C) \int_0^\infty \bar{u}^2 \rho\, d\rho,$$

$$P_2{}^f \underset{z\to\infty}{\longrightarrow} \bar{P}_2{}^f = \int_0^\infty \left[\int_0^{u^2} F(\xi)\, d\xi - F(\bar{u}^2) u^2 + 2k^2 C (1+C) \bar{u}^2 \right] \rho\, d\rho$$

$$= k^2 \int_0^\infty \left[C(4+3C) - \frac{\Phi(\bar{u}^2)}{\varepsilon_0} \right] \bar{u}^2 \rho\, d\rho, \qquad (11)$$

where, according to the notation introduced above, $\Phi(\xi) = \omega^{-2} F(\xi) = \epsilon_0 k^{-2} F(\xi)$.

The function $\bar{u}_C(\rho)$, which depends on C as a parameter, can be determined from Eq. (7a) for various values of C if the type of plane wave is known. Substituting $\bar{u}_C(\rho)$ in (11), we find the dependence of $\bar{P}_1^f$ and $\bar{P}_2^f$ on C, and by the same token, the parametric connection of $\bar{P}^f$ and $\bar{P}_2^f$. For example, let the relation between $\bar{P}_1^f$ and $\bar{P}_2^f$ have the form of curve 1 of Fig. 2. The point $(P_{10}, 0)$ corresponds to C equal to zero. (If $F''(\xi) < 0$, then the function $\bar{P}_2^f(\bar{P}_1^f)$ has the presented form in some range of variation of C.)

Let P_1 and P_2 have certain values $P_1^b > P_{10}$ and $P_2^b > 0$ on the boundary. Then the values $\bar{P}_1^f$ and $\bar{P}_2^f$ in the plane wave, corresponding to the data of these boundary conditions should on the oneehand lie on the curve 1 in Fig. 2, and on the other the quantities $P_1^b - \bar{P}_1 = \Delta\bar{P}_1$ and $P_2^b = \Delta\bar{P}_2$ are connected by the relation (10). Consequently, the values $\bar{P}_1^f$ and $\bar{P}_2^f$, which correspond to the boundary data P_1^b and P_2^b, will lie on the arc (N, M) of curve 1 in Fig. 2, where the position of points N and M is determined in the following way: $\tan\beta = 1$ and $\tan\alpha = \tilde{C}/(1 + \tilde{C})$, where $\tilde{C}$ is the value of the parameter C at the point N.

Thus, knowing the boundary conditions, the function $F(\xi)$, and the type of plane wave being formed, we can determine a certain section in which the values of C are located (from $\tilde{C}$ to the value of C corresponding to the point M). From this section we can calculate, starting with Eq. (7a), the corresponding section of the values of the amplitude and other parameters.

The exact values of $\bar{P}_1^f$ and $\bar{P}_2^f$ and the parameter C corresponding to them depend on the relative angular distribution of the amplitude of the scattered wave $Y(\theta)$. If this angular distribution can be measured with accuracy up to a multiplicative factor, then the linear relation between $\Delta\bar{P}_1$ and $\Delta\bar{P}_2$ can be found from (8) and (9). Let $\gamma = \arctan|\Delta\bar{P}_2/\Delta\bar{P}_1|$. Then the intersection point Q of curve 1 in Fig. 2 with the straight line passing through the point (P_1^b, P_2^b) at the angle γ gives the values of $\bar{P}_1^f$, $\bar{P}_2^f$ and C in the formed plane wave, and from them the values of the other parameters of the plane wave.

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