# Propagation of electromagnetic waves in an inhomogeneous nonlinear medium

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An investigation is made of the propagation of waves of the TE type along a spatially inhomogeneous layer of a nonlinear self-focusing or self-defocusing medium. It is shown that under certain conditions depending on the intensity of the electromagnetic field waveguide propagation of the TE field may be disrupted. Conditions for the localization of a strong field in a plasma layer region leading to oscillations of a weak field at infinity are elucidated for TE-type waves propagating along an inhomogeneous plasma layer.

1. A considerable number of papers has been devoted to investigations of self-focusing electromagnetic fields in a homogeneous nonlinear transparent medium. We indicate here a number of papers[1-6] in which the principal features of the structure of the electromagnetic field of self-focusing waveguides have been elucidated. However, in our opinion sufficient attention has not been paid to investigations of self-focusing fields in inhomogeneous nonlinear media. A dielectric waveguide produced in a nonlinear transparent medium can serve as an example which is also of practical interest.

It is shown below that propagation of an electromagnetic field along a layer of inhomogeneity in a nonlinear transparent medim possesses a number of important properties. Thus, as the intensity of the electromagnetic field in a dielectric waveguide increase a disruption occurs of propagation in the waveguide if the nonlinear medium is self-defocusing. Another limitation on the amplitude of the excited waves turns out to be the phenomenon of phase mismatching arising as the intensity of the field is increased in an optical dielectric waveguide the modes of which require phase matching with the radiation being introduced. Finally, in investigating the propagation of TE waves along an inhomogeneous plasma layer it has been shown that there exist solutions which correspond to the localization of a strong field in the region of the plasma layer and to oscillations of a weak field at infinity.

The dispersion relation for the above field distributions arises in seeking such solutions of the field in the symmetry plane of the layer which lead to a minimal amplitude of stable oscillations at infinity. Such field distributions are of interest in connection with efforts to create systems of controlled thermonuclear synthesis based on the "trapping" of powerful laser radiation in a long but thin plasma filament contained by a magnetic field[7].

The distributions of fields of the TE type in a nonlinear inhomogeneous medium in the case of plane geometry are determined by solutions of the equation

$$-d^2E/dx^2=[k^2\varepsilon(x,E^2)-k_z^2]E. \tag{1.1}$$

Here $k_z$ is the longitudinal wave number, $k=\omega/c$, $\epsilon(x, E^2)$ is the nonlinear dielectric permittivity at the point x which depends on the field at the given point E(x).

2. We consider the case when the nonlinear dielectric permittivity is of the form

$$\varepsilon(x,E^2)=\varepsilon(x)+\varepsilon_2(x)E^2. \tag{2.1}$$

Here $\epsilon(x)$ corresponds (in the case of a vanishingly weak field) to the inhomogeneous dielectric permittivity of an optically more dense layer, $\epsilon(0) > \epsilon(\infty)$; finally, $\epsilon_2(x)$ is the parameter of the inhomogeneous self-focusing (in the case of $\epsilon_2 > 0$) or self-defocusing (in the case of $\epsilon_2 < 0$) nonlinear medium. Taking into account (2.1) we write (1.1) in the form

$$d^2E/d\xi^2+[\varepsilon(\xi)-\chi_z^2]E=-\varepsilon_2(\xi)E^3,\quad \chi_z=k_z/k,\quad \xi=kx. \tag{2.2}$$

Let the inhomogeneity be such that

$$\varepsilon(\xi)=\varepsilon(\infty)+\frac{\Delta\varepsilon}{\mathrm{ch}^2(\xi/kl)}. \tag{2.3}$$

In this case Eq. (2.2) leads in the linear approximation to the following expression for the discrete spectrum of eigenvalues for the longitudinal wave number (cf., for example,[8]):

$$\chi_z^2(n)=\varepsilon(\infty)+(s-n)^2/(kl)^2. \tag{2.4}$$

Here the parameter s is determined by the relation

$$s(s+1)=(kl)^2\Delta\varepsilon, \tag{2.5}$$

while n are integers which do not exceed s. Relation (2.4) represents the dispersion relation for the constants characterizing the propagation in a plane dielectric waveguide of characteristic thickness $l$, while the greatest allowable value of n determines the number of waveguide modes of the TE type. Utilizing an expansion in terms of the small field amplitude

$$E=E_n+E_n^{(1)}+\dots,\quad \chi_z^2=\chi_z^2(n)+\delta^{(1)}(n)+\dots, \tag{2.6}$$

where $E_n$ and $\chi_z(n)$ are the field and the propagation constant for the n-th mode, we find that

$$d^2E_n^{(1)}/d\xi^2+[\varepsilon(\xi)-\chi_z^2(n)]E_n^{(1)}=-\varepsilon_2(\xi)E_n^3+\delta^{(1)}(n)E_n. \tag{2.7}$$

The condition for being able to solve the inhomogeneous equation (2.7) determines the value of $\delta^{(1)}(n)$ and leads, taking (2.6) into account, to the dispersion relation

$$\chi_z^2=\varepsilon(\infty)+\frac{(s-n)^2}{(kl)^2}+\int d\xi\,\varepsilon_2(\xi)E_n^4\Big/\int d\xi\,E_n^2+\dots \tag{2.8}$$

This dispersion relation takes into account the effect of the nonlinear properties of the medium and of the finite amplitude of the field on the constants describing the propagation of waveguide modes of the TE type. For example, for a single mode dielectric waveguide, when s = 1, the eigenfunction of the principal mode is equal to

$$E(\xi)=\frac{E(0)}{\mathrm{ch}(\xi/kl)} \tag{2.9}$$

and the relation (2.8) for the case when the parameter of an inhomogeneous nonlinear medium given by

$$\varepsilon_2(\xi)=\frac{\varepsilon_2(0)}{\mathrm{ch}^2(\xi/kl_2)},$$

assumes the form

$$\chi_z^2(0)=\varepsilon(\infty)+\frac{1}{(kl)^2}+\varepsilon_2(0)E^2(0)F\left(\frac{l}{l_2}\right). \quad (2.10)$$

Here F depends on the ratio of the characteristic dimensions of the inhomogeneities of the linear ($l$) and the nonlinear ($l_2$) contributions to the dielectric permittivity of the medium. In the case when the characteristic dimensions of the inhomogeneities are the same $F(l_2 = l) = 8/15$, while in the case $l_2 \gg l$ the function $F \to 2/3$.

We consider the case of a self-defocusing medium ($\epsilon_2 < 0$). The relation (2.10) shows that as the amplitude of the field E(0) increases the longitudinal wave number $\chi_z(0)$ diminishes and can become smaller than $\epsilon^{1/2}(\infty)$. However, Eq. (2.2) can not have localized solutions for $\chi_z^2 < \epsilon(\infty)$. Thus, the critical value of the amplitude of the field which leads to a disruption of the principal waveguide mode of the TE type of a dielectric waveguide due to the self-defocusing properties of the nonlinear medium is determined by the relation

$$E^2(0)=-\frac{\varepsilon_2^{-1}(0)}{(kl)^2F(l/l_2)}. \quad (2.11)$$

We note that for $(kl)^2 \gg 1$, and in virtue of (2.5) also $\Delta\epsilon \ll \epsilon(\infty)$, the critical amplitude is equal to

$$E(0)\approx\frac{1}{|\varepsilon_2(0)|^{1/2}}\frac{\lambda}{l}, \quad (2.12)$$

where $\lambda = k^{-1}$, and turns out to be smaller than the characteristic field of the nonlinearity in the ratio of the wavelength of the radiation $\lambda$ to the characteristic dimension of the inhomogeneity $l$. In the case of a non-single-mode dielectric waveguide (s > 2) in a self-defocusing medium as the intensity of the electromagnetic field increases at first modes of higher order will be subject to disruption, and the number of propagating localized modes will diminish as the field is increased.

In the excitation of optical dielectric waveguides (for example, by the method of spoiled total internal reflection[9]) resonance coupling is utilized which arises when the projection of the propagation vector of the radiation being introduced on the symmetry plane of the waveguide coincides with the propagation constant for one of the waveguide modes. For a dielectric waveguide in a nonlinear medium the condition of phase resonance depends in virtue of (2.8) on the amplitude of the field, and as the latter increases a mismatch arises which leads to a restriction on the amplitude of the mode being excited (with the projection of the propagation vector of the radiation being introduced remaining unchanged).

The example of inhomogeneity (2.3) considered above evidently has a universal significance since the same conclusions can be drawn also for other cases of inhomogeneous dielectric waveguides.

We note that in the case of propagation of waves of the TE type in a self-focusing medium along an optically less dense layer when the dielectric permittivity has the form (2.1) and $\epsilon(0) < \epsilon(\infty)$ waveguide distributions of the field localized with respect to the transverse variable x are possible only in the case when the amplitude of the field exceeds a certain critical value. Since one of the conditions for the existence of self-focusing states is $(k_z/k)^2 > \epsilon(\infty)$ then localization of the field in an optically less dense layer is possible only if

$$E(0)\geqslant\frac{1}{\varepsilon_2^{1/2}}\left(\frac{\varepsilon(\infty)-\varepsilon(0)}{\varepsilon(0)}\right)^{1/2}$$

If $\epsilon(\infty) - \epsilon(0) \ll \epsilon(0)$, the critical field E(0) is considerably smaller than the characteristic field of the nonlinearity $1/\epsilon_2^{1/2}$.

We note here that in a homogeneous transparent nonlinear medium localized solutions exist for any value of the amplitude of the field on the plane of symmetry of a self-focusing waveguide. It is evident that in the linear approximation, i.e., for fields with a vanishingly small amplitude, there are no localized solutions. The lower bound on the amplitude of the field corresponds essentially to the condition of total internal reflection of the radiation propagated in a layer of the nonlinear medium which is optically less dense only in a sufficiently weak field.

3. We consider the propagation of waves of the TE type along an inhomogeneous plasma layer taking into account the nonlinear properties of the plasma. The dielectric permittivity has the form

$$\varepsilon(x,E^2)=1-\frac{4\pi e^2}{m\omega^2}n\left(x,\frac{E^2}{E_n^2}\right). \quad (3.1)$$

Here $E_n$ is the characteristic field of the nonlinearity, n is the density of particles at the point x under the condition that the field at the given point is E(x).

For example, let

$$n\left(x,\frac{E^2}{E_n^2}\right)=n\left(\frac{x}{l}\right)\exp\left[-\left(\frac{E}{E_n}\right)^2\right], \quad (3.2)$$

where the explicit form of $n(x/l)$ is determined by the method of containment of plasma, $E_n^2 = 4Tm\omega^2/e^2$, while $l$ is the characteristic size of the inhomogeneity. In the case of a homogeneous nonlinear medium (n = const) equation (1.1) leads to self-focusing distributions of the TE-field in the plasma for $(k_z/k)^2 < 1$. Taking into account (3.1), (3.2) we have

$$\frac{d^2\mathscr{E}}{d\eta^2}+\left[1-A\rho\left(\frac{\eta}{\tau}\right)e^{-\mathscr{E}^2}\right]\mathscr{E}=0;$$

$$(1-\chi_z^2)A=\left(\frac{\omega_m}{\omega}\right)^2,\quad \tau=(1-\chi_z^2)^{1/2}kl,\quad \eta=(1-\chi_z^2)^{1/2}kx, \quad (3.3)$$

$$\mathscr{E}=\frac{E}{E_n},\quad \omega_m^2=\frac{4\pi e^2}{m}n(0),\quad \rho=\frac{n(\eta/\tau)}{n(0)}.$$

We consider the expression

$$\mathscr{H}=\left(\frac{d\mathscr{E}}{d\eta}\right)^2+\mathscr{E}^2-A\rho\left(\frac{\eta}{\tau}\right)[1-e^{-\mathscr{E}^2}]. \quad (3.4)$$

In the phase space $(d\mathscr{E}/d\eta, \mathscr{E}, \eta)$ the surfaces $\mathscr{H}$ = const for $|\eta| \ll \tau$ are close to the surfaces of the first integrals of the homogeneous nonlinear problem, while for $|\eta| \gg \tau$ they are close to the surfaces of the first integrals of the homogeneous problem for free space. Further we shall assume that the density of the particles of the plasma layer decreases monotonically from unity at x = 0 to zero as $|x| \to \infty$. Differentiating (3.4) we find that

$$\frac{d\mathscr{H}}{d\eta}=-A\frac{d\rho}{d\eta}[1-e^{-\mathscr{E}^2}]. \quad (3.5)$$

It is evident that for all $\eta \geq 0$ the derivative $d\mathscr{H}/d\eta$ is nonnegative.

Investigating the curves of the inflection points of the solutions of equation (3.3) for

$$\rho=\exp[-(\eta/\tau)^2], \quad (3.6)$$

and it is specifically for such a distribution of the particle density in the layer that the numerical calculations presented below have been carried out, it can be shown that for $|\eta| < \tau \ln^{1/2} A$ there exist three branches of the curve of inflection points:

$$\mathscr{E}_{\pm}(\eta)=\pm[\ln A-(\eta/\tau)^2]^{1/2}, \quad \mathscr{E}_0(\eta)\equiv 0, \tag{3.7}$$

while for $|\eta| > \tau \ln^{1/2} A$ there exists the single branch $\mathscr{E}_0(\eta) \equiv 0$. In phase space the surfaces $\mathscr{H} = 0$ and $\mathscr{H} = \text{const} < 0$ are closed (cf. Fig. 1), while the surfaces $\mathscr{H} = \text{const} > 0$ are open along the $\eta$ axis and for $|\eta| \gg \tau \ln^{1/2} A$ they are close to the surfaces of a circular cylinder

$$\left(\frac{d\mathscr{E}}{d\eta}\right)^2+\mathscr{E}^2=\text{const}>0. \tag{3.8}$$

We investigate the boundary conditions

$$\mathscr{E}_{\eta=0}=\mathscr{E}(0), \quad \left.\frac{d\mathscr{E}}{d\eta}\right|_{\eta=0}=0. \tag{3.9}$$

The values of $\mathscr{E}(0)$ belonging to the surfaces $\mathscr{H} = \text{const} > 0$ are of no interest to us. Indeed, in view of the nonnegative nature of the derivative $d\mathscr{H}/d\eta$ as $\eta$ increases the integral curve $\mathscr{E}[\eta, \mathscr{E}(0)]$ intersects the surfaces $\mathscr{H} = \text{const}$ with ever increasing values of the positive constant. As $\eta \to \infty$ the integral curve emerges on the surface of the circular cylinder (3.8). In virtue of the fact that $\mathscr{H}(\infty) > \mathscr{H}(0)$ the amplitude of the established oscillations $\mathscr{E}(\infty)$ will exceed the boundary value of the field $\mathscr{E}(0)$. Let $\mathscr{E}(0)$ belong to the surface $\mathscr{H} = \text{const} < 0$. In this case the integral curve will intersect the surface $\mathscr{H} = 0$ at the point which is separated from the boundary plane by a distance $< \tau \ln^{1/2} A$. If at the point of emergence of the integral curve on the surface $\mathscr{H} = 0$ the values of $(d\mathscr{E}/d\eta, \mathscr{E})$ turn out to be small, then the derivative $d\mathscr{H}/d\eta$ will also be small.

One should expect (and this is confirmed by results of numerical calculations), that as $\eta \to \infty$ the integral curve will emerge on the cylindrical surface (3.10) corresponding to oscillations of the field with the small amplitude $\mathscr{H}^{1/2}(\infty)$. The solution of the boundary problem (3.9) leads to the determination of the amplitude of the established oscillations of the field at infinity as a function of the parameters $(A, \tau)$ and of the amplitude of the field on the boundary plane $\mathscr{E}(0)$. Since in the case under consideration the field at infinity does not vanish, it is necessary to find such values of $\mathscr{E}(0)$ for which the amplitude of the oscillations of the field at infinity is minimal. The dependence of the amplitude of the established oscillations on the boundary field is shown in Fig. 2. For $A = 9$, $\tau = 10$ there eixts a sharply pronounced minimum for $\mathscr{E}(0) \approx 1.78$, which leads to $\mathscr{E}^2(\infty) \approx 0.007$, and a weak minimum for $\mathscr{E}(0) \approx 1.48$ which leads to established oscillations with a considerably greater amplitude. Figure 3 shows a distribution of the field corresponding to the minimal amplitude of

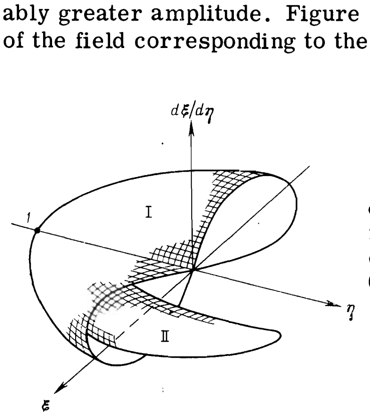


FIG. 1. The integral surfaces of $\mathcal{H}$ in phase space: I–the surface $\mathcal{H} = 0$. II–the surface $\mathcal{H} = \text{const} < 0$; 1–the point $d\xi/d\eta = 0$, $\eta = \tau \ln^{1/2} A$.

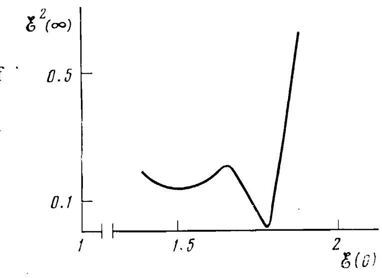


FIG. 2. The dependence of the amplitude of the oscillations of the field at infinity on the value of the field on the symmetry plane of the plasma layer for $A = 9$, $\tau = 10$.

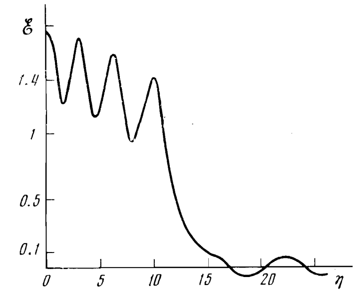


FIG. 3. The distribution of the electric field across the plasma layer for $A = 9$, $\tau = 10$ and $\xi(0) \approx 1.78$

established oscillations. For $\eta \lesssim 10$ the variations in the field correspond to oscillations near the curve of inflection points. However, for $\eta \gtrsim 10$ such oscillations are replaced by oscillations with a small amplitude about the zero value of the field.

Thus, in spite of the fact that in the propagation of TE waves along an inhomogeneous plasma layer there do not arise in the given case any self-focusing waveguide solutions, it turns out to be possible to construct solutions corresponding to a localization of a field of the TE type basically near the plane of symmetry of the inhomogeneous plasma layer with small established oscillations of the field at infinity. The dispersion relation which determines the longitudinal wave number as a function of the parameters of the inhomogeneous plasma layer and of the value of the field on the symmetry plane of the layer arises in seeking such a value of the field $\mathscr{E}(0)$ for which the amplitude of the field at infinity is minimal. Such a condition selects from the continuous spectrum of wave numbers $k_z$ only those which lead to the greatest permissible localization of the field in the region of the plasma layer.

In the geometrical optics approximation Steinhauer and Ahlstrom[7] have shown that a plasma filament which is the most favorable for the capture of laser radiation is such a distribution of plasma density, defining an inhomogeneous but linear dielectric permittivity of the medium, which has a local minimum on the plane or axis of symmetry of the plasma. The analysis carried out above has shown that the equations of nonlinear electrodynamics lead to solutions that correspond to a significant capture of radiation brought about by a redistribution of the plasma density in the field of a strong electromagnetic wave and leading to the formation of a local minimum of the density under conditions when the distribution of the plasma density in a weak field falls off monotonically as one recedes from the plane or axis of symmetry of the plasma layer (filament).

4. When dissipations are taken into account the equations of nonlinear electrodynamics for waves of the TE type assume the form[9]

$$\frac{d^2\mathscr{E}}{d\eta^2}+\left[1-\frac{\mu^2}{\mathscr{E}^4}-A\rho\left(\frac{\eta}{\tau}\right)e^{-\mathscr{E}^2}\right]\mathscr{E}=0,$$

$$\frac{d\mu}{d\eta} = -\gamma_s A \rho\left(\frac{\eta}{\tau}\right) e^{-\delta^2 \mathscr{E}^2};$$
$$(1-\chi_z^2)\mu^2 = M^2/E_n^4, \quad \gamma_s = \nu_{st}/\omega. \qquad (4.1)$$

Here M is the energy flux along the inhomogeneity, $\nu_{st}$ is the collision frequency.

We investigate the boundary conditions (3.9) supplemented by the condition $\mu(0) = 0$. Qualitative analysis shows that in this case for $\gamma_s \ll 1$ localization of a strong field turns out to be possible primarily in the region of the plasma layer. In this case at infinity there arise established oscillations of small amplitude about a weak nonzero value of the field. The absolute value of the energy flux as $\eta \to \infty$ increases and tends to a constant value. For constant plasma density the energy flux $\mu$ should grow indefinitely, and this also follows from results of the work of one of the authors[10]. In the case under consideration the plasma density falls off rapidly, and this limits the increase in the energy flux across the layer. Propagation of the TE waves along the plasma layer without a change in the amplitude of the field is maintained in the presence of dissipation at the expense of transverse energy fluxes $\mu(\pm\infty)$ directed from the external regions towards the symmetry plane of the plasma layer. The dispersion relation for such waves again arises from the condition of minimization of the field at infinity. Indeed, the asymptotic behavior of the solutions as $|\eta| \to \infty$ is completely determined by the expression

$$\mathscr{H}(\infty) = \left(\frac{d\mathscr{E}}{d\eta}\right)^2 + \mathscr{E}^2 + \frac{\mu^2(\infty)}{\mathscr{E}^2}.$$

Just as before, the problem consists of finding the minimum of $\mathscr{H}(\infty)$ as a function of the value of the amplitude of the field $\mathscr{E}(0)$. For low values of an established energy flux $\mu(\infty)$ this is what leads to the weak oscillations of the field at infinity about a nonzero solution of the average field which is likewise small compared to the value of the field on the symmetry plane of the plasma layer. Results of numerical calculations carried out for values of the parameters A = 10, $\tau$ = 10 and $\gamma_s = 0.001$ point to the existence of a weak minimum in the oscillations of the field at infinity for $\mathscr{E}(0) \approx 1.5$ and of a pronounced minimum at $\mathscr{E}(0) \approx 1.95$ (Fig. 4).

In conclusion we note that propagation of waves of the TE type along a plasma layer the density of which in a weak field is constant has been studied by Demchenko and Dolgopolov[11]. The distribution of the field across the layer investigated in this connection is such that the intensity of the electric field has a local minimum on the plane of symmetry of the plasma layer and increases towards the edges, while outside the plasma layer it oscillates with a constant amplitude greater than $\mathscr{E}(0)$. Solutions of such type also arise in the

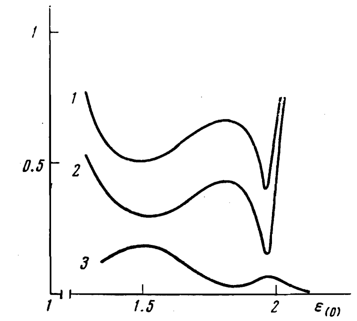


FIG. 4. The dependence of $\mathscr{E}_{max}(\infty)$ (curve 1), $\mathscr{H}(\infty)$ (curve 2) and $\mathscr{E}(\infty)$, (curve 3) on $\mathscr{E}(0)$. All curves correspond to the case A = 10, $\tau$ = 10 and $\gamma_s$ = 0.001.

problem which we have investigated above. However, our aim is the demonstration of solutions which correspond to the most complete capture of the radiation by the plasma layer, since it is just solutions that are of the greatest interest in connection with one of the new directions of work on controlled thermonuclear synthesis.


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