Self-stabilization of high-pressure plasma in toroidal traps

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We investigate the hydromagnetic stability of a high-pressure plasma in toroidal traps. The model of the toroidal trap is taken to be a helical pinch. It is shown that when the pressure of the plasma contained in the toroidal trap with a circular or near circular cross section is increased, the hydrodynamic stability is improved. By the same token it is demonstrated that hydromagnetic self-stabilization takes place in the case of high plasma pressure. It is concluded that in toroidal traps that are not suitable for stable maintenance of a low-pressure plasma, it is possible to confine stably only a plasma of sufficiently high pressure. The role of self-stabilization in toroidal traps of the Spitzer "figure-8" type is discussed, as well as the role in currentless traps with nonround cross sections and in traps of the tokamak type.

1. INTRODUCTION

In this paper we demonstrate the possibility of the following physical effect.

Assume that we produce with the aid of external conductors a magnetic field whose force lines constitute toroids that are embedded in one another, and let the integral characteristics of this field be such that a low-pressure plasma placed in it is hydromagnetically unstable. A "poor" trap of this type for a low-pressure plasma may turn out to be a "good" trap for a high-pressure plasma.

To reveal the principal aspects of this self-stabilization effect, we consider the simplest model of a toroidal trap - a plasma filament of helical symmetry. A quantitative analysis of this effect is carried out in Sections 2–4. The role of the self-stabilization effect in the case of toroidal traps of Spitzer's "figure-eight" type is discussed in Sec. 5 for round cross sections[1], in Sec. 6 for cross sections that are not round, and in Sec. 7 for the case of tokamaks.

2. FORMULATION OF PROBLEM AND INITIAL EQUATIONS

We assume that there is no longitudinal current at all or that it is weak enough so that the criterion of helical stability is satisfied. In this case the problem of hydromagnetic stability reduces to an analysis of perturbations localized near some particular magnetic surface. A particular class of localized perturbations are the so-called Mercier-type perturbations, the radial diameter of which is small in comparison with their dimensions along the small azimuth of the toroid. Perturbations of this type are stable if the criterion originally derived by Mercier[2] is satisfied:

$$\frac{\mu'^2}{4}+\frac{V'^3}{\Phi'^2}\left[\langle\alpha_s\rangle\frac{\Omega}{\Phi'^2}-\mu'\langle\gamma_s\rangle+\frac{V'^3}{\Phi'^2}(\langle\gamma_s\rangle^2-\langle\alpha_s\rangle\langle\beta_s\rangle)\right]>0, \quad (2.1)$$

where

$$\Omega=p'V''+J'\chi''-I'\Phi'',$$
$$\alpha_s=\mathbf{B}^2/|\nabla V|^2,\quad \beta_s=\mathbf{j}^2/|\nabla V|^2,\quad \gamma_s=\mathbf{jB}/|\nabla V|^2.$$

The symbol $\langle \dots \rangle$ denotes averaging over a closed force line on a rational magnetic surface near which the perturbation is localized; the prime denotes the derivative with respect to some surface quantity. The remaining notation is standard (see, e.g.,[3,4]): $\mu = \chi'/\Phi'$ (χ and Φ are the transverse and longitudinal magnetic fluxes); I and J are the transverse and longitudinal currents; V is the volume of the toroid bounded by the corresponding magnetic surface; p is the plasma pressure; **B** and **j** are the magnetic-field and current vectors.

It must be borne in mind that one can speak of a hydromagnetically stable plasma only if it is stable with respect to all the localized perturbations, and not only with respect to perturbations of the Mercier type. This was pointed out in[4]. An additional analysis carried out by one of us (A.B.M.) has shown that the localized perturbations that do not pertain to the Mercier type can be unstable even if the criterion (2.1) is satisfied. This additional instability, however, takes place only in the case of a very small magnetic shear, such that the terms with μ'^2 in (2.1) are negligibly small, and only in the region of peripheral magnetic surfaces, on which the function p(V) differs essentially from linear. In the case of high plasma pressure of interest to us, however, the magnetic shear in the peripheral region can become small only if a noticeable longitudinal current with a special distribution flows through the plasma. We assume below that there is no such possibility, all the more since $p'(V)$ will be assumed constant in the calculations that follow.

Under these assumptions, satisfaction of the criterion (2.1) guarantees hydromagnetic stability. By the same token, our problem reduces to an analysis of the criterion (2.1).

Our purpose will be to calculate the surface functions that enter into (2.1) for a concrete geometry of the magnetic field, and to take account in this calculation of sufficiently high degrees of plasma pressure. To analyze the stability of the peripheral region of the plasma, it is more convenient to perform the calculation by starting with a tensor form of the criterion (2.1); such an approach was developed, for example, in[3-5]. It is more convenient to investigate the stability of the central region by starting from a simplified criterion obtained from (2.1) by expanding all the surface quantities in powers of their "remoteness" from the magnetic axis[6]. We present these two forms of the criterion (2.1), which we shall need for the subsequent analysis.

1. Tensor Form of the Criterion (2.1)

We use the toroidal coordinates a, θ, and ζ, assuming that the surfaces a = const coincide with the magnetic surfaces and that θ and ζ are angular coordinates on the magnetic surface with period 2π, chosen such that the magnetic force lines are straight lines in these coordinates[3,4]. The metric tensor of this coordinate system will be designated g_{ik}, with det $g_{ik} \equiv g$. In this notation, the criterion (2.1) takes the form

$$\mu'^2/4+A_0W-A_1\mu'-A_0A_2+A_1^2>0, \qquad (2.2)$$

where

$$A_n=\left(\frac{2\pi}{\Phi'}\right)^2\left[\frac{\sqrt{g}\,\mathbf{B}^2}{g^{11}}(\alpha_0^{(1)})^n\right]^{(0)}, \qquad n=0,1,2;$$

$$\alpha_0^{(1)}=\frac{\mathbf{jB}}{\mathbf{B}^2}-\left(\frac{\mathbf{jB}}{\mathbf{B}^2}\right)^{(0)}, \qquad W=W_1+W_2+W_3,$$

$$W_1=\frac{p'\Phi'^2}{(\sqrt{g})^{(0)}\langle\mathbf{B}^2\rangle}\left\{\mu^2\left[(\sqrt{g})^{(0)}\left(\frac{g_{22}}{\sqrt{g}}\right)^{(0)}\right]'\right.$$

$$\left.+2\mu\left[(\sqrt{g})^{(0)}\left(\frac{g_{23}}{\sqrt{g}}\right)^{(0)}\right]'+\left[(\sqrt{g})^{(0)}\left(\frac{g_{33}}{\sqrt{g}}\right)^{(0)}\right]'\right\}, \qquad (2.3)$$

$$W_2=p'^2V'\left(\frac{1}{\langle\mathbf{B}^2\rangle}-\left\langle\frac{1}{\mathbf{B}^2}\right\rangle\right),$$

$$W_3=\frac{\mu'p'\Phi'^2}{(\sqrt{g})^{(0)}}\left[\frac{\langle(\mu g_{22}+g_{23})/g\rangle}{\langle\mathbf{B}^2\rangle}-\left\langle\frac{\mu g_{22}+g_{23}}{g\mathbf{B}^2}\right\rangle\right].$$

Here the prime denotes the derivative with respect to a. The symbol $(\ldots)^{(0)}$ denotes

$$(2\pi)^{-2}\int_0^{2\pi}(\ldots)d\theta\,d\zeta$$

Unlike the symbol $\langle(\ldots)\rangle\equiv[(\ldots)\sqrt{g}]^{(0)}/(\sqrt{g})^{(0)}$. The scalar products $\mathbf{j}\cdot\mathbf{B}$ and $\mathbf{B}^2$ are calculated in accordance with the well-known rule $\mathbf{a}\cdot\mathbf{b}=g_{ik}a^ib^k$ with $B^i = (0, \chi', \Phi')/2\pi\sqrt{g}$, $j^i = (0, I' - \partial\nu/\partial\zeta, J' + \partial\nu/\partial\theta)/2\pi\sqrt{g}$, where the function ν satisfies the relation

$$\mathbf{B}\nabla\nu=2\pi p'(V'/4\pi^2\sqrt{g}-1);$$
$$V'=4\pi^2(\sqrt{g})^{(0)}, \quad g^{11}=M_{11}/g, \quad M_{11}-\text{is the minor of } g_{11}. \qquad (2.3')$$

2. The Criterion (2.1) Near the Magnetic Axis in the Case of a Helical Plasma Filament

The recipe for using the criterion (2.1) near the magnetic axis of a helical plasma filament was first indicated in[6]. It consists in the following.

Let the equation of the family of magnetic surfaces near the magnetic axis be of the form

$$\psi=C[\rho_0^2(1+\varepsilon\cos 2\theta_0)+\rho_0^3(\alpha_1\cos\theta_0+\alpha_3\cos 3\theta_0)], \qquad (2.4)$$

where ψ is a surface function; C is a certain constant; $\theta_0 \equiv \omega_0 - \kappa_0 s$; ρ_0, ω_0 are quasistationary coordinates connected with the magnetic axis; κ_0 is the torsion of the axis; s is the length of the magnetic-axis arc reckoned from a certain point s = 0. According to[6], the criterion (2.2) can be expressed near the magnetic axis in terms of the parameters ϵ, α_1, α_3 and the parameters p_0, j_0, B_0, which characterize respectively the plasma pressure, the current density, and the magnetic field at the axis. If $\epsilon \ll 1$ (we are interested only in this case) this simplified criterion takes the form

$$\frac{\varepsilon^2\beta_0}{2}<\frac{\mu_J-\mu_0}{\mu_J+\mu_0}-\frac{\mu_J^2}{k_0^2R^2}+6\varepsilon\frac{q_1}{k_0}. \qquad (2.5)$$

Here

$$q_1=\frac{(2+\varepsilon)\alpha_3-\varepsilon\alpha_1}{4(1-\varepsilon^2)}, \quad \beta_0=\frac{p_0}{B_0^2}\left(\frac{\mu^2a_0^2}{R^2}\right)^{-1}, \quad \mu_J=\frac{j_0R}{2B_0},$$
$$\mu_0=-\varkappa_0R, \quad \mu=\mu_0+\mu_J, \quad R=(\varkappa_0^2+k_0^2)^{-1/2},$$

k_0 is the curvature of the magnetic axis and a_0 is the radius of the liner (it is assumed that the position of the liner coincides with a magnetic surface on which p = 0).

3. STABILITY CRITERION FOR THE PERIPHERAL REGION OF THE PINCH

We proceed to calculate the quantities g_{ik} which are needed by us. Assuming that the magnetic axis is a helical line of radius r_0 and pitch $2\pi\hbar$, and introducing the quasicylindrical coordinate system ρ (distance from the magnetic axis), ω (angle reckoned from the principal normal to the axis at $\zeta = 0$), and $\zeta \equiv s/R$ (s is the length of the arc of the axis from a certain fixed point), where $R = (r_0^2 + \hbar^2)^{1/2}$, we find in accordance with Sec. 3 of[10] that the square of the length element in this coordinate system is given by

$$dl^2=d\rho^2+\rho^2d\omega^2+2\rho^2\varkappa_0Rd\omega d\zeta+R^2[(1-k_0\rho\cos\Theta)^2+\varkappa_0^2\rho^2]d\zeta^2, \qquad (3.1)$$

where

$$\Theta\equiv\omega-\varkappa_0R\zeta, \quad k_0=r_0/R^2, \quad \varkappa_0=\hbar/R^2.$$

The quantity dl^2 should be expressed in terms of the coordinates a, θ, ζ referred to in Sec. 2. The transition from (ρ, ω, ζ) to (a, θ, ζ) is carried out in two steps. We first go from the polar coordinates (ρ, ω) connected with the magnetic axis to polar coordinates (ρ_0, ω_0) connected with the geometric center of the magnetic surface a = const. Denoting by $\xi(a)$ the displacement of the center of the cross section of this magnetic surface relative to the magnetic axis, and recognizing that by virtue of the assumed symmetry this displacement is directed along the principal normal to the magnetic axis, we obtain the connection between (ρ, ω) and (ρ_0, ω_0):

$$\rho\cos\Theta=\rho_0\cos\Theta_0+\xi(a),$$
$$\rho\sin\Theta=\rho_0\sin\Theta_0, \quad \Theta_0\equiv\omega_0-\varkappa_0R\zeta. \qquad (3.2)$$

We assume furthermore that ρ_0 and ω_0 are certain functions of a, θ, $\rho_0 = \rho_0(a, \theta)$, $\omega_0 = \omega_0(a, \theta)$. Without specifying the form of these functions, we find that by changing over to the variables $x^1 \equiv a$, $x^2 \equiv \theta$, $x^3 \equiv \zeta$ we transform expression (3.1) into $dl^2 = g_{ik}dx^idx^k$, where

$$g_{11}=\rho_0'^2+\rho_0^2\omega_0'^2+\xi'^2+2\xi'(\rho_0'\cos\Theta_0-\omega_0'\rho_0\sin\Theta_0),$$
$$g_{12}=\rho_0'\dot\rho_0+\rho_0^2\omega_0'\dot\omega_0+\xi'(\dot\rho_0\cos\Theta_0-\dot\omega_0\rho_0\sin\Theta_0),$$
$$g_{22}=\dot\rho_0^2+\rho_0^2\dot\omega_0^2,$$
$$g_{13}=\varkappa_0R[\rho_0^2\omega_0'+\xi(\rho_0'\sin\Theta_0+\rho_0\omega_0'\cos\Theta_0)-\xi'\rho_0\sin\Theta_0], \qquad (3.3)$$
$$g_{23}=\varkappa_0R[\rho_0^2\dot\omega_0+\xi(\dot\rho_0\sin\Theta_0+\rho_0\dot\omega_0\cos\Theta_0)],$$
$$g_{33}=R^2\{[1-k_0(\rho_0\cos\Theta_0+\xi)]^2+\varkappa_0^2(\rho_0^2+2\xi\rho_0\cos\Theta_0+\xi^2)\}.$$

Then

$$\sqrt{g}=\sqrt{g_{33}}[\rho_0(\rho_0'\dot\omega_0-\dot\rho_0\omega_0')+\xi'(\dot\omega_0\rho_0\cos\Theta_0+\dot\rho_0\sin\Theta_0)]. \qquad (3.4)$$

The prime in (3.3) and (3.4) denotes a derivative with respect to a, and the dot a derivative with respect to θ.

Assuming that the plasma is bounded by a linear of round cross section, and assuming that the magnetic surfaces inside the plasma have slightly elliptic cross sections, we put

$$\rho_0=a+\alpha(a)\cos 2\Theta_0, \quad \alpha/a\ll 1. \qquad (3.5)$$

We choose the function $\omega_0 = \omega_0(\theta)$ in a form such that the force lines in the coordinates θ and ζ are straight. To this end, in accordance with the procedure of[3], we introduce the "rectification" parameters $\bar\lambda(a)$ and $\bar\mu(a)$, putting

$$\omega_0=\theta+\bar\lambda(a)\sin\vartheta+\bar\mu(a)\sin 2\vartheta, \quad \vartheta\equiv\theta-\varkappa_0s. \qquad (3.6)$$

The equations for $\bar{\lambda}$ and $\bar{\mu}$ are obtained by using the relation (cf.[5])

$$g_{33}/\sqrt{g}-(g_{33}/\sqrt{g})^{(0)}=0. \tag{3.7}$$

We have left out here terms of order g_{23}/g_{33}, which are small quantities like $(a/R)^2$. Substitution of g_{33} and $\sqrt{g}$ calculated in the manner indicated above into (3.7) yields

$$\bar{\lambda}=-\left[k_0a+\xi'\left(1-\frac{\alpha}{4a}+\frac{\alpha'}{4}-\frac{\xi'^2}{8}\right)\right],$$
$$\bar{\mu}=-{}^1/{}_2\left(\alpha'+\frac{\alpha}{a}-\xi'^2\right) \tag{3.8}$$

Using these results, we express g_{ik} in terms of a, θ, ξ, and α. To find the explicit form of the functions $\xi(a)$ and $\alpha(a)$, we use a relation that follows from Maxwell's equations (cf.[5]):

$$\left\{\chi'\left[\frac{g_{22}}{\sqrt{g}}-\left(\frac{g_{22}}{\sqrt{g}}\right)^{(0)}\right]+\Phi'\left[\frac{g_{23}}{\sqrt{g}}-\left(\frac{g_{23}}{\sqrt{g}}\right)^{(0)}\right]\right\}'$$
$$-\frac{\partial}{\partial\theta}\left(\frac{g_{12}}{\sqrt{g}}\chi'+\frac{g_{13}}{\sqrt{g}}\Phi'\right)=-\frac{4\pi^2p'}{\chi'}[\sqrt{g}-(\sqrt{g})^{(0)}]. \tag{3.9}$$

Substituting here the corresponding expressions for g_{ik}, we obtain

$$\xi''+\frac{3\xi'}{a}+\frac{2\mu'}{\mu}\xi'-\frac{4R\varkappa_0k_0}{\mu}$$
$$=k_0\left[1-\frac{8\pi^2R^2ap'}{\chi'^2}\left(1+\frac{3}{2}\frac{\alpha}{a}+\frac{3}{2}\alpha'-\frac{9}{4}\xi'^2\right)\right] \tag{3.10}$$
$$+\frac{3\xi'}{a}\left(\frac{\alpha}{2}-\alpha'+\frac{3}{2}\xi'^2\right),$$
$$\alpha''+\frac{3\alpha'}{a}-\frac{3\alpha}{a^2}=-\frac{12\pi^2R^2k_0ap'}{\chi'^2}\xi'-3\frac{\xi'^2}{a}. \tag{3.11}$$

Using these results, we proceed to calculate the surface functions that enter in the left-hand side of (2.2). When g_{22} and g_{23} are substituted in W, it suffices to take into account in the latter only the terms of higher order in ξ:

$$g_{22}{}^0=a^2, \quad g_{23}{}^0=\varkappa_0Ra^2. \tag{3.12}$$

Then

$$W=\frac{2p'}{RB_0^2}\left(\mu^2+2\mu\varkappa_0R+\frac{1}{2a}g_{33}^{(0)\prime}\right). \tag{3.13}$$

In the expression for g_{33} we must take into account terms of order $k_0\xi$. Then

$$\frac{1}{2a}g_{33}^{(0)\prime}=-\frac{R^2k_0}{2a}(a\xi''+3\xi'+k_0a)+\varkappa_0{}^2R^2. \tag{3.14}$$

It follows from (3.13), (3.14), and (3.10) that

$$W=\frac{2p'}{RB_0^2}\left\{\mu_J{}^2-k_0{}^2R^2\frac{\mu_J-\mu_0}{\mu}+R^2k_0\left[\frac{\mu'}{\mu}\xi'\right.\right.$$
$$\left.\left.+\frac{4\pi^2R^2ap'k_0}{\chi'^2}\left(1+\frac{3}{2}\frac{\alpha}{a}+\frac{3}{2}\alpha'-\frac{9}{4}\xi'^2\right)-\frac{3\xi'}{a}\left(\frac{\alpha}{a}-\alpha'+\frac{3}{2}\xi'^2\right)\right]\right\}. \tag{3.15}$$

Expression (2.3) for $\alpha_0^{(1)}$ reduces to

$$\alpha_0^{(1)}=\frac{8\pi^2p'a^2k_0R}{\chi'\Phi'}\left[\left(1-\frac{3}{8}\xi'^2+\frac{\alpha'}{4}+\frac{3}{4}\frac{\alpha}{a}\right)\cos\vartheta-\frac{\xi'}{2}\cos2\vartheta\right], \tag{3.16}$$

whereas

$$\frac{\sqrt{g}\mathbf{B}^2}{g^{11}}=\frac{R}{a}\left(\frac{\Phi'}{2\pi}\right)^2\left[1+\frac{3}{2}\xi'^2+2\xi'\cos\vartheta+\left(2\alpha'-\frac{\xi'^2}{2}\right)\cos2\vartheta\right]. \tag{3.17}$$

With the aid of (3.16) and (3.17) we get

$$A_0=\frac{R}{a}, \quad A_1=\frac{8\pi^2p'ak_0R^2\xi'}{\chi'\Phi'},$$
$$A_2=\frac{R}{2a}\left(\frac{8\pi^2p'a^2k_0R}{\chi'\Phi'}\right)^2\left(1-\frac{\xi'^2}{4}+\frac{3}{2}\alpha'+\frac{3}{2}\frac{\alpha}{a}\right). \tag{3.18}$$

Taking (3.15) and (3.18) into account, we reduce (2.2) to the form

$$\frac{\mu'^2}{4}+\frac{2p'}{aB_0^2}\left[\mu_J{}^2-k_0{}^2R^2\frac{\mu_J-\mu_0}{\mu_J+\mu_0}\right.$$
$$\left.+\frac{3R^2k_0\xi'}{a}\left(\frac{\alpha}{a}-\alpha'+\frac{3}{2}\xi'^2\right)\right]>0. \tag{3.19}$$

In the case of a parabolic distribution of the pressure $p=p_0(1-a^2/a_0^2)$, it follows from (3.10) and (3.11) that $\xi'=\beta_\theta k_0a$, $\alpha=k_0^2\beta_\theta^2a(a^2-a_0^2)/4$, where $\beta_\theta=4\pi^2p_0R^2/\chi'^2$. Then (3.19) is equivalent to

$$\frac{1}{4}\left(\frac{\mu'}{\mu}\right)^2-4\frac{\beta_\theta}{R^2}\left[\mu_J{}^2-\frac{k_0{}^2R^2}{\mu}(\mu_J-\mu_0)-\frac{3}{2}\beta_\theta{}^3k_0{}^2a^2\right]>0. \tag{3.20}$$

To use this criterion it is also necessary to have expressions for μ', accurate to ξ'^2. Starting from the connection between the longitudinal current and the magnetic fluxes (see, e.g.,[5]),

$$J=\left(\frac{g_{22}}{\sqrt{g}}\right)^{(0)}\chi'+\left(\frac{g_{23}}{\sqrt{g}}\right)^{(0)}\Phi',$$

and from the expressions for g_{ik} calculated accurate to ξ'^2, we get

$$\mu\equiv\frac{\chi'}{\Phi'}=\left(1-\frac{\xi'^2}{2}\right)\left(\frac{RJ}{a\Phi'}-\varkappa_0R\right), \quad \frac{RJ}{a\Phi'}\equiv\mu_J. \tag{3.21}$$

This yields in the case of a parabolic distribution of the pressure

$$\mu'=\varkappa_0R\beta_\theta{}^2k_0{}^2a+[\mu_J(1-\beta_\theta{}^2k_0{}^2a^2/2)]'. \tag{3.22}$$

4. CRITERION FOR THE STABILITY OF THE CENTRAL REGION OF THE PLASMA FILAMENT

In accordance with the statements made in Sec. 2, we must find the equation for the magnetic surfaces of the helical plasma filaments and express it in the form (2.4). We derive this equation by starting from the results of Sec. 6 of[7].

Following[7], we introduce quasicylindrical coordinates ρ and ω connected with the geometrical axis of the filament, and an auxiliary coordinate $\theta\equiv\omega-\kappa_0s$. We designate the magnetic-field components in this system of coordinates B_ρ, B_ω, and B_s, and introduce the functions $\psi(\rho,\theta)$ and $I_B(\rho,\theta)$, defined by the relations

$$\partial\psi/\partial\rho=\varkappa_0\rho B_s-h_sB_\omega, \quad \partial\psi/\partial\theta=\rho h_sB_\rho,$$
$$I_B=h_sB_s+\varkappa_0\rho B_\omega, \tag{4.1}$$

where $h_s\equiv1-k_0\rho\cos\theta$. According to[7], ψ and I_B are surface functions, with ψ satisfying the equation

$$\frac{1}{\rho h_s}\frac{\partial}{\partial\rho}\left(\rho h_s\frac{\partial\psi/\partial\rho}{h_s{}^2+\varkappa_0{}^2\rho^2}\right)+\frac{1}{\rho^2h_s}\frac{\partial}{\partial\theta}\left(\frac{1}{h_s}\frac{\partial\psi}{\partial\theta}\right)$$
$$-\frac{2\varkappa_0I_B(\psi)}{(h_s{}^2+\varkappa_0{}^2\rho^2)^2}+\frac{I_BI_B'(\psi)}{h_s{}^2+\varkappa_0{}^2\rho^2}+4\pi p'(\psi)=0. \tag{4.2}$$

At $k_0\rho\ll1$, $\kappa_0\rho\ll1$ and under the conditions $p'(\psi)\approx$ const, $I_B(\psi)\approx$ const, $I_B'(\psi)\approx$ const. Eq. (4.2) reduces to

$$\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2}=2\varkappa_0I_B-j_0+2k_0p'\rho\cos\theta, \tag{4.3}$$

where $j_0\equiv p'+I_BI_B'$. The general solution of (4.3) is

$$\psi=\lambda_0+\lambda_1\rho\cos\theta+\lambda_2\rho^2+\lambda_3\rho^2\cos2\theta+\lambda_4\rho^3\cos\theta+\lambda_5\rho^3\cos3\theta. \tag{4.4}$$

The constants λ_2 and λ_4 are determined by the right-hand side of (4.3)

$$\lambda_2=C, \quad \lambda_4=k_0p'/4, \tag{4.5}$$

where $C\equiv(2\kappa_0I_B-j_0)/4$, and the remaining four constants are determined by the boundary conditions. We obtain them, assuming that the plasma is bounded by an ideally conducting liner with an ellipticity ϵ_0 and "triangularity" q_0, i.e., assuming in accordance with[6] that the coordinates ρ and θ on the boundary surface (liner) are connected by the relation

$$\rho^2(1+\varepsilon_0\cos 2\theta)+q_0[(2-\varepsilon_0)\cos 3\theta+3\varepsilon_0\cos\theta]=a_0^2. \quad (4.5')$$

Assuming, in addition, $\psi = 0$ on the liner and taking q_0 to be a small quantity of the order of $k_0\rho$, we obtain

$$\lambda_0=-Ca_0^2,\quad \lambda_1=-\frac{a_0^2}{2+\varepsilon_0}\frac{k_0p'}{2},\quad \lambda_3=\varepsilon_0 C,$$
$$\lambda_5=\frac{\varepsilon_0}{2+\varepsilon_0}\frac{k_0p'}{4}+4Cq_0\frac{1-\varepsilon_0^2}{2+\varepsilon_0}. \quad (4.6)$$

It is now necessary to change over in (4.4) to the coordinates ρ_0 and θ_0, connected with the magnetic axis (see Sec. 2). The connection between (ρ, θ) and (ρ_0, θ_0) is determined by the relations

$$\rho\cos\theta=\rho_0\cos\theta_0-\Delta,\quad \rho\sin\theta=\rho_0\sin\theta_0. \quad (4.7)$$

The quantity Δ is the displacement of the magnetic axis from the geometric axis and is determined from the condition that the function $\psi(\rho_0, \theta_0)$ contain no terms linear in ρ_0:

$$a_0^2\alpha-\Delta(1+\varepsilon_0)(2-3\alpha\Delta)=0. \quad (4.8)$$

As a result, (4.4) takes the form

$$\psi(\rho_0,\theta_0)=C_1[C_2+\rho_0^2(1+\varepsilon\cos 2\theta_0)+\rho_0^3(\alpha_1\cos\theta_0+\alpha_3\cos 3\theta_0)], \quad (4.9)$$

where C_1 and C_2 are certain constants, the explicit form of which is of no importance to us;

$$\varepsilon=\frac{\varepsilon_0-\alpha\Delta-2\varepsilon_0\alpha\Delta}{1-\alpha\Delta(2+\varepsilon_0)},\quad \alpha_1=\frac{\alpha(1+\varepsilon_0/2)}{1-\alpha\Delta(2+\varepsilon_0)},$$
$$\alpha_3=\frac{\alpha\varepsilon_0/2+4(1-\varepsilon_0^2)q_0/(2+\varepsilon_0)}{1-\alpha\Delta(2+\varepsilon_0)},\quad \alpha=\frac{k_0p'}{4C(1+\varepsilon_0/2)}. \quad (4.10)$$

From (2.6) and (4.10) follows an approximate ($\epsilon_0 \ll 1$) expression for q_1:

$$q_1=q_0+\alpha^2\Delta/4. \quad (4.11)$$

In the same approximation and at $\Delta/a_0 \ll 1$ we have

$$\varepsilon=\varepsilon_0-\alpha\Delta,\quad \Delta=k_0\beta_\theta a_0^2/2,\quad \alpha=-k_0\beta_\theta,$$
$$\beta_\theta=p_0/(\varkappa_0 aI_B-j_0a/2)^2. \quad (4.12)$$

Substitution of (4.11) and (4.12) in (2.5) leads to the sought criterion of the stability in the central region of the filament:

$$\frac{1}{2}\beta_\theta\left(\varepsilon_0+\frac{1}{2}k_0^2a_0^2\beta_\theta^2\right)^2<\frac{\mu_J-\mu_0}{\mu_J+\mu_0}-\frac{\mu_J^2}{k_0^2R^2}$$
$$+6\left(\varepsilon_0+\frac{1}{2}k_0^2a_0^2\beta_\theta^2\right)\left(\frac{q_0}{k_0}+\frac{1}{8}k_0^2a_0^2\beta_\theta^3\right). \quad (4.13)$$

5. STABILITY OF STELLARATORS OF SPITZER'S "FIGURE-EIGHT" TYPE

Spitzer[1] long ago proposed the use of toroidal traps of round cross section for plasma confinement. He considered particular types of such traps with the form of a figure-eight in space. These traps differ from tokamaks in that they carry no longitudinal current.

Johnson et al.[8] have analyzed the stability of a plasma in a trap of Spitzer's "figure-eight" type under the assumption that the plasma pressure is low. They have shown that toroidal traps of round cross section without the longitudinal current, filled with low-pressure plasma, are characterized by the absence of a magnetic well and by zero magnetic shear, and therefore do not satisfy the requirements of hydromagnetic stability. As a result, interest in round-section traps without longitudinal current has diminished, and experimental investigations of plasma confinement in such traps were stopped.

Taking the foregoing relations (3.20) and (4.13) into consideration, we conclude that the conclusion that a plasma in a toroidal round-section trap without a longitudinal current is unstable is, generally speaking, invalid in the interesting case of a plasma of sufficiently high pressure, when $\beta_\theta > 1$. Thus, it follows from (3.20) that the peripheral region of the plasma is stable if

$$-1+{}^{25}/_{16}k_0^2a^2\beta_\theta^3>0. \quad (5.1)$$

The central region of the plasma filament is also stable if, according to (4.13),

$$-1+{}^1/_4k_0^4a_0^4\beta_\theta^5>0. \quad (5.2)$$

Thus, our analysis offers evidence that hydromagnetically-stable confinement of a plasma of sufficiently high pressure in currentless toroidal round-section traps is feasible in principle. This conclusion was formulated by us in a brief communication[9].

In the practical utilization of our results it must be borne in mind that when relations (3.20) and (4.13) were derived it was assumed that Δ/a_0 is small. An additional analysis carried out for the central region of a plasma filament at finite Δ/a_0 has shown that the considered effect of self-stabilization of the central region of the filament takes place only when the length of the toroidal trap greatly exceeds its cross-section radius.

6. ROLE OF ELLIPTICITY AND "TRIANGULARITY" OF THE CROSS SECTION

It was shown earlier[6] that to attain hydromagnetic stability of a low-pressure plasma in a toroidal trap it is necessary to make the trap cross section elliptic, $\epsilon_0 \neq 0$, and slightly triangular, $q_0 \neq 0$. It follows from our formula (4.13) that in the case of confinement of a plasma with sufficiently large β_θ in a trap, the role of the ellipticity and "triangularity" is inessential if

$$\varepsilon_0<{}^1/_2(k_0a_0\beta_\theta)^2, \quad (6.1)$$

$$q_0/k_0<{}^1/_8k_0^2a_0^2\beta_\theta^3. \quad (6.2)$$

One more important result of (4.13) is the following. According to[6], the stabilizing role of the "triangularity" q_0 at low plasma pressure becomes manifest provided only that the liner is elliptical, $\epsilon_0 \neq 0$. On the other hand, if β_θ is large enough to satisfy the condition (6.1) and the "triangularity" q_0 is appreciable, so that

$$q_0/k_0>{}^1/_8k_0^2a_0^2\beta_\theta^3, \quad (6.3)$$

then the effect of stabilization by "triangularity" ceases to depend on the ellipticity. The stability criterion at $\epsilon_0 = 0$ and at a q_0 satisfying the condition (6.3) is

$$-1+3q_0k_0a_0^2\beta_\theta^2>0. \quad (6.4)$$

7. ROLE OF SELF-STABILIZATION EFFECT IN TOKAMAKS

In the case of a tokamak ($\mu_0 = 0$, $k_0R = 1$, $\mu_J = \mu$) with round liner cross section ($\epsilon_0 = 0$, $q_0 = 0$), the stability criteria (3.20) and (4.13) signify respectively that

$$1+\frac{3}{2}\beta_J{}^3k_0{}^2a^2+\frac{1}{16\beta_J}\left(\frac{R\mu'}{\mu}\right)^2>\frac{1}{q^2}, \tag{7.1}$$

$$1+{}^1/_4k_0{}^4a_0{}^4\beta_J{}^5>1/q^2, \quad \beta_J=\beta_0, \quad q=1/\mu. \tag{7.2}$$

It follows from (7.1) and (7.2) that if the condition of helical instability $q > 1$ is satisfied, then perturbations of the Mercier type are stable at arbitrary β_J.

The stability criterion (4.13), written out for a tokamak with a nonround liner cross section, takes the form

$$1+\frac{6q_0}{k_0}\left(\varepsilon_0+\frac{1}{2}k_0{}^2a_0{}^2\beta_J{}^2\right)>\frac{1}{q^2}+\frac{\beta_J}{2}\left(\varepsilon_0+\frac{1}{2}k_0{}^2a_0{}^2\beta_J{}^2\right)(\varepsilon_0-k_0{}^2a_0{}^2\beta_J{}^2). \tag{7.3}$$

We see that, just as in the case of stellarators with non-round cross section considered in Sec. 6, the ellipticity and triangularity cease to affect perturbations of the Mercier type if the conditions (6.1) and (6.2) are satisfied, and that "triangularity" of the liner cross section can lead to stabilization even in the absence of liner ellipticity.

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