Critical currents for resistive states in superconducting channels

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The Cooper instability with respect to formation of superconducting pairs of electrons in the normal state of a narrow superconducting channel included in a circuit with a direct-current source is studied. The difference, due to the special diamagnetic properties, in the responses of the superconductor to electric fields induced in a closed circuit by a varying magnetic flux and by switching on a direct-current source is analyzed. Unlike the former case, in which a transition to the superconducting state is impossible, in the case of a circuit with a current source, in an electric field lower that the critical field E_{c2}, superconducting centers develop (as in the transition to the mixed state in the field H_{c2} in type-II superconductors) and a transition occurs to a resistive state with nonzero superconducting order. The structure of the resistive state, with microscopic phase separation into layers of normal and superconducting regions alternating along the channel, is described.

The broad resistive region between zero and normal resistances that is observed experimentally in the volt-ampere characteristics of superconducting films (cf., e.g., [1]) evidently cannot be explained in the majority of cases, by the presence of an Abrikosov vortex structure[2,3] or by the structure of an intermediate state with current[4], in view of the small dimensions of the samples. Despite this, there are general theoretical arguments which show that such a resistive region should also exist in narrow superconducting channels. These arguments are based on the existence of a maximum uniform superconducting current j_c, determined by the dependence of the number of superconducting electrons on the velocity of the condensate[5,6]. On the other hand, in the normal state of the sample, reduction of the current below a certain critical value j_{c2} should lead to the development of a Cooper instability and to superconducting pairing of electrons, analogously to the manner in which this occurs in type-II superconductors below the critical field H_{c2}[2]. Because of the difference between the mechanisms determining the currents j_c and j_{c2}, these currents do not coincide in the general case. Since the superconducting current state is a thermodynamically stable equilibrium state, in contrast to the nonequilibrium resistive state, we should expect that $j_{c2} > j_c$. A natural limitation which arises here is that, in pure samples, the observation of the resistive states should be performed near the critical temperature T_c of the superconductor, when the currents are sufficiently small and do not damage the sample in the normal state.

It is clear from the above considerations that, for the elucidation of the microscopic structure of the resistive current states arising in a superconducting channel in the range of currents $j_c < j < j_{c2}$, it is necessary to elucidate first of all the character of the formation of superconducting centers in the normal state with a current, i.e., to calculate the pattern of the Cooper instability and to find the critical current j_{c2}. This is the purpose of the present article.

To calculate the instability, we shall make use of the kinetic equations proposed in[7–9] for the generalized electron-hole density matrix of a superconductor. These equations for the density matrix γ ($\gamma(\mathbf{r}_1, \mathbf{r}_2)$ in the coordinate representation) can be linearized in the small correction $\hat{\psi}$, which is off-diagonal in the "electron-hole" isotopic space:

$$\gamma=\hat{f}+\hat{\psi},$$
$$\mathrm{Tr}(\sigma_x\hat{f})=\mathrm{Tr}(\sigma_y\hat{f})=0,\quad \mathrm{Tr}\hat{\psi}=\mathrm{Tr}(\sigma_z\hat{\psi})=0.$$

As a result, they acquire the form of generalized Liouville equations with a self-consistent field:

$$i\frac{\partial\hat{f}}{\partial t}=[H+\sigma_z e\hat{V},\hat{f}]$$
$$i\frac{\partial\hat{\psi}}{\partial t}=[H+\sigma_z e\hat{V},\hat{\psi}]+[\sigma_x\hat{\Delta},\hat{f}], \tag{1}$$

where

$$H=\sigma_z(\hat{\xi}+U)+\mathbf{v}_s\hat{\mathbf{p}},$$
$$\xi(\mathbf{r}_1,\mathbf{r}_2)=(\hat{p}_1^2/2m-\varepsilon_F)\delta(\mathbf{r}_1-\mathbf{r}_2),$$
$$(\hat{\mathbf{v}}_s\mathbf{p})(\mathbf{r}_1,\mathbf{r}_2)=\mathbf{v}_s(\mathbf{r}_1)\hat{\mathbf{p}}_1\delta(\mathbf{r}_1-\mathbf{r}_2),\quad \hat{\Delta}(\mathbf{r}_1,\mathbf{r}_2)=\Delta(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2),$$
$$\hat{V}(\mathbf{r}_1,\mathbf{r}_2)=\hat{V}(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2),\quad \hat{\mathbf{p}}=-i\nabla\quad(\hbar=c=1). \tag{2}$$

Here e and m are the charge and mass of the electron, ϵ_F is the Fermi energy, σ_x, σ_y and σ_z are the Pauli matrices, Tr is the trace over the spin indices, and U is the potential for scattering of electrons by impurities; $\hbar = c = 1$.

The superconducting order parameter Δ is determined from the self-consistency equation

$$\Delta(\mathbf{r})={}^1/_2|g|\,\mathrm{Tr}\,(\sigma_x\hat{\psi}(\mathbf{r},\mathbf{r})) \tag{3}$$

($g < 0$ is the coupling constant of the electrons).

The electric and magnetic fields appear in Eqs. (1) and (2) in the form of gauge-invariant combinations

$$eV=\frac{1}{2}\frac{\partial\chi}{\partial t}+e\varphi,\quad \mathbf{p}_s=m\mathbf{v}_s=\frac{1}{2}\nabla\chi-e\mathbf{A}, \tag{4}$$

where φ and $\mathbf{A}$ are the scalar and vector potentials of the electro-magnetic field, $\chi + 2\epsilon_F t$ is the phase of the superconducting order parameter, V is the electrochemical potential, and $\mathbf{v}_s$ is the velocity of the superconducting condensate. The phase χ is determined from the continuity equation

$$\mathrm{div}\,\mathbf{j}=0,$$
$$\mathbf{j}(\mathbf{r})=Ne\mathbf{v}_s(\mathbf{r})+e\frac{\hat{\mathbf{p}}-\hat{\mathbf{p}}'}{2m}\mathrm{Tr}\left[\frac{1}{2}\delta(\mathbf{r}-\mathbf{r}')-\hat{f}(\mathbf{r},\mathbf{r}')\right]_{\mathbf{r}'=\mathbf{r}} \tag{5}$$

(N is the electron density), which is equivalent to the condition that the parameter Δ (3) be real:

$$0=\frac{1}{2}|g|\,\mathrm{Tr}(\sigma_y\hat{\psi}(\mathbf{r},\mathbf{r})). \quad (6)$$

From the definitions (4) and the Maxwell equations, the generalized London equations follow:

$$\mathrm{rot}\,\mathbf{p}_s=-e\mathbf{H},\quad \mathrm{rot}\,\mathbf{H}=4\pi\mathbf{j},\quad \mathrm{div}\,\mathbf{H}=0, \quad (7)$$

in which, in view of the high electron density in metals, the displacement current is omitted. For the same reason, the Maxwell equation $\mathrm{div}\,\mathbf{E}=4\pi e\delta N$ is written in the form of the electro-neutrality condition

$$\delta N=0,$$
$$N(\mathbf{r})=\mathrm{Tr}[\tfrac{1}{2}\delta(\mathbf{r}-\mathbf{r}')-\sigma_z\hat{f}(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'=\mathbf{r}}. \quad (8)$$

Equations (1), (3), (5), (7) and (8) form a complete system of equations for the quantities $\hat{f}$, $\hat{\psi}$, $\mathbf{v}_S$, $\mathbf{H}$ and V. The electric field intensity $\mathbf{E}$ can be found, using (4) from the following relation:

$$e\mathbf{E}=\frac{\partial\mathbf{p}_s}{\partial t}-e\nabla V. \quad (9)$$

We emphasize that this expression is just a formula for the calculation of the electric field $\mathbf{E}$ in terms of previously determined $\mathbf{p}_S$ and V, and is not an equation for, say, the condensate velocity $\mathbf{v}_S=\mathbf{p}_S/m$. This is manifest, in particular, in the fact that the last Maxwell equation curl $\mathbf{E}=-\partial\mathbf{H}/\partial t$, as can easily be seen, is satisfied identically by virtue of the relations (9) and (7).

Before proceeding directly to the calculations, we shall make the formulation of the problem more precise and discuss certain questions associated with the phase of the order parameter, which are important for what follows. First of all, we shall discuss results from the work of Gor'kov[10] and Kulik[11] that are relevant to the problem under consideration. In these papers, the fluctuational correction to the electric current in the normal state of a superconductor at temperatures below the critical temperature T_c was calculated. Although the negative differential conductivity which they obtained, which is due to fluctuations of the order parameter, does indicate a certain instability, these fluctuations turn out to be finite for all currents and electric fields and do not display critical behavior.

At first sight, these results directly contradict the above assumption of the existence of a critical current j_{c2}. In fact, as will be shown below, this is not the case. We note, first, the fact that the calculations in the papers mentioned were performed in the gauge $\mathbf{A}=-\mathbf{E}t$, $\varphi=0$. Physically meaningful results should not, of course, depend on the gauge of the electromagnetic potentials. However, in a superconductor, the phase of a macroscopic quantity—the superconducting order parameter—is related to the gauge transformations, and this compels a more careful analysis of the situation.

Decisive in the present case is the fact that, in the problem under consideration, there is a direct normal current in a closed conductor. There are two physically different possibilities for the creation of the electromotive force (EMF) necessary to support such a current. In the first case, the EMF is created by a magnetic flux, varying with a constant rate, through the closed contour of the conductor. In the second case, which corresponds to the experimental situation, a current source with a so-called "external" EMF is included in the closed circuit. Because of its special diamagnetic properties a superconductor is extremely sensitive to changes of magnetic field and we can therefore expect in advance that, unlike the normal metal, the superconductor will respond differently to these two possibilities.

From a formal angle, the case of a varying magnetic flux is a particular case of a varying electromagnetic field and is contained in the equations given above. We note that these equations could also be written with a complex order parameter and an arbitrary gauge for the potentials, without separating out the phase χ (cf. [7]).

We now consider those refinements which arise in the case when current sources are included in the closed circuit. The explicit introduction of sources into the formal framework of the theory is easily realized by adding to the Hamiltonian of the system (cf. [7]) the Hamiltonian of the sources[1]:

$$\mathcal{H}_{\mathrm{sou}}=\int dV[J(\mathbf{r})\psi_\uparrow^+(\mathbf{r})\psi_\downarrow^+(\mathbf{r})+J^*(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})], \quad (10)$$

where $\psi^+(\mathbf{r})$ and $\psi(\mathbf{r})$ are operators creating and annihilating electrons at the point $\mathbf{r}$, and $J(\mathbf{r})$ is the given power density of the localized (and distant) sources. Since the sources are associated with charges of the opposite sign (and by virtue of the total charge conservation), the phase χ_{sou} of the sources ($J(\mathbf{r})=|J(\mathbf{r})|\exp(i\chi_{\mathrm{sou}}(\mathbf{r}))$) is subject to the usual gauge transformations:

$$\mathbf{A}\to\mathbf{A}+\nabla f,\quad \varphi\to\varphi-\frac{\partial f}{\partial t},\quad \chi_{\mathrm{sou}}\to\chi_{\mathrm{sou}}+2ef.$$

Calculating (in a manner analogous to the way in which this was done in [7]) the change per unit time of the mean number of electrons in a system with a total Hamiltonian which includes the source Hamiltonian (10), it is not difficult to obtain the following result:

$$e\frac{\partial N}{\partial t}+\mathrm{div}\,\mathbf{j}\sim J(\mathbf{r})\Delta^*(\mathbf{r})-J^*(\mathbf{r})\Delta(\mathbf{r}),$$
$$\Delta(\mathbf{r})\sim\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle,\quad \Delta^*(\mathbf{r})\sim\langle\psi_\uparrow^+(\mathbf{r})\psi_\downarrow^+(\mathbf{r})\rangle. \quad (11)$$

An independent requirement imposed on the sources, and stemming from the formulation of the problem, is that the sources should not lead to accumulation of electron charge in the conductor (the "source" and "sink" should be balanced). Hence, according to (11), we obtain

$$\chi=\chi_{\mathrm{sou}}+\pi n,\quad n=0,\pm1,\pm2,\ldots$$

Thus, the inclusion of a source in the circuit leads to the result that the source "ties" its phase to the superconducting order parameter. In these conditions, it turns out that Eqs. (1)–(9) are most naturally written in terms of the gauge-invariant quantities $\mathbf{p}_S$ and V, and this form, in fact, enables us to distinguish clearly the two above-mentioned possibilities of creating a constant EMF in a closed circuit. Integrating the expression (9) around the circuit and taking the relations (4) and (7) into account, we obtain

$$\oint\mathbf{E}\,d\mathbf{l}=-\frac{\partial\Phi}{\partial t}+\mathscr{E}_{\mathrm{ext}},$$
$$\mathscr{E}_{\mathrm{ext}}=-\frac{1}{2e}\frac{\partial}{\partial t}\delta\chi=-\oint\nabla V\,d\mathbf{l}, \quad (12)$$

where Φ is the magnetic flux through the circuit and $\delta\chi$ is the given discontinuity in the phase of the source and determines the source EMF.

Another important consequence stems from the arguments presented: in a problem with a current source, the phase χ should be determined independently from the more fundamental condition of charge conservation, i.e., from Eq. (5) and not from the formally equivalent Eq. (6).

The above account enables us to understand the results of [10,11]. The choice of gauge ($\mathbf{A} = -\mathbf{E}t$, $\varphi = 0$) and the character of the approximations made (which are unavoidable in the calculations) predetermine that these results refer to the case of a closed conductor with a magnetic flux increasing in time[2]. But in these conditions the stability of the normal current state with respect to a transition to the superconducting state is obvious, since here a genuine "prescribed-voltage circuit" is realized. In an electric field specified independently of the nature of the conductor and of its state, and determined only by the rate of increase of the magnetic flux, a transition to a state with zero resistance is clearly impossible.

In contrast to this, no real circuit with current sources is a "prescribed-voltage circuit." In this case, only the EMF's of the sources are specified, and the distribution of currents and voltages in the circuit is determined by Kirchhoff's rules (which are a consequence of conservation of current (5) and of the second Eq. (12)) and thus depends on the nature of the conductors and their electronic states.

Returning to the instability problem, and having in mind the practically important case of the inclusion of a narrow superconducting channel in a circuit with a current source, we put $\partial p_s/\partial t = 0$ in Eqs. (1)–(9) and, neglecting next the weak effect of the magnetic field due to the current, we drop the small quantity p_s in these equations.

The first Eq. (1) determines the density matrix $\hat{f}$ of the normal electrons. As already remarked, in view of the nonequilibrium character of the normal current state and the dissipation of energy, this state is possible, generally speaking, only in the case when the presence of the electric field weakly perturbs the equilibrium state. Thus, in the leading approximation, the matrix $\hat{f}$ is the equilibrium matrix:

$$[H, \hat{f}^{(0)}] = 0.$$

With relaxation at impurities taken into account, the equilibrium matrix $\hat{f}^{(0)}$ has the following form:

$$\hat{f}^{(0)} = \int d\omega \frac{1}{2}\left(1 + \operatorname{th}\frac{\omega}{2T}\right)\delta(\omega - H),$$
$$\delta(\omega - H) = \frac{1}{2\pi i}\left(\frac{1}{\omega - H - i0} - \frac{1}{\omega - H + i0}\right). \tag{13}$$

Using the procedure of adiabatic switching-on of the interaction, well-known in linear-response theory[12], it is not difficult to find from Eq. (1) the correction to the equilibrium density matrix that is linear to the potential V, and then to calculate the current $\mathbf{j}$ and the change δN of electron density from formulas (5) and (8). Omitting the detailed calculations, which are analogous to those presented below, we write out the well-known final result:

$$\mathbf{j} = -\sigma_n \nabla V, \quad \sigma_n = \frac{Ne^2\tau_{tr}}{m}, \quad \delta N = 0, \tag{14}$$

(σ_n is the conductivity of the normal metal). Thus, Eq. (8) is satisfied identically. Under conditions of linear homogeneity, the current conservation equation (5) leads to constancy of the current over the conductor. Equating the expression (14) for the current to the prescribed external current $\mathbf{j}_{ext}$, we find the gradient of the potential V and, from formula (9) (with $\partial p_s/\partial t = 0$), the electric field intensity $\mathbf{E}$:

$$\mathbf{E} = -\nabla V = \mathbf{j}_{ext}/\sigma_n. \tag{15}$$

To calculate the instability, we substitute the matrix $\hat{f}$ in the leading approximation (13) into the second Eq. (1) and solve this equation by means of a Laplace transformation with respect to the time:

$$\hat{\psi}(z) = \int_0^\infty dt\, e^{-zt}\hat{\psi}(t), \quad \hat{\Delta}(z) = \int_0^\infty dt\, e^{-zt}\hat{\Delta}(t).$$

An equation for the quantities $\hat{\psi}(z)$ and $\hat{\Delta}(z)$ follows from (1):

$$[H + \sigma_z e\hat{V}, \hat{\psi}] + [\sigma_x\hat{\Delta}, \hat{f}^{(0)}] - iz\hat{\psi} = -i\hat{\psi}_0, \tag{16}$$

where $\hat{\psi}_0$ is the small initial perturbation of $\hat{\psi}$.

Being interested only in the real positive pole (Re z > 0, Im z = 0, corresponding to the unstable solution) of the resolvent of Eq. (16) and of the self-consistency equation (3), we consider the corresponding homogeneous equation (16). The solution of this equation has the form

$$\hat{\psi} = \int \frac{d\omega}{2\pi i}(\omega + i\zeta - H - \sigma_z e\hat{V})^{-1}[\sigma_x\hat{\Delta}, \hat{f}^{(0)}](\omega - i\zeta - H - \sigma_z e\hat{V})^{-1},$$
$$z = 2\zeta > 0. \tag{17}$$

It is necessary to substitute the solution (17) found into the self-consistency condition (3), and this will establish a linear homogeneous equation for Δ. To simplify the following calculations, we confine ourselves here to temperatures close to the critical temperature: $(T_c - T)/T_c \ll 1$. In this case, the "frequency" z, the potential V and gradients of all quantities are small and it is sufficient to confine oneself to the first nonvanishing terms in the expansion of $\hat{\psi}$ in these quantities. In the zeroth approximation in the potential V, simple transformations in the expression (17), with (13) taken into account, give

$$\int \frac{d\omega}{2\pi i}\frac{1}{\omega + i\zeta - H}[\sigma_x\hat{\Delta}, \hat{f}^{(0)}]\frac{1}{\omega - i\zeta - H} = \int \frac{d\omega}{2\pi i}\frac{1}{2}\left(1 + \operatorname{th}\frac{\omega}{2T}\right)$$
$$\times[\mathcal{G}(\omega + iz)\sigma_x\hat{\Delta}\mathcal{G}(\omega - i0) - \mathcal{G}(\omega + iz)\sigma_x\hat{\Delta}\mathcal{G}(\omega + i0)$$
$$+\mathcal{G}(\omega - i0)\sigma_x\hat{\Delta}\mathcal{G}(\omega - iz) - \mathcal{G}(\omega + i0)\sigma_x\hat{\Delta}\mathcal{G}(\omega - iz)], \tag{18}$$
$$\mathcal{G}(\omega) = (\omega - H)^{-1}.$$

When this expression is substituted into the self-consistency equation (8), averaging takes place over the positions of the impurities that scatter the electrons. According to [12], as a result of such averaging the pair products of Green functions $\mathcal{G}\times\mathcal{G}$ must be replaced by the following expression:

$$\mathcal{G}(\omega_1)\sigma_x\hat{\Delta}\mathcal{G}(\omega_2) \to \Pi(\omega_1, \omega_2),$$
$$\Pi(\mathbf{r}_1, \mathbf{r}_2; \omega_1, \omega_2) = \left(\frac{p_F}{2\pi}\right)^2 \int do\, \exp(ip_F\mathbf{n}(\mathbf{r}_1 - \mathbf{r}_2))\,\Pi(\mathbf{n}, \boldsymbol{\rho}; s_1, s_2; \omega_1, \omega_2), \tag{19}$$
$$\mathbf{r} = \mathbf{n}s + \boldsymbol{\rho},$$

where the function Π satisfies the equation

$$\Pi(\mathbf{n}, \boldsymbol{\rho}; s_1, s_2; \omega_1, \omega_2) = \int ds\, G(s_1, s; \omega_1)\Big[\sigma_x\Delta(\boldsymbol{\rho}, s)$$
$$+ nv_F^2 \int do' |f(\mathbf{n}, \mathbf{n}')|^2 \sigma_z \Pi(\mathbf{n}', \boldsymbol{\rho}'; s', s'; \omega_1, \omega_2)\sigma_z\Big] G(s, s_2; \omega_2), \tag{20}$$
$$\mathbf{r} = \mathbf{n}s + \boldsymbol{\rho} = \mathbf{n}'s' + \boldsymbol{\rho}'.$$

Here $\mathbf{n}$ is the unit vector defining the direction of motion of an electron with the large Fermi momentum $p_F = mv_F$, $\epsilon_F = p_F^2/2m$; $s = \mathbf{n}\cdot\mathbf{r}$ is the path traversed by the electron in the direction of motion ($\mathbf{n}\cdot\boldsymbol{\rho} = 0$); $f(\mathbf{n}, \mathbf{n}')$ is the electron scattering amplitude at an impurity, and n is the impurity concentration. The free-electron Green function, averaged over the impurities, G is represented, in the spatially uniform case, in the form:

$$G(s_1, s_2; \omega) = \int \frac{d\xi}{2\pi v_F} \exp\left(\frac{i\xi}{v_F}(s_1 - s_2)\right) G(\xi, \omega),$$
$$G(\xi, \omega \pm i0) = \left(\omega \pm \frac{i}{2\tau} - \sigma_z \xi\right)^{-1}, \tag{21}$$

where $1/\tau = n\nu_F \int do' |f(\mathbf{n}, \mathbf{n}')|^2$ is the frequency of collisions of the electron with impurities.

On substitution of the expressions (17) and (18) into Eq. (3), it is sufficient to take the function $\Pi(\mathbf{r}_1, \mathbf{r}_2)$ (19) into account for equal arguments $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$ only. Defining a Fourier transform with respect to the coordinate $\mathbf{r}$:

$$\Pi(\mathbf{n}, \rho; s, s; \omega_1, \omega_2) = -\frac{1}{v_F} \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\mathbf{r}} \Delta(\mathbf{q}) Q(\mathbf{n}, \mathbf{q}; \omega_1, \omega_2) \sigma_x, \tag{22}$$

we obtain from (20) and (21) the following equation for the Fourier transform Q:

$$Q(\mathbf{n}, \mathbf{q}; \omega_1, \omega_2) = -\int \frac{d\xi}{2\pi} G(\xi_+, \omega_1) G(-\xi_-, \omega_2)$$
$$\times \left(1 + n\nu_F \int do' |f(\mathbf{n}, \mathbf{n}')|^2 Q(\mathbf{n}', \mathbf{q}; \omega_1, \omega_2)\right), \quad \xi_\pm = \xi \pm \frac{1}{2} v_F \mathbf{n}\mathbf{q}. \tag{23}$$

As can be seen from the expression (21) for the Green function $G(\xi, \omega)$, the function $Q(\omega_1, \omega_2)$ determined by Eq. (23) is non-zero only under the condition $\mathrm{Im}\,\omega_1 = \mathrm{Im}\,\omega_2$. Taking this fact into account and collecting together the formulas (18), (19) and (22), we find the contribution of the zeroth approximation in the potential V to the right-hand side of the self-consistency equation (3):

$$\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\mathbf{r}} \Delta(\mathbf{q}) \frac{|g|\nu_F}{4} \int do \int \frac{d\omega}{2\pi i} \cdot \frac{1}{2} \left(1 + \mathrm{th}\frac{\omega}{2T}\right)$$
$$\times \frac{1}{2} \mathrm{Tr}\left(Q(\omega + iz, \omega + i0) - Q(\omega - i0, \omega - iz)\right)(\mathbf{n}, \mathbf{q}), \tag{24}$$

where $\nu_F = mp_F/\pi^2$ is the density of states at the Fermi surface.

After the integration over ξ in the right-hand side of Eq. (23), this equation determines the functions $Q(\omega + iz, \omega + i0)$ and $Q(\omega - i0, \omega - iz)$, which are analytic in the upper and lower half-planes of ω respectively. In connection with this, it is convenient to substitute into the expression (24) an expansion of $\tanh(\omega/2T)$ in simple fractions:

$$\frac{1}{2}\left(1 + \mathrm{th}\frac{\omega}{2T}\right) = T \sum_{\omega_n} \frac{e^{-i\omega_n 0}}{\omega - i\omega_n},$$
$$\omega_n = \pi T(2n+1), \quad n = 0, \pm 1, \pm 2, \ldots \tag{25}$$

After this, integrating over the frequency in the expression (24) we obtain

$$\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\mathbf{r}} \Delta(\mathbf{q}) \frac{|g|\nu_F}{4} \int do\, T \sum_{\omega_n} \frac{1}{2} \mathrm{Tr}\, Q(\mathbf{n}, \mathbf{q}; \omega_n, z), \tag{26}$$

where the function $Q(\mathbf{n}, \mathbf{q}; \omega_n, z)$, according to (23), satisfies the following equation:

$$\left(2|\omega_n| + \frac{1}{\tau} + z + i\sigma_z \operatorname{sgn}\omega_n \cdot v_F \mathbf{n}\mathbf{q}\right) Q(\mathbf{n}, \mathbf{q}; \omega_n, z)$$
$$= 1 + n\nu_F \int do' |f(\mathbf{n}, \mathbf{n}')|^2 Q(\mathbf{n}', \mathbf{q}; \omega_n, z).$$

We shall find the solution of the latter equation by expanding it in powers of the small vector $\mathbf{q}$:

$$Q^{(0)} = \frac{1}{2|\omega_n| + z} \approx \frac{1}{2|\omega_n|} - \frac{z}{(2\omega_n)^2},$$
$$Q^{(1)} = -\frac{i\sigma_z \operatorname{sgn}\omega_n \cdot v_F \mathbf{n}\mathbf{q}}{2|\omega_n|(2|\omega_n| + 1/\tau_{tr})},$$
$$\int \frac{do}{4\pi} Q^{(2)}(\mathbf{n}) \approx -\frac{(v_F q)^2}{3(2\omega_n)^2(2|\omega_n| + 1/\tau_{tr})},$$

$$\frac{1}{\tau_{tr}} = \frac{1}{\tau} - \frac{1}{\tau_1}, \quad n\nu_F \int do' |f(\mathbf{n}, \mathbf{n}')|^2 \mathbf{n}' = \frac{1}{\tau_1}\mathbf{n}.$$

Substituting these formulas into the expression (26), we finally find the contribution of the zeroth approximation in the potential V to the right-hand side of the self-consistency equation (8):

$$\frac{|g|\nu_F}{2} \cdot 2\pi T \sum_{\omega_n} \left(\frac{1}{2|\omega_n|} - \frac{z}{(2\omega_n)^2} + \frac{{}^1/_3 (v_F \nabla)^2}{(2\omega_n)^2(2|\omega_n| + 1/\tau_{tr})}\right) \Delta(r). \tag{27}$$

The calculation of the term linear in V in the expression (17) proceeds analogously. In this case, in the expansion in the gradients in this term, it is sufficient to confine oneself to terms of not higher than first order. As the calculations show, the contribution of this term to the self-consistency equation (3) equals zero. Moreover, it should be noted that it turns out that $\mathrm{Tr}(\sigma_y \hat{\psi}(\mathbf{r}, \mathbf{r})) \neq 0$ in this approximation (cf. Eqs. (6) and (11)). From a physical point of view, this corresponds to the accumulation of small ($\sim\Delta^2$) charges, compensating the electric field inside the superconducting center. From a formal point of view, there are no contradictions here, since in a problem with a given current the potential V (and with it the phase χ(4)) is determined from the continuity equation: $j = j_{ext}$ (i.e., in view of the relation (14), from Eq. (15)), which is an equation of zeroth order in Δ, unlike the condition (6), which is linear in Δ [3)].

It must be emphasized that this point of the calculations, in accordance with the arguments expounded earlier, is fundamental in character. With the aim of obtaining the so-called temporal generalization of the gauge-invariant Ginzburg-Landau equations [13], one usually combines [14] the pair of equations (3) and (6), on an equal footing, into one equation for the complex order parameter. Hence, near the critical temperature T_c for sufficiently small frequencies, gradients and fields, in the linear approximation in Δ an equation is obtained [14] which describes the dynamics of free fluctuations of the order parameter inside a superconductor in the vicinity of an equilibrium normal state. It was with the aid of precisely such an equation, supplemented by random forces, that the fluctuational conductivity was calculated, and the result obtained that the normal state is stable in an electric field, in the work of Kulik[11]. However, as already remarked, such a formulation of the problem does not correspond to the usual conditions of an experiment with inclusion of a superconductor in a circuit with a current source. In these conditions, the situation turns out to be nonequilibrium from the very beginning, and the equation of [14] loses its meaning. This nonequilibrium character is manifested, in particular, in the fact that the phase χ, together with the potential V, is determined from Eq. (15), in which the dissipative normal current (14) appears.

In the last term, quadratic in V, in the expression (17), it is sufficient to confine ourselves to the local approximation. When account is taken of the commutation properties of the Pauli matrices, this term is equal to:

$$\Delta(eV)^2 \frac{1}{2}\left(\frac{\partial}{\partial\alpha}\right)^2 \int \frac{d\omega}{2\pi i} (\omega + i\zeta - H - \sigma_z\alpha)^{-1} [\sigma_x, \hat{f}^{(0)}] (\omega - i\zeta - H - \sigma_z\alpha)^{-1}$$
$$= \frac{1}{2}\Delta(eV)^2 \frac{1}{2}\left(\frac{\partial}{\partial\alpha}\right)^2 (i\zeta - H - \sigma_z\alpha)^{-1} [\sigma_x, \hat{f}^{(0)}] \quad (\alpha \to 0, \ \zeta \to +0).$$

Substituting formula (13) for $\hat{f}^{(0)}$ into this, we obtain the expression

$$\frac{1}{2}\Delta(eV)^2 \cdot \frac{1}{2}\left(\frac{\partial}{i\partial\zeta}\right)^2 \int d\omega \cdot \frac{1}{2}\left(1 + \mathrm{th}\frac{\omega}{2T}\right)$$
$$\times \left(\frac{\sigma_x \delta(\omega - H)}{\omega + i\zeta} + \frac{\delta(\omega - H)\sigma_x}{\omega - i\zeta}\right), \quad \zeta \to +0. \tag{28}$$

We shall make use of the relation

$$\operatorname{Tr}\delta(\omega-H)(\mathbf{r},\mathbf{r})=\frac{1}{\Omega}\operatorname{Sp}\delta(\omega-H)=\nu_F$$

(Ω is the normalization volume), which is valid by virtue of the inequality $\omega \ll \epsilon_F$. Taking this relation and formula (25) into account, we find that the contribution of (28) to the right-hand side of Eq. (3) is equal to

$$-\frac{|g|\nu_F}{2}2\pi T\sum_{\omega_n}\frac{1}{|\omega_n|^3}\cdot\frac{1}{2}(eV)^2\Delta. \qquad (29)$$

Collecting the contributions (27) and (29) in Eq. (3) and performing well-known [12] transformations, we finally obtain the following equation for Δ:

$$\left[-D\left(\frac{\partial}{\partial x}\right)^2+\frac{14\zeta(3)}{\pi}T_c\left(\frac{eEx}{\pi T_c}\right)^2\right]\Delta(x)=\left(\frac{8}{\pi}(T_c-T)-z\right)\Delta(x), \qquad (30)$$

$$D=\frac{8v_F^2}{3\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\left(2\pi T_c(2n+1)+\frac{1}{\tau_{tr}}\right)^{-1}. \qquad (31)$$

In Eq. (30), we have taken into account the fact that the instability develops in the direction of the electric field $\mathbf{E}$, which is taken to be along the x axis. The obtained equation (30) clearly corresponds to the temporal equation of [14], but for the real parameter Δ:

$$\frac{\partial\Delta}{\partial t}=\left[\frac{8}{\pi}(T_c-T)+D\nabla^2-\frac{14\zeta(3)}{\pi}T_c\left(\frac{eV}{\pi T_c}\right)^2\right]\Delta.$$

As can be seen from Eq. (30), for large electric fields E we have $z<0$. The point of instability in the critical field E_{c2} corresponds to $z=0$. From a formal point of view, the problem of determining the form of the superconducting center and of the critical field E_{c2} coincides completely with the analogous problem for the transition to the mixed state in the field H_{c2} [2]. Hence, comparing Eq. (30) with the corresponding equation of the paper [2], we can obtain the relation

$$E_{c2}=\pi H_{c2}\sqrt{2\pi DT_c/7\zeta(3)}.$$

It must be noted, however, that this relation is formal in character, since really the field H_{c2} has meaning only for a bulk type-II superconductor.

Expressing the field E_{c2} directly in terms of the coefficients of Eq. (30), we obtain the following formulas (in the usual units) for E_{c2} and the critical current j_{c2}:

$$E_{c2}=\frac{4k(T_c-T)}{e}\sqrt{\frac{2\pi kT_c}{7\zeta(3)\hbar D}}$$
$$j_{c2}=\sigma_n E_{c2} \qquad (32)$$

(k is Boltzmann's constant).

The result (32) obtained has a simple physical meaning. The instability begins to develop when the energy acquired by an electron in the electric field over the coherence length $\xi(T)\sim\sqrt{D/(T_c-T)}$ becomes less than or of the order of the condensation energy

$$\Delta(T)\sim\sqrt{T_c(T_c-T)}$$
$$(eE\xi(T)\leqslant\Delta(T)).$$

It is not difficult to convince oneself that, as was assumed at the beginning of the paper, the critical current j_{c2} is always greater than the "pair-breaking" current j_c. We note also that the temperature dependence of j_{c2} near T_c turns out to be linear, in contrast to the current $j_c\sim(T_c-T)^{3/2}$.

The coefficient D (31) can be calculated in two limiting cases: a "clean" ($\tau_{tr}T_c \gg 1$) and a "dirty" ($\tau_{tr}T_c \ll 1$) superconductor:

$$\tau_{tr}T_c\gg 1:\ D=\frac{7\zeta(3)v_F^2}{6\pi^3 T_c},\qquad \tau_{tr}T_c\ll 1:\ D=\frac{1}{3}\tau_{tr}v_F^2$$

($\zeta(3)$ is a particular value of the Riemann zeta-function). Correspondingly, the field E_{c2} (32) in these cases has the form

$$\tau_{tr}T_c\gg 1:\ E_{c2}=\frac{8\sqrt{3}(\pi k)^2T_c}{7\zeta(3)e\hbar v_F}(T-T)\quad\left(E_{c2}=\frac{v_F}{\sqrt{3}c}H_{c2}\right);$$
$$\tau_{tr}T_c\ll 1:\ E_{c2}=\frac{4k(T_c-T)}{ev_F}\sqrt{\frac{6\pi kT_c}{7\zeta(3)\hbar\tau_{tr}}} \qquad (33)$$

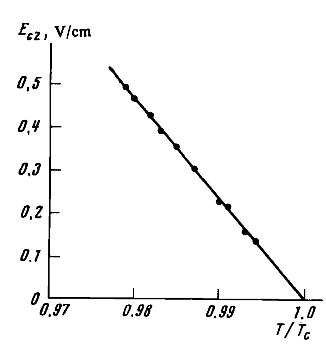
From the latter formulas (33), we can obtain an estimate of the proximity of T to T_c necessary for the observation of resistive states (in the whole range of currents $j_c<j<j_{c2}$) in "clean" superconductors. Since the energy acquired by a normal electron over the mean free path $l\sim v_F\tau$ in the field E_{c2} should be small compared with the temperature, according to (33) we obtain the following inequality in the case $\tau T_c \gg 1$:

$$(T_c-T)/T_c\ll\xi_0/l\quad(\xi_0\sim v_F/T_c).$$

The figure shows the experimental temperature dependence of the field E_{c2}, obtained by Dmitriev and Churilov[15,16]. In addition to the clear linear dependence, the estimate of the fields E_{c2} from formula (32) also agrees with the experimental data in order of magnitude.

In conclusion, we shall discuss the description of the microscopic structure of the resistive state [16], which arises as a result of the development of the instability considered. Clearly, in a resistive state in a superconducting channel, a microscopic phase separation, with alternating normal and superconducting regions along the channel, should occur. Because of the nonequilibrium nature of the resistive state, we should expect that this structure will be dynamic, and this, in turn, should lead to the experimentally observed[15,16] generation of electromagnetic waves.

A detailed calculation of this picture will be published later, but here we make the following comment. For non-zero (on the average) superconducting order along the channel, an unlimited increase of the electrochemical potential, which would "drain off" electrons from the Fermi surface, in the vicinity of which the superconducting pairing occurs, is impossible. From this, according to Eq. (9), it follows that, in the presence of a constant electric-field component (i.e., resistive character) in the channel, it is necessary in the regions $\Delta=0$ to introduce discontinuities of the phase χ and, correspondingly, of the potential V. These discontinuities of phase are related to the vortex singularities in the theory of the mixed state [2], and from a formal point of view their origin is associated with the fact that the superconducting condensate is, in itself, a self-consistent



"source" and "sink" of electrons. The latter remark permits a deeper understanding, from the energy point of view, of the difference in the response of superconducting electrons to electric fields created by a varying magnetic field and by a direct-current source. From this point of view, work can only be performed on the superconducting condensate by varying the magnetic field. As regards the current source, this, as must be the case by the nature of the phenomenon of superconductivity, does not do work during the passage of the superconducting current.

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[1)]Since both directions of the electron spin are equivalent in the theory under consideration, it is convenient, as is done here, to introduce immediately a source of pairs of electrons with opposite spins, without limiting the generality of the treatment by doing this.

[2)]This is clear, if only from the fact that integration around a long closed conductor gives: $\oint \mathbf{A}\cdot d\mathbf{l} = -t\oint \mathbf{E}\cdot d\mathbf{l} = \Phi$, whence $\partial\Phi/\partial t \neq 0$.

[3)]Here there is an analogy with the electro-neutrality condition $\delta N = 0$ (8), which can, generally speaking, establish electric fields such that $\operatorname{div}\mathbf{E} \neq 0$ in the next approximation.

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