Microgeons with spin

A. Ya. Burinskiĭ

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A geon with angular momentum is considered. The corresponding approximate solutions of the Einstein–Maxwell system of equations are obtained, solutions which are wave generalizations of the stationary Kerr–Newman metric. The metric does not lead to causality-violating trajectories for $a^2+e^2>m^2$.

1. INTRODUCTION

The Wheeler geons are known to be macroscopic objects. Removing two of Wheeler's restrictions [a) admitting singular wave fields, b) considering fields with angular momentum], one can obtain a self-consistent construction of a geon of arbitrary size. In[1] a toroidal geon with Kerr metric has been considered. The Kerr gravitational field is concentrated near a singular ring of radius a. This ring can serve as a gravitational waveguide, confining photons on a ring-like orbit of radius a. In a remarkable way the electromagnetic field of the photon which is singular on the same ring, produces itself a gravitational field of the Kerr type with the required parameters, guaranteeing the self-consistency of the construction. We shall indicate below an approximate solution of the corresponding system of Maxwell-Einstein equations.

We shall adopt the following values of the parameters of the Kerr metric ($\hbar = c = 1$):

$$e^2\approx 1/137,\quad m\approx 10^{-22},\quad a\approx 10^{22},\quad ma={}^1/_2. \tag{1.1}$$

As was first indicated by Carter[2], when these three parameters of the Kerr metric are adopted, one obtains a model for the four parameters of the electron: charge, mass, spin and magnetic moment, since the electromagnetic field of the Kerr metric exhibits the anomalous Dirac gyromagnetic ratio. The microgeon considered by us also simulates in a certain respect the properties of the electron, as will be discussed below.

If the parameter a, which has the meaning of angular momentum density per unit mass, becomes larger than m, the horizons of the Kerr metric vanish, exposing the naked ring-shaped singularity and a region which contains causality-violating closed timelike curves. In the nonstationary generalization considered by us the relation a > m between the parameters does not lead to causality violation.

2. THE KERR-NEWMAN METRIC: THE MAXWELL EQUATIONS ON THE BACKGROUND OF THE KERR METRIC

The metric tensor of the Kerr field and of the charged version of the Kerr-Newman field[3] can be written in the form

$$g^{\mu\nu}=\eta^{\mu\nu}-2Hk^\mu k^\nu, \tag{2.1}$$

where $\eta^{\mu\nu}$ is the metric tensor of Minkowski space with the Cartesian coordinates (x, y, z, t), the 4-vector k^μ is isotropic

$$k^\mu k_\mu=0, \tag{2.2}$$

and is tangent to an isotropic congruence with rotation but no displacement. The scalar function H has the form

$$H=(mr_c-e^2/2)/(r_c^2+a^2\cos^2\theta_c), \tag{2.3}$$

where e is the charge, and r_c and θ_c are oblate spheroidal coordinates related to the Cartesian coordinates by

$$x+iy=(r_c+ia)e^{i\varphi}\sin\theta_c,\quad z=r_c\cos\theta_c. \tag{2.4}$$

For $r_c = 0$, $\theta_c = \pi/2$ the function H becomes singular. This leads to a genuine singularity of the metric on a ring of radius a:

$$z=0,\quad x^2+y^2=a^2. \tag{2.5}$$

The electromagnetic field has the form

$$(E_x-iH_x, E_y-iH_y, E_z-iH_z)=\frac{e}{(r_c+ia\cos\theta_c)^3}(x,y,z+ia). \tag{2.6}$$

We shall show below that (2.6) formally coincides with the corresponding flat-space solution. The reason for this lies mainly in the fact that the metric (2.1) in Cartesian coordinates has

$$\sqrt{-g}=1. \tag{2.7}$$

Indeed, using the self-dual tensor

$$\mathcal{F}^{\mu\nu}=F^{\mu\nu}+i\mathring{F}^{\mu\nu}, \tag{2.8}$$

where

$$\mathring{F}^{\mu\nu}={}^1/_2\varepsilon^{\mu\nu\alpha\beta}\sqrt{-g}F_{\alpha\beta}, \tag{2.9}$$

one can rewrite the system of Maxwell equations in the form

$$\mathcal{F}^{\mu\nu}{}_{;\nu}=(\sqrt{-g}\mathcal{F}^{\mu\nu})_{,\nu}/\sqrt{-g}=0, \tag{2.10}$$

which together with (2.7) leads to an expression which coincides formally with the Maxwell equations in flat space:

$$\mathcal{F}^{\mu\nu}{}_{,\nu}=F^{\mu\nu}{}_{,\nu}+i\mathring{F}^{\mu\nu}{}_{,\nu}=0. \tag{2.11}$$

Thus, the Kerr metric opens up the possibility of obtaining solutions which take into account the gravitational field from the corresponding solutions in flat space.

Further, we shall consider on the background of the Kerr metric the wave functions of Izmest'ev[4], which are obtained from the usual spherical harmonics by means of a complex shift. In particular, the Kerr-Newman field (2.6) is obtained as a result of a complex shift of the Coulomb field, fact which was pointed out in[1,5].

3. THE COMPLEX SHIFT

The method of complex shifts in gravitation and electrodynamics has been discovered independently by several authors[1)]. The essence of the method consists in using the translation invariance of the equations to show that after the complex shift z → z + ia the solutions of the equations continue to satisfy the equations, i.e., one can obtain new solutions from the old ones.

Under the complex shift D_a: z → z + ia the radius vector b becomes complex. We have

$$(\tilde{r})^2=(D_a r)^2=x^2+y^2+(z+ia)^2=(\xi+i\eta)^2. \quad (3.1)$$

This implies

$$\xi^2+\eta^2=x^2+y^2+z^2-a^2, \quad \xi\eta=az. \quad (3.2)$$

Making the substitution

$$\xi=r_c, \quad \eta=a\cos\theta_c, \quad (3.3)$$

we obtain oblate spheroidal coordinates, which are used in writing the Kerr metric and are related to the Cartesian coordinates by means of the relation (2.4).

The complex radius has the form

$$\tilde{r}=D_a r=r_c+ia\cos\theta_c. \quad (3.4)$$

Similarly, the angle θ becomes complex, so that

$$D_a\cos\theta=(z+ia)/\tilde{r}, \quad D_a\sin\theta=(x^2+y^2)^{1/2}/\tilde{r}. \quad (3.5)$$

Making use of these relations one can obtain from known solutions new solutions by replacing z, r, cos θ, and sin θ by the corresponding functions which have been subjected to the complex shift. Applying this rule to the simplest Coulomb field yields immediately the Kerr-Newman electromagnetic field:

$$\tilde{\mathbf{E}}-i\tilde{\mathbf{H}}=D_a\nabla(-e/r)=D_a[(e/r^3)(x,y,z)]=(e/\tilde{r}^3)(x,y,z+ia). \quad (3.6)$$

Taking (3.4) into account, this expression coincides with (2.6).

After the complex shift the solution which has a point singularity at r = 0 acquires a singularity on the ring $\tilde{r} = 0 = r_c + i\, a\cos\theta_c$, coinciding with the singular ring of the Kerr metric, (2.5). This also refers to the wave solutions; for instance, the shifted spherical harmonics acquire a singularity on the singular ring and can serve as the source of a metric which has a structure close to the Kerr metric.

4. THE STRUCTURE OF THE FIELD NEAR THE SINGULAR RING

Near the singular ring filament we introduce the following coordinates in a two-dimensional plane orthogonal to the ring, with the coordinate origin on the ring:

$$u=z, \quad v=\rho-a=\sqrt{x^2+y^2}-a. \quad (4.1)$$

The spatial part of the vector k^μ near the ring is parallel to the ring[6], so that

$$k_u=k_v=0. \quad (4.2)$$

From (3.1) we obtain

$$\tilde{r}=(\rho^2-a^2+z^2+2iaz)^{1/2}\approx[2a(v+iu)]^{1/2}. \quad (4.3)$$

Similarly, the relations (3.5) yield

$$D_a\cos\theta\approx ia/\tilde{r}, \quad D_a\sin\theta\approx a/\tilde{r}; \quad (4.4)$$

$$D_a\exp(i\theta)\approx i2a/\tilde{r}, \qquad D_a\exp(-i\theta)\approx(u-iv)/\tilde{r}\approx -i\tilde{r}/2a. \quad (4.5)$$

The expression (3.6) has the form

$$(E_v-iH_v,\ E_u-iH_u)\approx(ea/\tilde{r}^3)(1,\ i), \quad (4.6)$$

or, in coordinate notation

$$-H_u=E_v=\mathrm{Re}\,\Phi, \quad E_u=H_v=-\mathrm{Im}\,\Phi, \quad (4.7)$$

$$\Phi=ae/\tilde{r}^3. \quad (4.8)$$

Thus, near the ring, (3.6) describes the background electromagnetic field

$$\mathbf{EH}=0, \quad \mathbf{E}^2=\mathbf{H}^2. \quad (4.9)$$

Taking (4.4) into account, Φ takes on the form

$$\Phi=ae/[2a(v+iu)]^{3/2}. \quad (4.10)$$

The fractional power in the denominator shows that the field changes sign if one goes around the point u = v = 0, i.e., if the path surrounds the singular ring. This is the well-known "two-sheetedness" of the Kerr field, for which there is so far no satisfactory physical interpretation.

Taking (3.4) into account, the function H is of the form

$$H=\frac{m}{2}\left(\frac{1}{\tilde{r}}+\overline{\frac{1}{\tilde{r}}}\right)-\frac{e^2}{2}\frac{1}{|\tilde{r}|^2} \quad (4.11)$$

where the bar denotes complex conjugation.

Let us discuss the question of causality violation. As one approaches the singular ring, $|\tilde{r}|\to 0$, the function H increases. Carter[2] has noted that for 2H > 1 the coordinate φ in the Kerr metric becomes timelike and there appear timelike closed curves ξ (parametrized by the variation of φ from 0 to 2π), which violate causality. Taking into account the relation of the parameters (1.1) considered by us, the region 2H > 1 can be described by the condition $|\tilde{r}|^2 < e^2$, or

$$u^2+v^2<e^4/4a^2. \quad (4.12)$$

This is a tubelike neighborhood around the singular ring. One can take as an example of a closed timelike curve the curve u = v = const., where u and v satisfy (4.12).

Let us consider the shifted spherical wave function

$$\tilde{\Psi}_{11}=D_a\Psi_{11}=A\tilde{r}^{-1/2}H_{3/2}(k\tilde{r})P_1^1(D_a\cos\theta)\exp[i(\varphi-kt)]. \quad (4.13)$$

Near the singular ring the Hankel function $H_{3/2}$ has a singularity of order 3/2 in 1/r, the Legendre polynomial $P_1^1(D_a\cos\theta)$ has a pole of first order, so that near the ring (4.13) takes on the form

$$\tilde{\Psi}_{11}=A\tilde{r}^{-3}\exp[i(\varphi-kt)].$$

Setting

$$\Phi=ae\tilde{r}^{-3}\exp[i(\varphi-kt)], \quad (4.14)$$

the relations (4.7) yield again the electromagnetic null field satisfying the condition

$$k_\mu F^{\mu\nu}=0. \quad (4.15)$$

The energy-momentum tensor of this field can be written in the form

$$T^{\mu\nu}={}^1/_2\mathcal{F}^{\mu\sigma}\overline{\mathcal{F}}_\sigma{}^\nu=2k^\mu k^\nu|\Phi|^2. \quad (4.16)$$

Thus, near the singular filament both the Kerr-Newman stationary electromagnetic field and the wave solution (4.7), (4.14) have the same form of the energy-momentum tensor

$$T^{\mu\nu}=2k^\mu k^\nu a^2e^2/|\tilde{r}|^6. \quad (4.17)$$

5. THE MICROGEON: APPROXIMATE SIMULTANEOUS SOLUTION OF THE SYSTEM OF MAXWELL-EINSTEIN EQUATIONS

As we already remarked, the relation of the parameters (1.1) is such that the gravitational field is concentrated on an extremely thin filament ring, outside of which one practically has a realization of Minkowski space. This filament serves as a gravitational waveguide along which an electromagnetic wave propagates, the singularity of which coincides with the gravitational singularity. The wave is described by the electromagnetic null-field (4.7), (4.14) and near the filament this field can be considered a plane-front field with direction of propagation k^{μ}. According to (4.2) the following relation holds

$$\Phi_{,\nu}k^{\nu}=0. \quad (5.1)$$

A metric of the form (2.1) with an electromagnetic field satisfying (5.1) leads us to a class of solutions considered by Peres[7]. The Ricci tensor can be written in the form

$$R^{\mu\nu}=-k^{\mu}k^{\nu}\Box H, \quad (5.2)$$

where $\Box$ is the flat-space D'Alembertian. Comparing (5.2) and (4.16) and taking into account the vanishing of the curvature scalar one sees that the Einstein equations reduce to a scalar equation relating H and Φ:

$$\Box H=-2|\Phi|^2. \quad (5.3)$$

The solution of this equation is composed of the solution of the homogeneous equation, H^0, corresponding to $\Phi = 0$ plus a particular solution of the inhomogeneous equation.

If one selects as solution of the homogeneous equation

$$H^0=\frac{m}{2}\left(\frac{1}{[2a(v+iu)]^{1/2}}+\frac{1}{[2a(v-iu)]^{1/2}}\right)\approx\frac{m}{2}\left(\frac{1}{\tilde{r}}+\frac{1}{\bar{\tilde{r}}}\right), \quad (5.4)$$

we obtain the neutral Kerr field. The charge (4.8) induces the particular solution

$$H^{\cdot}\approx\frac{-e^2}{4a(v^2+u^2)^{1/2}}\approx\frac{-e^2}{2|\tilde{r}|^2}. \quad (5.5)$$

Thus, the Kerr-Newman solution is the sum of these two solutions.

The electromagnetic wave field (4.14) induces, according to ξ (4.17), the same particular solution (5.5) near the filament, leading, as we have seen, to a violation of causality.

Let us now consider an electromagnetic field which is the superposition of the "charge" (4.8) and the wave field (4.14):

$$\Phi=\frac{ea}{\tilde{r}^3}(1+e^{i\phi})=\frac{ea}{\tilde{r}^3}\cos\frac{\phi}{2}e^{i\phi/2}. \quad (5.6)$$

Then

$$|\Phi|^2=(2e^2a^2/|\tilde{r}|^6)(1+\cos\phi), \quad (5.7)$$

where we have introduced the notation $\phi = \varphi - kt$. This electromagnetic field induces a metric with the function

$$H^{\cdot}=\frac{-e^2}{|\tilde{r}|^2}(1+\cos\phi)=\frac{-e^2}{|\tilde{r}|^2}2\cos^2\frac{\phi}{2}. \quad (5.8)$$

Owing to the time-dependence introduced by means of (5.8) the metric varies periodically, oscillating with the frequency k. Averaging the metric over a period of the oscillation yields the Kerr-Newman metric, i.e., this metric is only "in the mean" equal to the Kerr-Newman metric.

In distinction from the cases considered before, here there are no closed timelike trajectories. Indeed, the condition $2H > 1$ leads to the relation

$$u^2+v^2<(e^4/4a^2)\cos^2(\phi/2). \quad (5.9)$$

A tubular neighborhood of variable thickness is formed around the singular ring. At $\phi = \pi$ the tube becomes disconnected, preventing the formation of closed timelike curves.

Thus, we have described an approximate solution of the simultaneous system of Maxwell-Einstein equations, solution which is schematically represented as a geon with a Kerr metric. The ability of the Kerr metric to simulate charge, spin and the anomalous gyromagnetic ratio of the electron has already been noted in a series of papers[2,3,5,8]. The consideration of the geon gives additional possibilities, since it has wave properties and, using Wheeler's expression, it simulates "mass without mass," allowing one to describe the mutual transformations of massive particles and massless fields.

A microgeon with the parameters of the electron can also be considered as a wave packet, "bound" by the gravitational field into the form of an extremely thin ring, of radius equal to the Compton wavelength. This recalls the limits of localization of an electron known from quantum electrodynamics.

[1)]It seems that the first to use this operation was P. Appel in 1887. Newman and co-workers have used this approach to obtain the charged version of the Kerr metric from the Reissner-Nordström metric. The complex shift has also been used by Synge, Trautman, Kerr, Israel, and Keres. The wave functions have been indicated by Izmest'ev[4]; the relations (3.1)–(3.5) have been appropriated from his paper.

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42