Three-dimensional solitons[1)]

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We show that three-dimensional ion-sound solitons can exist in a low-pressure magnetized plasma. We prove their stability.

The role played by solitons—solitary waves propagating without distortion of their shape—in the physics of non-linear waves is well known (see, e.g.,[2]). So far one-dimensional solitons have been studied in hydrodynamics and plasma physics. We show in the present paper that in a low-pressure magnetized plasma ($\beta = 8\pi nT/H^2 \ll 1$) three-dimensional solitons, which decrease in all directions, can propagate along the magnetic field.

We consider slow motions of a non-isothermal plasma ($T_e \gg T_i$) which is situated in a uniform magnetic field H_0 with characteristic frequencies $\omega \lesssim \omega_{Hi}$ (ω_{Hi} is the ion cyclotron frequency). We can describe such motions in the framework of the hydrodynamic set of equations for the density n and the ion velocity v. We shall assume the electrical field to be a potential field. One can easily verify that the requirement that the field is potential is the same as the condition $\beta \ll 1$. Therefore,

$$\partial n/\partial t + \operatorname{div} n\mathbf{v} = 0, \tag{1}$$

$$\partial \mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v} = -e\nabla\varphi/M + [\mathbf{v}\times\boldsymbol{\omega}_{Hi}], \tag{2}$$

$$\nabla^2\varphi = -4\pi e(n - n_0\exp(e\varphi/T_e)). \tag{3}$$

This set of equations describes two kinds of oscillations—ion-sound and cyclotron oscillations, which in the long-wavelength limit have the dispersion laws:

$$\omega_1(k) = k_z c_s(1 - {}^1/_2 k_\perp^2 r_H^2 - {}^1/_2 k^2 r_d^2),$$
$$\omega_2(k) = \omega_{Hi}(1 + {}^1/_2 k_\perp^2 r_H^2).$$

Here r_d is the Debye radius, $r_H = c_s/\omega_{Hi}$, and c_s is the sound velocity.

We can from (3) find the electrical potential for long-wavelength oscillations ($kr_d \ll 1$) and weak non-linearity ($\delta n/n_0 \ll 1$):

$$\frac{e\varphi}{T_e} = (1 + r_d^2\nabla^2)\frac{\delta n}{n_0} - \frac{1}{2}\left(\frac{\delta n}{n_0}\right)^2.$$

Eliminating φ from the equations of motion (2) we get then

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -c_s^2\nabla\left\{(1 + r_d^2\nabla^2)\frac{\delta n}{n_0} - \frac{1}{2}\left(\frac{\delta n}{n_0}\right)^2\right\} + [\mathbf{v}\times\boldsymbol{\omega}_{Hi}]. \tag{4}$$

We split off from the set of Eqs. (1) to (4) only the low-frequency motions—the ion-sound oscillations. Omitting the derivation of the equations for them (the derivation for analogous equations is contained in[3]) we emphasize solely that for ion-sound waves $\omega_2(k) \gg \omega_1(k)$ and that then the ion velocity is to a good approximation along the z-axis—the direction of the magnetic field. This is the fact which enables us to split off from (1) to (4) a set of equations for δn and v_z which describes only the ion-sound oscillations:

$$\frac{\partial \delta n}{\partial t} + n_0\frac{\partial}{\partial z}(1 + r_H^2\nabla_\perp^2)v_z + \frac{\partial}{\partial z}\delta n v_z = 0,$$
$$\frac{\partial v_z}{\partial t} + v_z\frac{\partial v_z}{\partial z} = -c_s^2\frac{\partial}{\partial z}\left\{(1 + r_d^2\nabla^2)\frac{\delta n}{n_0} - \frac{1}{2}\left(\frac{\delta n}{n_0}\right)^2\right\}. \tag{5}$$

The group velocity of the ion-sound oscillations is directed along the magnetic field; in that case waves propagating in opposite directions interact weakly with one another. This enables us to reduce the set (4) and (5) to a single equation:

$$\frac{\partial v_z}{\partial t} + c_s\frac{\partial}{\partial z}\left\{1 + \frac{1}{2}(r_H^2 + r_d^2)\nabla_\perp^2 + \frac{1}{2}r_d^2\frac{\partial^2}{\partial z^2} + \frac{1}{2}\frac{v_z}{c_s}\right\}v_z = 0, \tag{6}$$

which describes ion-sound waves propagating in one direction along the magnetic field. Equation (6) is a generalization of the well known Korteweg-de Vries (KdV) equation.

Changing to a system of coordinates which moves with the sound velocity along the magnetic field and introducing the variables

$$\xi_z = r_d^{-1}(z - c_s t), \quad \boldsymbol{\xi}_\perp = (r_H^2 + r_d^2)^{-1/2}\mathbf{r}_\perp,$$
$$\tau = {}^1/_2\omega_{pi}t, \quad u = v_z/2c_s,$$

we can write Eq. (5) in dimensionless form:

$$\frac{\partial u}{\partial \tau} + \frac{\partial}{\partial \xi_z}(\nabla^2_{\xi\xi} + u)u = 0. \tag{7}$$

Moreover, we can write Eq. (7) in the form

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial \xi_z}\frac{\delta\mathscr{H}}{\delta u},$$

where the Hamiltonian is

$$\mathscr{H} = \int\{{}^1/_2(\nabla_\xi u)^2 - {}^1/_3 u^3\}d\xi.$$

The conservation of $\mathscr{H}$ follows immediately from this form. One verifies easily that Eq. (7) has also other integrals of motion:

$$M(\xi_\perp) = \int u\,d\xi_z, \quad P = \int u^2\,d\xi,$$
$$\mathbf{I} = \int \boldsymbol{\xi} u\,d\xi - t\mathbf{n}\int u^2\,d\xi.$$

The meaning of the first integral is the conservation law for the "mass" on the line $\boldsymbol{\xi}_\perp$ = constant. The second integral is the momentum conservation law, and the third one the conservation law for the center of mass. From the latter it follows, in particular, that the center of mass velocity is equal to $P/\int M d\boldsymbol{\xi}_\perp$ and is directed along the magnetic field ($\mathbf{n} = \mathbf{H}_0/H_0$).

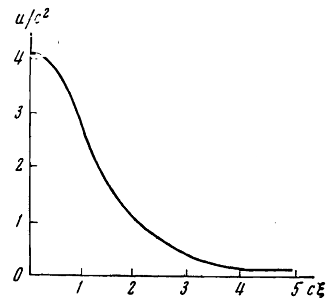
Furthermore, we shall consider the stationary solutions of Eq. (7) of the form $u = u(\xi_z - \lambda t)$ which satisfy the equation

$$\Delta_{\xi\xi}u - (\lambda - u)u = 0. \tag{8}$$

If $\lambda = c^2 > 0$ it has solutions which decrease exponentially as $|\xi| \to \infty$. The simplest one—the spherically symmetric one—satisfies the equation

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\xi^2\frac{\partial u}{\partial \xi} - (c^2 - u)u = 0.$$

This solution is a three-dimensional soliton with size $c^{-1}(r_H^2 + r_d^2)^{1/2}$ in the transverse and size r_d/c in the longitudinal direction. We give in the figure a graph of this solution $u = u(\xi)$, calculated with a computer.



We shall study the problem of the stability of a three-dimensional soliton. We note that Eq. (8) can be written in the form

$$\frac{\delta}{\delta u}\left(\frac{\lambda}{2}P+\mathcal{H}\right)=0,$$

so that all its solutions are stationary points of the functional $\mathcal{H}$ for fixed P. From Lyapunov's theorem (see, e.g., [4]) it follows that the soliton is stable if it corresponds to an absolute minimum of the functional $\mathcal{H}$. One verifies easily that among the values at all stationary points the lowest is realized for the spherically symmetric soliton. We note also that we obtain from Hölder's inequality the estimate

$$\int u^3\,d\xi \leqslant \left(\int u^2\,d\xi\right)^{1/2}\left(\int u^4\,d\xi\right)^{1/2}.$$

For $\int u^4 d\xi$ we can use the interpolation inequality:[5]

$$\int u^4\,d\xi \leqslant 4\left(\int u^2\,d\xi\right)^{1/2}\left(\int (\nabla u)^2 d\xi\right)^{3/2}.$$

We finally have

$$\mathcal{H} \geqslant \int \frac{(\nabla u)^2}{2}\,d\xi - \frac{2}{3}\left(\int u^2\,d\xi\right)^{3/4}\left(\int (\nabla u)^2\,d\xi\right)^{3/4} \geqslant -\frac{1}{6}\left(\int u^2\,d\xi\right)^3.$$

From this it follows that the functional $\mathcal{H}$ is bounded from below and thus attains an absolute minimum for the spherical soliton. This then proves its stability.

It is well known (see [6]) that one can show for the one-dimensional soliton that it is stable and that the asymptotic state as $t \to \infty$ of any initial condition is a set of solitons. This derivation is based upon the formalism of the inverse scattering problem for the Schrödinger operator. It follows, clearly, from the fact that the three-dimensional soliton is stable that the asymptotic state as $t \to \infty$ for Eq. (7) is also a set of solitons. One must note that usually in a plasma $r_H > r_d$; the soliton is under those conditions quasi-planar, being elongated in the transverse direction. One can then clearly study the interaction between solitons using perturbation theory, starting from the one-dimensional KdV equation.

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1) The results of the present paper were given before in [1].

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