The parametric excitation of the electron cyclotron oscillations of a plasma located in an alternating electric field

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The parametric excitation of electron cyclotron oscillations in a plasma is studied in a wide range of pump-field strengths when the growth rate of the oscillations significantly exceeds the ion cyclotron frequency. The dependence of the maximum growth rate on the pump-field amplitude is determined. The effective collision rate at which saturation of the parametric instability occurs is found.

1. INTRODUCTION

As is well known, the oscillations of the electrons relative to the ions of a plasma located in an alternating electric field lead to numerous parametric instabilities in the plasma[1-3]. If the pumping electric field is perpendicular to the magnetic field, then parametric excitation of electrostatic electron cyclotron waves turns out to be possible. This possibility has been pointed out before[4,5]. The thresholds and regions of these instabilities were determined in [4-7], where the growth rates in certain limiting cases were also found. The phenomenon of anomalous pump-wave absorption accompanied by the acceleration of the plasma electrons and a strong epithermal plasma radiation, which can be related to the parametric excitation of the electron cyclotron harmonics, was discovered in experimental investigations [8-10], while the decay of the pump wave into an electron cyclotron wave and ion sound was experimentally discovered in [11].

In the present paper we undertake a theoretical study of the parametric excitation of electron cyclotron oscillations in a wide range of pump-field strengths when the amplitude of the electron oscillations relative to the ions lies between the electron and ion thermal velocities (and the ion-sound velocity). In this region the growth rate of the oscillations is considerably greater than the ion cyclotron frequency. In the region of weak fields, when $T_i \gtrsim T_e$, this instability is kinetic: the oscillations are built up by the individual resonant ions. If, however, $T_e \gg T_i$, then there occurs at low field strengths a coherent excitation of coupled ion-acoustic and electron-cyclotron oscillations. In the region of high field strengths the electron cyclotron oscillations are excited coherently. The maximum growth rate of the oscillations under consideration increases monotonically with increasing amplitude of the current velocity. Only in a dense plasma, when $\omega_{pe} \gg \omega_{Be}$, does it attain a maximum value at $u \sim v_{Te}\omega_{Be}/\omega_{pe}$, and then slowly decrease as $u^{-1/3}$.

Assuming that the scattering of the electrons and ions by the growing unstable fluctuations in the electric field is small-angle scattering that is determined by the Fokker-Planck equation, we consider the nonlinear phase of the instabilities under consideration, and determine the effective collision rate at which saturation of the parametric instability occurs. The obtained results are very similar to the results obtained in the case of a purely beam excitation of electron cyclotron oscillations in a plasma with a transverse current, when the pump field has a frequency significantly less than the growth rate of the oscillations (see [12-13] and the literature cited there).

We shall consider the oscillations propagating perpendicularly to the magnetic field. In this case the contribution of the electrons to the longitudinal permittivity is given by

$$\delta\varepsilon_e(\omega)=\frac{\omega_{pe}^2}{k^2v_{Te}^2}\left[1-\sum_{s=-\infty}^{\infty}A_s(k^2\rho_e^2)\frac{\omega}{\omega-s\omega_{Be}}\right], \tag{1.1}$$

where

$$\omega_{pe}=(4\pi e^2n_0/m_e)^{1/2},\quad v_{Te}=(T_e/m_e)^{1/2},$$
$$\rho_e=v_{Te}/\omega_{Be},\quad \omega_{Be}=eB_0/m_ec\quad(\omega_{Be}>0),\quad A_s=e^{-x}I_s(x),$$

and $I_s(x)$ is the modified Bessel function. The exponentially small term

$$\sim\exp\left[-\left(\frac{\omega-s\omega_{Be}}{\sqrt{2}k_\parallel v_{Te}}\right)^2\right],$$

which is responsible for the Cerenkov and cyclotron absorption of the waves by the electrons, will be neglected ($k_\parallel \to 0$). Since $\gamma \gg \omega_{Bi} = eB_0/m_ic$ and, furthermore, $k\rho_i \gg i(\rho_i = v_{Ti}/\omega_{Bi})$, the contribution of the ions to the longitudinal permittivity is determined by its expression at $B_0 = 0$:

$$\delta\varepsilon_i(\omega)=\frac{\omega_{pi}^2}{k^2v_{Ti}^2}[1+i\sqrt{\pi}z_iW(z_i)], \tag{1.2}$$

where

$$\omega_{pi}=(4\pi e^2n_0/m_i)^{1/2},\quad v_{Ti}=(T_i/m_i)^{1/2},$$
$$W(z_i)=e^{-z_i^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^{z_i}e^{t^2}dt\right),\quad z_i=\frac{\omega}{\sqrt{2}kv_{Ti}}.$$

The development of the oscillations in the presence of an alternating electric field $\mathbf{E}=\mathbf{E}_0\sin\omega_0t$ that was adiabatically switched on at $t=t_0\to-\infty$ is determined by the equation [14,15]

$$[1+\delta\varepsilon_\alpha(\omega,k)]\varphi(\omega)+\sum_{n=-\infty}^{\infty}a_n(\omega)\varphi(\omega-n\omega_0)=Q(\omega), \tag{1.3}$$

where

$$a_n(\omega)=\sum_{m=-\infty}^{\infty}J_n(a_E)J_{n+m}(a_E)\delta\varepsilon_\beta(\omega-m\omega_0), \tag{1.4}$$

$$a_E=\left\{\left[\sum_{\alpha=e,i}\frac{|e_\alpha|\mathbf{k}_\perp\mathbf{E}_{0\perp}}{m_\alpha(\omega_0^2-\omega_{B\alpha}^2)}\right]^2+\left[\sum_{\alpha=e,i}\frac{e_\alpha\omega_{B\alpha}[\mathbf{kE}_0]z}{m_\alpha\omega_0(\omega_0^2-\omega_{B\alpha}^2)}\right]^2\right\}^{1/2} \tag{1.5}$$

$\varphi(\omega)$ is a quantity proportional to the Fourier transform of the oscillation potential, and $Q(\omega)$ is a quantity proportional to the initial perturbation of the electron and ion distribution function. For $\omega_{Bi} \lesssim \omega_0 \lesssim \omega_{Be}$ we obtain from (1.5) the following estimate: $a_E \sim ku/\omega_0$, where

$$u\sim c\frac{E_0}{B_0}\frac{\omega_{Be}}{\omega_0-\omega_{Be}}$$

is the amplitude of the drift velocity of the particles in the crossed $\mathbf{E}_0$ and $\mathbf{B}_0$ fields. We shall assume that $v_{Ti} \ll u \ll v_{Te}$.

It follows from (1.3) that the dispersion equation determining the complex frequency ω has the form of an infinite determinant

$$\det\left\|\delta_{m,n}+\frac{a_{n-m}(\omega-m\omega_0)}{1+\delta\varepsilon_\alpha(\omega-n\omega_0)}\right\|=0. \quad (1.6)$$

Below we shall investigate Eq. (1.6) in a number of limiting cases.

2. THE KINETIC INSTABILITY ($T_i \gg T_e$)

Let us consider a plasma with hot ions and cold electrons ($T_i \gg T_e$). If the velocity u is sufficiently small (see below), then the generated parametric instability is, as in the adiabatic case ($\omega_0 \ll \gamma$), a kinetic instability, and the oscillations are built up by resonant ions.

Since $T_i \gg T_e$, the terms $\sim a_n(\omega) \sim \delta\epsilon_i$ in Eq. (1.3) are small, and we can retain in the sum over n in (1.3) only the term with $n=0$. Then we obtain the following dispersion equation:

$$1+\delta\varepsilon_e(\omega)+a_0(\omega)=0. \quad (2.1)$$

In the zeroth approximation $1+\delta\epsilon_e(\omega)=0$. This equation determines the frequencies of the electron cyclotron oscillations with $\omega=\omega(k)$ lying in the interval $l\omega_{Be} < \omega^{(l)} < (l+1)\omega_{Be}$ (the Bernstein modes). In [16,17], $\omega^{(l)}(k)$ versus $k\rho_e$ plots obtained by means of a numerical solution of the equation $1+\delta\epsilon_l(\omega)=0$ are given for different values of the ratio ω_{pe}/ω_{Be}.

If the frequency $\omega(k)$ is close to $n\omega_{Be}$, then

$$\omega(k) \approx n\omega_{Be}\left(1+\frac{A_n(k^2\rho_e^2)}{1+k^2\lambda_{De}^2}\right), \quad (2.2)$$

where $\lambda_{De}=v_{Te}/\omega_{pe}$. The formula (2.2) is valid for arbitrary values of n if $k\rho_e \ll 1$ or $k\rho_e \gg 1$, and for arbitrary $k\rho_e$ if $n \gg 1$.

Setting $\omega=\omega(k)+\Delta\omega+i\gamma$, where $\Delta\omega \ll kv_{Ti}$ and $\gamma \ll kv_{Ti}$, and taking the small term a_0 in (2.1) into account, we obtain

$$\Delta\omega=-\sum_{m=-\infty}^{\infty}\frac{\omega_{pi}^2}{k^2v_{Ti}^2}J_m^2(a_E)\left(1-2z_me^{-z_m^2}\int_0^{z_m}e^{t^2}dt\right)\left(\frac{d\delta\varepsilon_e}{d\omega}\Big|_{\omega=\omega^{(l)}}\right)^{-1}, \quad (2.3)$$

$$\gamma(k)=-\frac{\omega_{pi}^2}{k^2v_{Ti}^2}\sum_{m=-\infty}^{\infty}J_m^2(a_E)\sqrt{\pi}z_me^{-z_m^2}\left(\frac{d\delta\varepsilon_e}{d\omega}\Big|_{\omega=\omega^{(l)}}\right)^{-1}, \quad (2.4)$$

where $z_m=(\omega(k)+m\omega_0)\sqrt{2}k\,v_{Ti}$ and

$$\frac{d\delta\varepsilon_e}{d\omega}=4\frac{\omega_{pe}^2}{k^2v_{Te}^2}\sum_{s=1}^{\infty}A_s(k^2\rho_e^2)\frac{s^2\omega_{Be}^2\omega}{(\omega^2-s^2\omega_{Be}^2)^2}.$$

Since $kv_{Ti} \ll \omega_0$, ω_{Be}, we need consider in the sum over m in the expression (2.4) only the term with the resonance value of m, when $\omega(k)\approx -m\omega_0$ ($m<0$). The instability develops if $z_m<0$. Then for $|z_m| \lesssim 1$ we find, using the expression (2.2), that

$$\gamma(k)\sim\frac{T_e}{T_i}\frac{A_n(k^2\rho_e^2)J_m^2(a_E)}{(1+k^2\lambda_{De}^2)^2}\omega_{Be}. \quad (2.5)$$

The magnitude of the growth rate (2.5) depends in a complicated manner on the quantities k, m, and n. Let us estimate the maximum value of the expression (2.5) and the value of k at which this maximum value is attained for different values of the numbers m and n. If $m=1$ and $\omega_{pe} \lesssim \omega_{Be}$, then

$$\frac{\gamma}{\omega_{Be}}\sim\frac{T_e}{T_i}\left(\frac{\omega_{pe}}{\omega_{Be}}\right)^4\left(\frac{u}{v_{Te}}\right)^2 \quad (k\rho_e\sim1). \quad (2.6)$$

If $m=1$, $n \gtrsim 1$, and $\omega_{pe} \gtrsim \omega_{Be}$, then

$$\frac{\gamma}{\omega_{Be}}\sim\frac{T_e}{T_i}\frac{\omega_{pe}}{\omega_{Be}}\left(\frac{u}{v_{Te}}\right)^2 \quad \left(k\lambda_{De}\sim1,\ \frac{u}{v_{Te}}\frac{\omega_{pe}}{\omega_{Be}}\leqslant1\right). \quad (2.7)$$

If $m=2$ and $\omega_{Be} \lesssim \omega_{pe}$, then

$$\frac{\gamma}{\omega_{Be}}\sim\frac{T_e}{T_i}\left(\frac{\omega_{pe}}{\omega_{Be}}\right)^3\left(\frac{u}{v_{Te}}\right)^4 \quad \left(k\lambda_{De}\sim1,\ \frac{u}{v_{Te}}\frac{\omega_{pe}}{\omega_{Be}}\leqslant1\right). \quad (2.8)$$

If $m=2$ and $\omega_{Be} \gtrsim \omega_{pe}$, then

$$\frac{\gamma}{\omega_{Be}}\sim\frac{T_e}{T_i}\left(\frac{\omega_{pe}}{\omega_{Be}}\right)^4\left(\frac{u}{v_{Te}}\right)^4 \quad (k\rho_e\sim1). \quad (2.9)$$

If $m \geq 3$, and also if $m=1$, 2 and $n \geq 1$ when $\omega_{pe} \gg \omega_{Be}$ and $u\omega_{pe}/V_{Te}\omega_{Be} \gtrsim 1$, then

$$\frac{\gamma}{\omega_{Be}}\sim\frac{T_e}{T_i}\left(1+\frac{v_{Te}^2}{u^2}\frac{\omega_{Be}^2}{\omega_{pe}^2}\right)^{-2}\frac{u}{v_{Te}} \quad \left(k\rho_e\sim\frac{v_{Te}}{u}>1\right). \quad (2.10)$$

If $m \gg 1$ ($\omega_0 \ll \omega_{Be}$) and $n\sim1$, then $a_E \sim m$

$$\frac{\gamma}{\omega_{Be}}\sim\frac{T_e}{T_i}\frac{u}{v_{Te}}\left(1+\frac{v_{Te}^2}{u^2}\frac{\omega_{Be}^2}{\omega_{pe}^2}\right)^{-2}\left(\frac{\omega_0}{\omega_{Be}}\right)^{2/3} \quad \left(k\rho_e\sim\frac{v_{Te}}{u}\right). \quad (2.11)$$

It should be noted that the numerical coefficients that were dropped in the derivation of these formulas are very sensitive to the choice of the numbers m and n. The quantities (2.6)–(2.11) increase with the velocity u. However, the applicability of these expressions at large values of u is restricted by the condition $\gamma < kv_{Ti}$. This condition leads to the following inequalities, which determine the condition of applicability of the estimates (2.6)–(2.11):

$$u^2 \leqslant u_0^2(\omega_{Be}/\omega_{pe})^4, \quad (2.6a)$$

$$u^2 \leqslant u_0^2, \quad (2.7a)$$

$$u^4 \leqslant u_0^2v_{Te}^2(\omega_{Be}/\omega_{pe})^2, \quad (2.8a)$$

$$u^4 \leqslant u_0^2v_{Te}^2(\omega_{Be}/\omega_{pe})^4, \quad (2.9a)$$

$$u^2 \leqslant u_0^2(1+(v_{Te}/u)^2(\omega_{Be}/\omega_{pe})^2)^2, \quad (2.10a)$$

$$u^2 \leqslant u_0^2(1+(v_{Te}/u)^2(\omega_{Be}/\omega_{pe})^2)^2(\omega_{Be}/\omega_0)^{2/3}, \quad (2.11a)$$

where $u_0^2=(T_i/T_e)v_{Ti}v_{Te}$.

The expressions obtained for $\gamma(k)$ decrease with the velocity, so that the quantity $\gamma(k)$ quickly becomes smaller than ω_{Bi}. The quantity u is bounded from below by the condition $\gamma > \omega_{Bi}$ and, furthermore, by the condition $\omega_0 > kv_{Ti}$, which allows us to separate out from the sum (2.4) the resonance term. The latter condition for $m\sim1$ is easily fulfilled, and can prove to be important only when $\omega_0 \ll \omega_{Be}$ ($m \gg 1$).

If the frequency of the binary Coulomb collisions is sufficiently high, then the influence of the collisions on the development of the instability being studied will have to be taken into account before the violation of the condition $\gamma > \omega_{Bi}$.

Let us find the value $u=u_{cr}$ that determines the threshold for the appearance of the instability being studied. To take the Coulomb collisions into account, we must, when $\omega \approx n\omega_{Be}$, add to the expression (1.1) a term equal to (see [13])

$$\delta\varepsilon'=i\eta\frac{\omega_{pe}^2}{k^2v_{Te}^2}\frac{\omega k\rho_e\nu_{ei}}{(\omega-n\omega_{Be})^2} \quad (k\rho_e\gg1), \quad (2.12)$$

$$\delta\varepsilon'\sim i\frac{\omega_{pe}^2}{k^2v_{Te}^2}\frac{\omega\nu_{ei}}{k\rho_e(\omega-n\omega_{Be})^2} \quad (k\rho_e\leqslant1),$$

where

$$\eta=\frac{3}{8\sqrt{2\pi}}\left(1+\frac{7}{2}\pi\right),\quad \nu_{ei}=\frac{4\sqrt{2\pi}e^4n_0L}{3\sqrt{m_e}T_e^{3/2}},$$

Allowance for the collisions leads to a growth rate equal to $\gamma=\gamma(k)-\gamma_{st}$, where $\gamma(k)$ is given by the formula (2.4) and

$$\gamma_{st}=\nu_{ei}(k\rho_e)^2\sqrt{2\pi}\cdot\eta \quad (k\rho_e\gg1), \qquad (2.13)$$
$$\gamma_{st}\sim\nu_{ei} \quad (k\rho_e\leqslant1).$$

The condition $\gamma(k)=\gamma_{st}$ determines the magnitude of the critical velocity above which the oscillations with the given values of k, m, and n are excited. Determining from this condition the maximum value of the quantity

$$\frac{\nu_{ei}}{\omega_{Be}}\sim\frac{T_e}{T_i}\frac{A_n(k^2\rho_e^2)J_m^2(a_E)}{(k\rho_e)^2(1+k^2\lambda_{De}^2)^2}, \qquad (2.14)$$

we find the threshold value of u.

Fixing the values of m, n, and ω_{pe}/ω_{Be}, we find from this that the maximum value of (2.14) and the values of k at which this maximum is attained are given by the following formulas.

If $m=1$, then

$$\frac{\nu_{ei}}{\omega_{Be}}\sim\frac{T_e}{T_i}\frac{1}{(1+\omega_{Be}^2/\omega_{pe}^2)^2}\left(\frac{u}{v_{Te}}\right)^2 \quad (k\rho_e\sim1). \qquad (2.15)$$

If $m=2, 3$ and $\omega_{Be}>\omega_{pe}$, then

$$\frac{\nu_{ei}}{\omega_{Be}}\sim\frac{T_e}{T_i}\left(\frac{\omega_{pe}}{\omega_{Be}}\right)^4\left(\frac{u}{v_{Te}}\right)^{2m-3} \quad (k\rho_e\sim1). \qquad (2.16)$$

If $m=2, 3$ and $\omega_{pe}>\omega_{Be}$, then for $u\omega_{pe}/v_{Te}\omega_{Be}<1$

$$\frac{\nu_{ei}}{\omega_{Be}}\sim\frac{T_e}{T_i}\left(\frac{\omega_{pe}}{\omega_{Be}}\right)^{2m-3}\left(\frac{u}{v_{Te}}\right)^{2m} \quad (k\lambda_{De}\sim1). \qquad (2.17)$$

If $m\geq4$, then

$$\frac{\nu_{ei}}{\omega_{Be}}\sim\frac{T_e}{T_i}\left(\frac{u}{v_{Te}}\right)^3\left(1+\frac{v_{Te}^2}{u^2}\frac{\omega_{Be}^2}{\omega_{pe}^2}\right)^{-2}\left(\frac{\omega_0}{\omega_{Be}}\right)^{4/3} \quad \left(k\rho_e\sim\frac{v_{Te}}{u}\right). \qquad (2.18)$$

The formula (2.18) is also applicable when $m=2, 3$, $\omega_{pe}\gg\omega_{Be}$, and $u\omega_{pe}>v_{Te}\omega_{Be}$.

The development of parametric instability leads to the appearance of intense pulsations in the intensity of the electric field of the electron cyclotron oscillations. We shall assume that the saturation of the growth of these oscillations is due to their nonlinear interaction as a result of the appearance of a strong nonlinearity in the kinetic equation for the electrons. The scattering of the electrons by the turbulent pulsations leads to the appearance of an effective scattering frequency ν_{eff}. Assuming that this scattering leads to the deflection of the electrons through small angles, we can describe it with the aid of a Fokker-Planck type of collision integral. Therefore, there will arise in the nonlinear phase of the development of the oscillations an effective turbulent collision rate ν_{eff}, whose presence will lead to the decrease of the growth rate. The growth rate will be determined by the expression $\gamma_{NL}=\gamma(k)-\gamma_{st}$, where $\gamma(k)$ is determined by the expression (2.4) and γ_{st} by the formula (2.13) in which ν_{ei} should be replaced by ν_{eff}.

As the amplitude of the oscillations grows, the quantity ν_{eff} will grow. The condition $\gamma(k)=\nu_{eff}k^2\rho_e^2$ determines the value of ν_{eff} for which the oscillations with a growth rate less than $\gamma(k)$ will attain saturation. The maximum of the expression $\nu_{eff}=\gamma(k)/k^2\rho_e^2$ determines the value of ν_{eff} for which all the oscillations reach the stationary turbulent level. Using for $\gamma(k)$ the expression (2.5), we find that the quantity ν_{eff} is determined for a given value of the magnitude of the velocity u by the formulas (2.14)–(2.18), in which ν_{ei} should be replaced by ν_{eff}.

These expressions allow us to estimate the rate of damping of the pump wave and the rate of turbulent heating of the plasma.

In conclusion, let us note that the estimates obtained in this section are also valid for $T_i\sim T_e$. However, to obtain exact expressions for the growth rate in this case, the dispersion equation (1.6) must be solved numerically.

3. HYDRODYNAMIC INSTABILITY ($u>u_1$)

In the region of high velocities, when the inequalities (2.6a)–(2.11a) are replaced by the inverse inequalities, the instability under consideration becomes hydrodynamic. Assuming in this case that $z_i\gg1$, and taking into account the fact that $\omega\ll\omega_0, \omega_{Be}$, we obtain the following dispersion equation[6]:

$$1-\frac{\omega_{pi}^2}{\omega^2}\sum_{m'=-\infty}^{\infty}\frac{J_m^2(a_E)}{1+\delta\varepsilon_e(\omega+m'\omega_0)}=0. \qquad (3.1)$$

We shall assume that the frequency $m\omega_0$ is close to the natural frequency $\omega(k)$. Then, considering in (3.1) only the terms with $m'=m$ and $m'=-m$ [1)], we obtain the following expression for ω^2:

$$\omega^2={}^1\!/_2\delta^2\pm{}^1\!/_2\sqrt{\delta^4-4\delta\gamma_0^3}, \qquad (3.2)$$

where $\delta=m\omega_0-\omega(k)$ and

$$\gamma_0^3=2\omega_{pi}^2J_m^2(a_E)\,(d\delta\varepsilon_e/d\omega)^{-1}_{\omega=\omega(k)}.$$

For $\omega(k)\approx n\omega_{Be}$, we have

$$\gamma_0^3=2\frac{m_e}{m_i}\frac{A_n(k^2\rho_e^2)J_m^2(a_E)}{(1+k^2\lambda_{De}^2)^2}n\omega_{Be}^3. \qquad (3.3)$$

For $\delta<0$, the generated instability is aperiodic. For $\delta>0$, the instability develops if $\delta^3>4\gamma_0^3$. A characteristic value of the growth rate is equal to

$$\gamma\sim\gamma_0 \quad (|\delta|\sim\gamma_0). \qquad (3.4)$$

The maximum value of $\gamma_0(k)$ and the value of k for which this maximum is attained are determined by the following estimates.

If $m=1$ and $\omega_{Be}\gtrsim\omega_{pe}$, then

$$\frac{\gamma}{\omega_{Be}}\sim\left(\frac{m_e}{m_i}\right)^{1/3}\left(\frac{\omega_{pe}}{\omega_{Be}}\right)^{4/3}\left(\frac{u}{v_{Te}}\right)^{2/3} \quad (k\rho_e\sim1). \qquad (3.5)$$

If $m=1$ and $\omega_{Be}\lesssim\omega_{pe}$, then for $u\omega_{pe}/v_{Te}\omega_{Be}<1$

$$\frac{\gamma}{\omega_{Be}}\sim\left(\frac{m_e}{m_i}\frac{v_{Te}}{u}\right)^{1/3}\frac{\omega_{pe}}{\omega_{Be}}\frac{u}{v_{Te}} \quad (k\lambda_{De}\sim1). \qquad (3.6)$$

If $m\geq2$, then

$$\frac{\gamma}{\omega_{Be}}\sim\left(\frac{m_e}{m_i}\frac{v_{Te}}{u}\right)^{1/3}\left(1+\frac{v_{Te}^2}{u^2}\frac{\omega_{Be}^2}{\omega_{pe}^2}\right)^{-2/3}\quad\left(k\rho_e\sim\frac{v_{Te}}{u}\right). \qquad (3.7)$$

The estimate (3.7) is also valid for $m=1$ and $\omega_{Be}\ll\omega_{pe}$ when $u\omega_{pe}/v_{Te}\omega_{Be}>1$.

If $m\gg1$ ($\omega_0\ll\omega_{Be}$), then

$$\frac{\gamma}{\omega_{Be}}\sim\left(\frac{m_e}{m_i}\frac{v_{Te}}{u}\right)^{1/3}\left(1+\frac{v_{Te}^2}{u^2}\frac{\omega_{Be}^2}{\omega_{pe}^2}\right)^{-2/3}\left(\frac{\omega_0}{\omega_{Be}}\right)^{4/9}\quad\left(k\rho_e\sim\frac{v_{Te}}{u}\right). \qquad (3.8)$$

The growth rates (3.5)–(3.8) increase monotonically with increasing u; it is only in the region $u\omega_{pe}/v_{Te}\omega_{Be}>1$ that they begin to decrease slowly as $u^{-1/3}$.

The obtained expressions are applicable if $\gamma\gg kv_{Ti}$. This condition limits the applicability of the estimates

(3.5)–(3.8) with sufficiently large values of $u > u_1$:

$$u_1^2 \sim u_0^2 (\omega_{Be}/\omega_{pe})^4, \qquad (3.5a)$$

$$u_1^2 \sim u_0^2, \qquad (3.6a)$$

$$u_1^2 \sim u_0^2 \left(1 + \frac{v_{Te}^2}{u^2} \frac{\omega_{Be}^2}{\omega_{pe}^2}\right)^2, \qquad (3.7a)$$

$$u_1^2 \sim u_0^2 \left(1 + \frac{v_{Te}^2}{u^2} \frac{\omega_{Be}^2}{\omega_{pe}^2}\right)^2 \left(\frac{\omega_{Be}}{\omega_0}\right)^{2/3}. \qquad (3.8a)$$

Furthermore, for the formulas (3.2)–(3.8) to be applicable, it is necessary that the inequality

$$|\omega(k) - n\omega_{Be}| \sim \frac{\omega_{Be} A_n(k^2\rho_e^2)}{1 + k^2\lambda_{De}^2} \gg \gamma, \qquad (3.9)$$

which also imposes limitations on the value of u, be fulfilled. Substituting in place of γ in (3.9) the estimates (3.5)–(3.8), we obtain

$$u^2 < \frac{m_i}{m_e} \left(\frac{\omega_{pe}}{\omega_{Be}}\right)^2 v_{Te}^2, \qquad (3.5b)$$

$$u^2 < \frac{m_i}{m_e} \left(\frac{\omega_{Be}}{\omega_{pe}}\right)^6 v_{Te}^2, \qquad (3.6b)$$

$$u > \frac{m_e}{m_i} v_{Te} \left(\frac{\omega_{pe}}{\omega_{Be}}\right)^3 \left(1 + \frac{v_{Te}^2}{u^2} \frac{\omega_{Be}^2}{\omega_{pe}^2}\right), \qquad (3.7b)$$

$$u > \frac{m_e}{m_i} v_{Te} \left(\frac{\omega_{pe}}{\omega_{Be}}\right)^3 \left(1 + \frac{v_{Te}^2}{u^2} \frac{\omega_{Be}^2}{\omega_{pe}^2}\right) \left(\frac{\omega_0}{\omega_{Be}}\right)^{1/3}. \qquad (3.8b)$$

In the nonlinear phase of the development of the hydrodynamic instability there appears effective electron scattering by the turbulent pulsations in the growing oscillations. We take this effect into account by introducing into the resonance denominators in $\delta\epsilon_e$ an effective collision rate ν_{eff}:

$$\omega - n\omega_{Be} \to \omega - n\omega_{Be} + ik^2\rho_e^2\nu_{eff}. \qquad (3.10)$$

When the quantity $k^2\rho_e^2\nu_{eff}$ exceeds the growth rate (3.3), the hydrodynamic instability does not become saturated, and the growth of the oscillations in this (nonlinear) phase continues, although with a significantly smaller growth rate. Taking the substitution (3.10) into account, we obtain for the nonlinear growth rate in this case the estimate

$$\gamma_{NL} \sim \frac{\gamma_0^{3/2}}{(k^2\rho_e^2\nu_{eff})^{1/2}}, \qquad (3.11)$$

where γ_0 is given by the formula (3.3). In this phase, the parametric instability is dissipative. Saturation of the nonlinear dissipative instability sets in when the growth rate (3.11) decreases so much so that it becomes of the order of kv_{Ti}. In this case there develops strong ion-induced damping of the oscillations, which leads to the saturation of the oscillations. The value of the effective scattering frequency, which is determined from the condition $\gamma_{NL} \sim kv_{Ti}$, coincides literally with the quantity ν_{eff}, (2.14), determined for the case of kinetic instability in the preceding section.

If there are sufficiently frequent Coulomb collisions, so that $k^2\rho_e^2\nu_{ei} > \gamma_0$, then the hydrodynamic instability with the growth rate (3.3) is not excited. In this case the growth rate is determined by the formula (3.11), in which ν_{eff} should be replaced by ν_{ei}.

4. ION-SOUND INSTABILITY ($u < u_1$)

The hydrodynamic instability investigated in the preceding section develops if the oscillation velocity u is sufficiently high ($u > u_1$) both when $T_i > T_e$ and when $T_i < T_e$. If $T_i \gtrsim T_e$, then this instability develops into a kinetic instability in the low-velocity region (see Sec. 2). If, however, $T_i \ll T_e$, then ion-acoustic oscillations are parametrically excited in the region of small u. The excitation of these oscillations is due to the fact that they are resonantly coupled to the electron cyclotron oscillations, and it occurs coherently. Let us represent Eq. (1.3) in the form

$$\varepsilon(\omega)\varphi(\omega) + \sum_{n=-\infty}^{\infty} b_n(\omega)\varphi(\omega - n\omega_0) = Q(\omega), \qquad (4.1)$$

where

$$\varepsilon(\omega) = 1 + \frac{1}{k^2\lambda_{De}^2} - \frac{\omega_{pi}^2}{\omega^2} \quad (\omega \gg kv_{Ti}), \qquad (4.2)$$

$$b_n(\omega) = -\sum_{m=-\infty}^{\infty} J_n(a_E) J_{n+m}(a_E) \frac{\omega_{pe}^2}{k^2 v_{Te}^2} \sum_{s=-\infty}^{\infty} A_s(k^2\rho_e^2) \frac{\omega + m\omega_0}{\omega + m\omega_0 - s\omega_{Be}},$$

We shall assume that $k\rho_e$ is so large that even when $|\omega + m\omega_0| \approx s\omega_{Be}$ the quantity $b_n(\omega)$ in (4.1) can be considered to be small. Then, discarding in (4.1) the terms with $n \neq 0$, we obtain the following dispersion equation:

$$\varepsilon(\omega) + b_0(\omega) = 0. \qquad (4.3)$$

In the zeroth approximation $\epsilon(\omega) = 0$, whence we find the frequency of the oscillations:

$$\omega = \omega_s(k) = \frac{kv_s}{\sqrt{1 + k^2\lambda_{De}^2}}, \quad v_s = \left(\frac{T_e}{m_i}\right)^{1/2}. \qquad (4.4)$$

The term $b_0(\omega)$ in (4.3) proves to be important only at the resonance

$$\omega_s(k) + m\omega_0 = -n\omega_{Be}. \qquad (4.5)$$

Setting in this case $\omega = \omega_s + i\gamma$ ($\gamma \ll \omega_s$), and taking into account in $b_0(\omega)$ only the resonance terms, we obtain

$$\gamma = f(k)\,(m_e/m_i)^{1/4}\omega_{Be}, \qquad (4.6)$$

where

$$f(k) = \left[\frac{nk\rho_e A_n(k^2\rho_e^2) J_m^2(a_E)}{2(1 + k^2\lambda_{De}^2)^{3/2}}\right]^{1/2} \quad (n > 0).$$

The maximum value of the growth rate (4.6) and the values of k at which this maximum occurs are determined by the following expressions.

If $m = 1$ and $\omega_{Be} > \omega_{pe}$, then

$$\gamma \sim \omega_{Be} \left(\frac{m_e}{m_i}\right)^{1/4} \left(\frac{\omega_{pe}}{\omega_{Be}}\right)^{3/2} \frac{u}{v_{Te}} \quad (k\rho_e \sim 1,\ \omega_0 \sim \omega_{Be}). \qquad (4.7)$$

If $m = 1$ and $\omega_{Be} < \omega_{pe}$, then for $u\omega_{pe}/v_{Te}\omega_{Be} < 1$

$$\gamma \sim \omega_{Be} \left(\frac{m_e}{m_i}\right)^{1/4} \frac{\omega_{pe}}{\omega_{Be}} \frac{u}{v_{Te}} \quad (k\lambda_{De} \sim 1,\ \omega_0 \sim \omega_{Be}). \qquad (4.8)$$

If $m \geq 2$ and $n \sim 1$, then for $m\omega_0 \sim m\omega_{Be}$

$$\frac{\gamma}{\omega_{Be}} \sim \left(\frac{m_e}{m_i}\right)^{1/4} \left(1 + \frac{v_{Te}^2}{u^2} \frac{\omega_{Be}^2}{\omega_{pe}^2}\right)^{-3/4} \left(\frac{\omega_0}{\omega_{Be}}\right)^{1/3} \quad (a_E \sim 1). \qquad (4.9)$$

As the velocity u increases, the growth rates (4.7)–(4.9) increase, but these expressions are applicable for u less than some value $u \sim u_2$. The condition of applicability of the approximate dispersion equation (4.3) and of the expression (4.6) for γ has the form

$$\frac{J_m^2(a_E) A_n(k^2\rho_e^2)}{\gamma(1 + k^2\lambda_{De}^2)} n\omega_{Be} \sim \left(\frac{n\omega_{Be} A_n J_m^2}{kv_s\sqrt{1 + k^2\lambda_{De}^2}}\right)^{1/2} \ll 1. \qquad (4.10)$$

The cutoff velocity $u = u_2$ is determined by this condition. In particular, for the expressions (4.7)–(4.9) we find from this inequality that

$$u_2 \sim \left(\frac{m_e}{m_i}\right)^{1/4} v_{Te} \left(\frac{\omega_{Be}}{\omega_{pe}}\right)^{1/2}, \qquad (4.7a)$$

$$u_2 \sim \left(\frac{m_e}{m_i}\right)^{1/4} v_{Te}, \qquad (4.8a)$$

$$u_2 \sim \left(\frac{m_e}{m_i}\right)^{1/4} v_{Te} \left(1+\frac{v_{Te}^2}{u^2}\frac{\omega_{Be}^2}{\omega_{pe}^2}\right)^{1/4}. \qquad (4.9a)$$

In the nonlinear phase of the development of the ion-sound instability under consideration, when electron scattering by the ion-sound vibrations appears, the nonlinear growth rate is determined from Eq. (4.3), in which ω should be replaced by $\omega + ik^2\rho_e^2\nu_{eff}$. Taking the resonance term into account in the expression for $b_0(\omega)$, we obtain [2)]

$$\gamma_{NL} = \frac{\gamma^2(k)}{\gamma_{NL} + \nu_{eff} k^2 \rho_e^2}, \qquad (4.11)$$

where $\gamma(k)$ is determined by the formula (4.6). When the quantity ν_{eff} becomes larger than $\gamma(k)/k^2\rho_e^2$, the growth of the oscillations is slowed down:

$$\gamma_{NL} \approx \gamma^2(k)/\nu_{eff} k^2 \rho_e^2 < \gamma(k). \qquad (4.12)$$

When the quantity γ_{NL} becomes of the order of kv_{Ti}, the oscillations become stabilized. The effective scattering frequency of the stabilized oscillations is equal to

$$\nu_{eff} \sim \max \frac{\gamma^2(k)}{kv_{Ti} k^2 \rho_e^2}. \qquad (4.13)$$

We find from this that if $m = 1$, then

$$\frac{\nu_{eff}}{\omega_{Be}} \sim \left(\frac{u}{v_{Te}}\right)^2 \left(1+\frac{v_{Te}^2}{u^2}\frac{\omega_{Be}^2}{\omega_{pe}^2}\right)^{-1/2} \left(\frac{T_e}{T_i}\right)^{1/2}. \qquad (4.14)$$

If $m = 2, 3$ and $\omega_{Be} > \omega_{pe}$, then

$$\frac{\nu_{eff}}{\omega_{Be}} \sim \left(\frac{\omega_{pe}}{\omega_{Be}}\right)^3 \left(\frac{u}{v_{Te}}\right)^{2m} \left(\frac{T_e}{T_i}\right)^{1/2}. \qquad (4.15)$$

If $m = 2, 3$ and $\omega_{pe} > \omega_{Be}$, then for $u\omega_{pe}/v_{Te}\omega_{Be} < 1$

$$\frac{\nu_{eff}}{\omega_{Be}} \sim \left(\frac{\omega_{pe}}{\omega_{Be}}\right)^{2m-3} \left(\frac{u}{v_{Te}}\right)^{2m} \left(\frac{T_e}{T_i}\right)^{1/2}. \qquad (4.16)$$

If $m \geq 4$ and $m\omega_0 \sim \omega_{Be}$, and also when $m = 2, 3$ $\omega_{pe} \gg \omega_{Be}$, and $u\omega_{pe}/v_{Te}\omega_{Be} > 1$, we have

$$\frac{\nu_{eff}}{\omega_{Be}} \sim \left(\frac{u}{v_{Te}}\right)^3 \left(1+\frac{v_{Te}^2}{u^2}\frac{\omega_{Be}^2}{\omega_{pe}^2}\right)^{-5/2} \left(\frac{\omega_0}{\omega_{Be}}\right)^{-2/3} \left(\frac{T_e}{T_i}\right)^{1/2}. \qquad (4.17)$$

The formulas (4.13)–(4.17) are valid only if the quantity $\nu_{eff}k^2\rho_e^2$ is smaller than ω_{Be}. In the opposite case the development of the oscillations in the nonlinear phase proceeds differently. After the quantity $\nu_{eff}k^2\rho_e^2$ has become greater than $\gamma(k)$, the development of the oscillations proceeds more slowly with the nonlinear growth rate (4.12). When the quantity $\nu_{eff}k^2\rho_e^2$ attains a value of the order of ω_{Be}, the separation of the individual harmonics in the sum over s in the expression for $b_0(\omega)$ becomes impossible, since the individual cyclotron harmonics overlap and the spectrum of the oscillations becomes continuous, and the instability under consideration develops in this phase into a nonlinear ion-sound instability for which the magnetization of the electrons is unimportant (see [13,18]).

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1) In the case when $m\omega_0 \approx n\omega_{Be}$ and the difference $|\omega(k) - n\omega_{Be}| \leq \gamma$, we must take into account in the sums over m' and s entering into (3.1) an infinite number of terms with m' = rm and s = rn (r = ±1, ±2,... ...); the expression for ω in this case was obtained in [6].

2) If the Columb-collision rate is sufficiently high ($\nu_{ei}k^2\rho_e^2 > \gamma(k)$), then in the nonlinear phase the growth rate is determined by the relation (4.11), in which ν_{eff} should be replaced by ν_{ei}.

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