Investigation of photocounting statistics without the use of perturbation theory

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Glauber's model of the polyatomic photodetector, which reduces to a harmonic oscillator interacting with the field, is considered. The photocount generating function for this model is computed exactly by the functional method in the rotating-wave approximation without the use of perturbation theory. In the case of coherent-field detection this distribution is a Poisson distribution whose parameter is expressible in terms of the solution to a system of two linear integro-differential equations. The effect of the thermal fluctuations in the photodetector is considered, and it is shown that the photocount-probability distribution in coherent-field detection by a heated detector deviates from the Poisson distribution, the deviation increasing with the detector temperature.

INTRODUCTION

Photocounting statistics is usually investigated with the aid of perturbation theory and some additional hypotheses of probabilistic nature (see[1-4]). In this case for any steady state of the electro-magnetic field the computation leads to the Poisson distribution. This distribution arises as a result of arguments (quite justified in the framework of perturbation theory) of probabilistic nature used in its derivation. But the polyatomic detector-field interaction Hamiltonian is proportional to the number of atoms, and for a macroscopic detector the perturbation-theory calculation may turn out to be incorrect if its quantum yield is not low. Since photodetectors with high quantum yields have now been developed[5], this question has assumed practical importance.

To carry out in the general case the corresponding computations for the polyatomic photon detector without recourse to perturbation theory is hardly possible. However, we can use the model proposed by Glauber[1], which treats the photodetector as consisting of "two-level" atoms with a level spacing of $\hbar\omega$. To the excitation of n atoms of this system corresponds an excitation energy of $n\hbar\omega$, so that such a system is, to a high degree of accuracy, equivalent to a harmonic oscillator. The equivalent oscillator-field interaction Hamiltonian is proportional to the total number N of detector atoms[1], as a result of which the interaction can be fairly strong.

Although it is not quite clear to what extent this model can account for the properties of a real photodetector, it is of interest to study it, since in such a formulation of the problem it proves to be possible to derive without the use of perturbation theory explicit expressions for the probabilities of transition of the oscillator to the n-th excited state, which, in the present model, is interpreted as the probability of detecting n photons.

The direct solution of the problem is preceded by Sec. 1, in which a convenient representation, which considerably reduces the number of independent variables, is introduced for the electromagnetic-field operators. In the second section, the Schrödinger equation of the problem under consideration is written in the form of an equation with functional derivatives, and its solution is found in the form of a Gaussian functional. For the coefficients of the quadratic form of this functional, a system of integro-differential equations is derived. In the third section the general form of the probability distribution for the post-interaction states of the photodetector is found by integrating over all the final states of the field. The closing fourth section of the paper is devoted to the elucidation of the possible role of the thermal noise of the detector.

1. Λ-REPRESENTATION OF THE ELECTROMAGNETIC FIELD

Let $a_\nu^+(\mathbf{k})$ and $a_\nu(\mathbf{k})$ be the creation and annihilation operators for a photon of wave vector $\mathbf{k}$ and polarization $\mathbf{e}^\nu(\mathbf{k})$, operators which satisfy the commutation relation

$$[a_\nu(\mathbf{k}),\ a_\mu^+(\mathbf{k}')]=\delta_{\nu\mu}\delta(\mathbf{k}-\mathbf{k}'). \tag{1.1}$$

The vectors $\mathbf{e}^\nu(\mathbf{k})$ ($\nu = 1, 2$) satisfy the relations

$$\begin{gathered}\mathbf{e}^\nu(\mathbf{k})\mathbf{e}^\mu(\mathbf{k})=\delta_{\nu\mu},\\ \mathbf{k}\mathbf{e}^\nu(\mathbf{k})=0,\quad e_i^\nu(\mathbf{k})e_j^\nu(\mathbf{k})=\delta_{ij}-k_ik_j/k^2\end{gathered} \tag{1.2}$$

(a repeated Latin index implies summation with respect to that index over the range from 1 to 3, while a repeated Greek polarization index implies summation over the range from 1 to 2).

Let us consider the operator

$$\begin{aligned}\mathbf{I}(\mathbf{r})=\frac{1}{8\pi^2}\iint d\Omega(\mathbf{n})\iint d\Omega(\mathbf{m})\int_0^\infty k^4\,dk\,(\mathbf{n}+\mathbf{m})\,(\mathbf{e}^\nu(\mathbf{n})\mathbf{e}^\mu(\mathbf{m}))\\ \times\exp[-ik(\mathbf{n}-\mathbf{m})\mathbf{r}]a_\nu^+(k\mathbf{n})a_\mu(k\mathbf{m}),\end{aligned} \tag{1.3}$$

where $\mathbf{n}$ and $\mathbf{m}$ are unit vectors and $d\Omega(\mathbf{n})$ is an element of solid angle. The operator $\mathbf{I}$, which was introduced in[6], has the meaning of an operator for the number of photons that cross a unit of area around the point $\mathbf{r}$ during an infinite period of time. The operator $\mathbf{I}$ commutes with the free-field Hamiltonian

$$H_r=\frac{1}{2}\int d^3k\hbar ck[a_\nu^+(\mathbf{k})a_\nu(\mathbf{k})+a_\nu(\mathbf{k})a_\nu^+(\mathbf{k})], \tag{1.4}$$

but different components $I_j(\mathbf{r})$ do not commute with each other. We use the eigenvectors of the operator I_3 for the construction of the basis. For this purpose, let us consider the eigenvalue equation

$$I_i(\mathbf{r})\,|I\rangle=I_i(\mathbf{r})\,|I\rangle,\quad i=3. \tag{1.5}$$

We shall seek the solution to this equation among the one-photon Fock states:

$$|I\rangle=\int f_\lambda(\varkappa)a_\lambda^+(\varkappa)\,|0\rangle d^3\varkappa. \tag{1.6}$$

Substituting (1.6) into (1.5) and multiplying from the left by the state $\langle 0\,|\,a_\nu{}'(\mathbf{k}'\mathbf{n}')$, we obtain an integral equation for the function $F_\nu(kn) = \exp(ik\mathbf{n}\cdot\mathbf{r})f_\nu(kn)$, an equation which has the form

$$e_j^\mu(\mathbf{n})[n_iB_j+B_{ij}]=I_iF_\mu(k\mathbf{n}),\quad i=3, \tag{1.7}$$

where

$$B_j(k)=\frac{k^2}{8\pi^2}\iint d\Omega(\mathbf{m})e_j^\mu(\mathbf{m})F_\mu(k\mathbf{m}),$$
$$B_{ij}(k)=\frac{k^2}{8\pi^2}\iint d\Omega(\mathbf{m})m_ie_j^\mu(\mathbf{m})F_\mu(k\mathbf{m}). \tag{1.8}$$

The dependence of F_μ on $\mathbf{n}$ is determined by the left-hand side of the formula (1.7). Substituting the left-hand side of (1.7) into the relations (1.8) multiplied by I_i, and evaluating the integrals, we obtain a system of equations for B_j and B_{ij}:

$$I_iB_j=\frac{k^2}{3\pi}B_{ij},\quad I_iB_{ij}=\frac{k^2}{30\pi}[4\delta_{il}B_j-\delta_{ij}B_l-\delta_{lj}B_i]. \tag{1.9}$$

Setting $i = 3$ and equating to zero the determinant of the resulting system of 12 equations, we find the eigenvalues:

$$I^{\pm1}(k)=\pm\frac{k^2}{3\pi\sqrt{5}},\quad I^{\pm2}(k)=I^{\pm3}(k)=\pm\frac{k^2\sqrt{2}}{3\pi\sqrt{5}},\quad I^7=\ldots=I^{12}=0. \tag{1.10}$$

In a spherical system of coordinates, where

$$\mathbf{n}=(\cos\varphi\sin\theta,\ \sin\varphi\sin\theta,\ \cos\theta),$$
$$\mathbf{e}^1(n)=(\cos\varphi\cos\theta,\ \sin\varphi\cos\theta,\ -\sin\theta),\ \mathbf{e}^2(n)=(-\sin\varphi,\ \cos\varphi,\ 0),$$

the functions $F_\nu^q(kn)$ (q is the number of the eigenvalue) have the following form:

$$F_\nu^q(k\mathbf{n})=\frac{\delta(k-k_0)}{k_0}Z_\nu^q(\mathbf{n}),$$
$$Z_1^{\pm1}(\theta,\ \varphi)=\sqrt{3/16\pi}\sin\theta(1\pm\sqrt{5}\cos\theta),\quad Z_2^{\pm1}=0,$$
$$Z_1^{\pm2}(\theta,\ \varphi)=\sqrt{3/16\pi}\cos\varphi\cos\theta(1\pm\sqrt{5/2}\cos\theta),$$
$$Z_2^{\pm2}(\theta,\ \varphi)=-\sqrt{3/16\pi}\sin\varphi(1\pm\sqrt{5/2}\cos\theta), \tag{1.11}$$
$$Z_1^{\pm3}(\theta,\ \varphi)=\sqrt{3/16\pi}\sin\varphi\cos\theta(1\pm\sqrt{5/2}\cos\theta),$$
$$Z_2^{\pm3}(\theta,\ \varphi)=\sqrt{3/16\pi}\cos\varphi(1\pm\sqrt{5/2}\cos\theta).$$

The functions $Z_\nu^q(\mathbf{n})$ determined by the formulas (1.11) satisfy the condition

$$\iint Z_\nu^q(\mathbf{n})Z_\nu^{*q'}(\mathbf{n})d\Omega(\mathbf{n})=\delta_{qq'} \tag{1.12}$$

The solutions corresponding to the zero values of I are, as follows from (1.7), determined by the conditions $B_3 = 0$ and $B_{3j} = 0$, or, in accordance with (1.8) (for $q \neq \pm1, \pm2$, or ±3), by the conditions

$$\mathbf{n}_q\equiv\iint d\Omega(\mathbf{m})\mathbf{e}^\mu(\mathbf{m})Z_\mu^q(\mathbf{m})=0,$$
$$\iint d\Omega(\mathbf{m})m_3e_j^\mu(\mathbf{m})Z_\mu^q(\mathbf{m})=0. \tag{1.13}$$

The system (1.13) can be solved easily by expanding Z_μ^q in series in trigonometric functions and the Legendre polynomials in φ and θ. It can also be easily shown that the functions $Z_\mu^q(\theta, \varphi)$ form a complete system of functions on a sphere if q assumes the values corresponding to $I = 0$, as well as the values $-3 \leq q \leq 3$. We shall assume that the functions Z_μ^q are orthonormalized by the condition (1.12) not only for $-3 \leq q \leq 3$, but for all q. Then the completeness condition for this system assumes the form

$$\sum_q Z_\nu^q(\mathbf{n})Z_\mu^{*q}(\mathbf{m})=\delta_{\nu\mu}\delta_2(\mathbf{n}-\mathbf{m}), \tag{1.14}$$

where $\delta_2(\mathbf{n} - \mathbf{m})$ is the two-dimensional delta function on the sphere.

Let us consider the operators

$$\Lambda_q^+(k,\mathbf{r})=k\iint d\Omega(\mathbf{n})\exp(-ik\mathbf{n}\mathbf{r})Z_\nu^q(\mathbf{n})a_\nu^+(k\mathbf{n}),$$
$$\Lambda_q(k,\mathbf{r})=k\iint d\Omega(\mathbf{n})\exp(ik\mathbf{n}\mathbf{r})Z_\nu^{*q}(\mathbf{n})a_\nu(k\mathbf{n}). \tag{1.15}$$

Using the orthogonality condition (1.12), we can easily show that

$$[\Lambda_q(k,\ \mathbf{r}),\ \Lambda_{q'}^+(k',\ \mathbf{r})]=\delta_{qq'}\delta(k-k'). \tag{1.16}$$

Multiplying (1.15) by $Z_\mu^{*q}(\mathbf{m})$, summing over all q, and using (1.14), we easily obtain the inverse (to 1.15) transformations:

$$a_\mu^+(k\mathbf{m})=k^{-1}\exp(ik\mathbf{m}\mathbf{r})\sum_q Z_\mu^{*q}(\mathbf{m})\Lambda_q^+(k,\mathbf{r}),$$
$$a_\mu(k\mathbf{m})=k^{-1}\exp(-ik\mathbf{m}\mathbf{r})\sum_q Z_\mu^q(\mathbf{m})\Lambda_q(k,\mathbf{r}), \tag{1.17}$$

which express the operators a_μ^+ and a_μ in terms of Λ_q^+ and Λ_q.

Substituting (1.17) into the expressions for the various operators in the representation of second quantization, we can express the operators in the Λ-representation. Thus, the Hamiltonian of the field has the form

$$H_r=\frac{1}{2}\sum_q\int_0 dk\hbar ck[\Lambda_q^+(k,\mathbf{r})\Lambda_q(k,\mathbf{r})+\Lambda_q(k,\mathbf{r})\Lambda_q^+(k,\mathbf{r})]. \tag{1.18}$$

Let us now consider the vector-potential operator, which, in the Heisenberg representation with the Coulomb gauge, is of the form

$$\mathbf{A}(r,t)=\frac{\sqrt{\hbar c}}{2\pi}\int\frac{d^3k}{\sqrt{k}}\mathbf{e}^\nu(\mathbf{k})[e^{i\mathbf{kr}-ickt}a_\nu(\mathbf{k})+e^{-i\mathbf{kr}+ickt}a_\nu^+(\mathbf{k})].$$

Substituting (1.17) into this expression, and going over to spherical coordinates $\mathbf{k} = k\mathbf{n}$, $d^3k = k^2dkd\Omega(\mathbf{n})$, we obtain

$$\mathbf{A}(\mathbf{r},t)=\frac{\sqrt{\hbar c}}{2\pi}\sum_{q=-3}^{3}\mathbf{n}_q\int_0^\infty dk\sqrt{k}[e^{-ikct}\Lambda_q(k,\mathbf{r})+e^{ikct}\Lambda_q^+(k,\mathbf{r})]. \tag{1.19}$$

Here $\mathbf{n}_q$ is a vector determined by the left-hand side of (1.13) for $q = \pm1, \pm2, \pm3$. Since, according to (1.13), $\mathbf{n}_q = 0$ for the remaining values of q, the sum in (1.19) is extended only to the values $-3 \leq q \leq 3$. The advantage of the representation under consideration consists precisely in the fact that for the given point $\mathbf{r}$ it separates out in explicit form the states with electric fields that do not vanish at this point. The nonzero values of $\mathbf{n}_q$ computed with the aid of (1.11) have the form

$$\mathbf{n}_{\pm1}=(0,0,-\sqrt{4\pi/3}),\ \mathbf{n}_{\pm2}=(\sqrt{4\pi/3},0,0),\ \mathbf{n}_{\pm3}=(0,\sqrt{4\pi/3},0). \tag{1.20}$$

If we differentiate (1.15) with respect to $\mathbf{r}$ and then replace a_ν^+ by the right-hand side of (1.17), then it is easy to obtain the equation determining the dependence of Λ_q on the coordinates:

$$\frac{\partial\Lambda_q(k,\mathbf{r})}{\partial x_i}=ik\sum_{q'}B_i^{qq'}\Lambda_{q'}(k,\mathbf{r}),\quad B_i^{qq'}=\iint m_iZ_\mu^{*q}(\mathbf{m})Z_\mu^{q'}(\mathbf{m})d\Omega(\mathbf{m}). \tag{1.21}$$

The solution to this equation (in matrix form) has the form

$$\Lambda(k,\ \mathbf{r}+\mathbf{r}')=\exp\{ik\mathbf{B}\mathbf{r}'\}\Lambda(k,\ \mathbf{r}). \tag{1.22}$$

Since $B_i^{qq'} \sim 1$, it follows from (1.22) that for $kr' \ll 1$ the solution $\Lambda(k, \mathbf{r} + \mathbf{r}') \approx \Lambda(k, \mathbf{r})$. Consequently, the representation of $\mathbf{A}(\mathbf{r}, t)$ in the form (1.19) can also be used in the vicinity of the point $\mathbf{r}$ in a region of linear dimension of the order of a wavelength.

2. SOLUTION OF THE SCHRÖDINGER EQUATION FOR AN OSCILLATOR INTERACTING WITH A FIELD

Let us denote by b^+ and b the creation and annihilation operators for the oscillator. In the Heisenberg representation for the free oscillator, they have the form $b^+(t) = e^{i\omega t}b^+$, $b(t) = e^{-i\omega t}b$, where

$$[b, b^+]=1. \tag{2.1}$$

Let us choose the equivalent-oscillator-field interaction Hamiltonian in the form[1]

$$H_{int}=-\frac{eN}{mc}pA_x, \quad p=i\left(\frac{\hbar\omega m}{2}\right)^{1/2}(b^+(t)-b(t)). \tag{2.2}$$

In (2.2) we have introduced the simplifying assumption that the detector responds only to the x component of the field. We shall assume that the dimensions of the detector do not exceed the wavelength of the radiation. Then the representation (1.19) can be used in the entire region where the detector is located. According to (1.19) and (1.20), the x component is different from zero only for $q = \pm 2$. Furthermore, since $n_2 = n_{-2}$,

$$A_x(r,t)=\frac{\sqrt{\hbar c}}{2\pi}\sqrt{\frac{4\pi}{3}}\int_0^\infty dk\sqrt{k}\{e^{-ikct}[\Lambda_2(k,\mathbf{r})+\Lambda_{-2}(k,\mathbf{r})] + e^{ikct}[\Lambda_2^+(k,\mathbf{r})+\Lambda_{-2}^+(k,\mathbf{r})]\}. \tag{2.3}$$

Let us introduce the new operators:

$$\alpha(k)=\frac{1}{\sqrt{2}}[\Lambda_2(k,\mathbf{r})+\Lambda_{-2}(k,\mathbf{r})], \quad \alpha^+(k)=\frac{1}{\sqrt{2}}[\Lambda_2^+(k,\mathbf{r})+\Lambda_{-2}^+(k,\mathbf{r})]. \tag{2.4}$$

Then, on account of (1.16),

$$[\alpha(k), \alpha^+(k')]=\delta(k-k'). \tag{2.5}$$

Substituting the expressions (2.3) and (2.4) into (2.2), we obtain the interaction Hamiltonian in the interaction representation:

$$H_{int}(t)=ieN\hbar\sqrt{\omega/3\pi mc}[e^{-i\omega t}b-e^{i\omega t}b^+]\int_0^\infty dk\sqrt{k}[e^{-ikct}\alpha(k)+e^{ikct}\alpha^+(k)]. \tag{2.6}$$

The Schrödinger equation in the interaction representation

$$i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t}=H_{int}(t)|\Psi(t)\rangle \tag{2.7}$$

after the introduction of the dimensionless variables $\tau = \omega t$, $\varkappa = ck/\omega$ and the operators $\beta^+(\varkappa)$, $\beta(\varkappa)$ defined by the relations

$$\alpha(k)=\sqrt{c/\omega}\beta(\varkappa), \ \alpha^+(k)=\sqrt{c/\omega}\beta^+(\varkappa), \ [\beta(\varkappa), \beta^+(\varkappa')]=\delta(\varkappa-\varkappa'), \tag{2.8}$$

assumes the form

$$\frac{\partial|\Psi(\tau)\rangle}{\partial\tau}=g[e^{-i\tau}b-e^{i\tau}b^+]\int_0^\infty d\varkappa\sqrt{\varkappa}[e^{-i\varkappa\tau}\beta(\varkappa)+e^{i\varkappa\tau}\beta^+(\varkappa)]|\Psi(\tau)\rangle, \tag{2.9}$$

where $g = Ne\sqrt{\omega/3\pi mc^3}$ is the dimensionless interaction constant.

The interaction Hamiltonian in (2.9) contains the term $[e^{-i(\varkappa+1)\tau}b\beta(\varkappa) - e^{i(\varkappa+1)\tau}b^+\beta^+(\varkappa)]$. The presence of this term in the Hamiltonian leads to a situation in which the ground energy level of the interacting (with each other) oscillator-field system turns out to be lower than the sum of the energies of the isolated components of the system (this can be verified on the example of two oscillators interacting through a Hamiltonian of this type). This leads to certain difficulties in the solution of the problem. For example, if at the initial moment of time the system is in the state $|0, 0\rangle$ (the product of the vacuum states of the oscillator and the field), then the excited states of the oscillator will eventually appear. On the other hand, the term under consideration contains rapidly oscillating factors and, therefore, its contribution is negligible. Below we shall employ the rotating-wave approximation, which eliminates from the complete interaction Hamiltonian terms of the type under consideration. In this case the state $|0, 0\rangle$ is also the ground state of the interacting system, so that the above-noted difficulty no longer arises. In the rotating-wave approximation the Schrödinger equation (2.9) assumes the form

$$\frac{\partial|\Psi(\tau)\rangle}{\partial\tau}=g\int_0^\infty d\varkappa\sqrt{\varkappa}\{e^{i(\varkappa-1)\tau}b\beta^+(\varkappa)-e^{-i(\varkappa-1)\tau}b^+\beta(\varkappa)\}|\Psi(\tau)\rangle. \tag{2.10}$$

We realize the operators b and b^+ with the aid of differential operators, the operators $\beta(\varkappa)$ and $\beta^+(\varkappa)$ with the aid of the corresponding functional-differentiation operators[7]:

$$b=\frac{i}{2}\frac{\partial}{\partial x}+ix, \quad b^+=\frac{i}{2}\frac{\partial}{\partial x}-ix,$$
$$\beta(\varkappa)=\frac{i}{2}\frac{\delta}{\delta u(\varkappa)}+iu(\varkappa), \quad \beta^+(\varkappa)=\frac{i}{2}\frac{\delta}{\delta u(\varkappa)}-iu(\varkappa). \tag{2.11}$$

As can easily be verified, the thus introduced operators satisfy the requisite commutation relations. The state vector $|\Psi(\tau)\rangle$ is then realizable in the form of a function of x and τ and a functional of $u(\cdot)$: $|\Psi(\tau)\rangle = \Psi[x, u(\cdot); \tau]$.

We define a scalar product by the formula

$$\langle\Phi|\Psi\rangle=\sqrt{2/\pi}\int_{-\infty}^\infty dx\int Du\Phi^*[x,u(\cdot)]\Psi[x,u(\cdot)], \tag{2.12}$$

where

$$Du=\prod_{\varkappa=0}^\infty\left(\frac{2d\varkappa}{\pi}\right)^{1/2}du(\varkappa). \tag{2.13}$$

The scalar product (2.12) has been chosen such that the normalized (to unity) vacuum state of the system has the form

$$|0\rangle=\exp\{-x^2-\int_0^\infty u^2(\varkappa)d\varkappa\}, \quad \langle 0|0\rangle=1,$$

since, on account of (2.13), the relation

$$\int Du\exp\{-2\int_0^\infty u^2(\varkappa)d\varkappa\}=1. \tag{2.14}$$

is fulfilled. We shall subsequently need the explicit form of the coherent states of the oscillator and the field. They are given by the formulas

$$\zeta_{coh}(x)=\exp\{-[x^2+2i\zeta x+(\mathrm{Im}\,\zeta)^2]\},$$
$$b\zeta_{coh}(x)=\zeta\zeta_{coh}(x), \quad \sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty\zeta^*_{coh}(x)\zeta_{coh}(x)dx=1, \tag{2.15}$$

$$\Phi_{coh}[u]=\exp\{-\int_0^\infty[u^2(\varkappa)+2iz(\varkappa)u(\varkappa)+(\mathrm{Im}\,z(\varkappa))^2]d\varkappa\},$$
$$\beta(\varkappa)\Phi_{coh}[u]=z(\varkappa)\Phi_{coh}[u], \quad \int Du\Phi^*_{coh}[u]\Phi_{coh}[u]=1. \tag{2.16}$$

Equation (2.10), after the substitution of the operators (2.11), assumes the form

$$\frac{1}{g}\frac{\partial\Psi[x,u;\tau]}{\partial\tau}=\int_0^\infty d\varkappa\sqrt{\varkappa}\left\{\cos[(\varkappa-1)\tau]\left[u(\varkappa)\frac{\partial}{\partial x}-x\frac{\delta}{\delta u(\varkappa)}\right] +2i\sin[(\varkappa-1)\tau]\left[xu(\varkappa)-\frac{1}{4}\frac{\partial}{\partial x}\frac{\delta}{\delta u(\varkappa)}\right]\right\}\Psi. \tag{2.17}$$

This equation can be satisfied if we seek $\Psi[x, u; \tau]$ in the form of a Gaussian functional of the following form:

$$\Psi[x,u;\tau]=\exp\left\{-\int_0^\infty u^2(\varkappa)d\varkappa-x^2-2i\eta(\tau)x-2i\int_0^\infty\xi(\varkappa,\tau)u(\varkappa)d\varkappa-v(\tau)\right\}. \tag{2.18}$$

Substituting (2.18) into (2.17), we easily obtain the following equations for the functions η, ξ, and v:

$$\frac{1}{g}\frac{d\eta(\tau)}{d\tau}=-\int_0^\infty d\varkappa\,\sqrt{\varkappa}\exp[-i(\varkappa-1)\tau]\xi(\varkappa,\tau), \tag{2.19}$$

$$\frac{1}{g}\frac{\partial\xi(\varkappa,\tau)}{\partial\tau}=\sqrt{\varkappa}\exp[i(\varkappa-1)\tau]\eta(\tau), \tag{2.20}$$

$$\frac{1}{g}\frac{dv(\tau)}{d\tau}=-2i\eta(\tau)\int_0 d\varkappa\,\sqrt{\varkappa}\sin[(\varkappa-1)\tau]\xi(\varkappa,\tau). \tag{2.21}$$

If the introduced functions satisfy Eqs. (2.19)–(2.21), then the functional (2.18) is a solution to the Schrödinger equation (2.17). Notice that Eqs. (2.19) and (2.20) are linear, and do not depend on Eq. (2.21), which guarantees the preservation of the normalization.

In order to formulate the initial conditions for the obtained equations, we must consider initial states of the same form as (2.18). As the initial states, we can consider the coherent states of the oscillator and the field. Since, as noted in[8], a coherent state can be treated as the generating function for n-particle states, we can express the solution to the problem in the general case in terms of the solution to the problem with such initial conditions (in the fourth section of the paper we consider, using such a procedure, the influence of the thermal fluctuations).

We shall first consider the case when the initial field is in a coherent state of the form (2.16) and the oscillator is in the vacuum state:

$$\Psi[x,u;0]=\exp\left\{-x^2-\int_0^\infty u^2(\varkappa)d\varkappa-2i\int_0^\infty z(\varkappa)u(\varkappa)d\varkappa-\int_0^\infty[\operatorname{Im}z(\varkappa)]^2d\varkappa\right\}. \tag{2.22}$$

Then the initial conditions for the Eqs. (2.19)–(2.21) assume the form

$$\eta(0)=0,\quad \xi(\varkappa,0)=z(\varkappa),\quad v(0)=\int_0^\infty[\operatorname{Im}z(\varkappa)]^2d\varkappa. \tag{2.23}$$

3. THE PROBABILITY DISTRIBUTION OF THE OSCILLATOR STATES

Let us suppose that the functions η and ξ have been found. Let $|\Phi_m\rangle$ be a complete set of basis field states. The amplitude of the probability of transition of the system to the state $|f_{coh},\Phi_m\rangle$, where $f_{coh}(x) = \exp(-x^2-2ifx-(\operatorname{Im} f)^2)$ is the final coherent state of the oscillator and $f=f_0e^{i\tau}$, is equal to

$$\langle\Psi|f_{coh},\Phi_m\rangle=\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty dx\int Du\,\Psi^*[x,u;\tau]f_{coh}(x)\Phi_m[u]. \tag{3.1}$$

Then the probability of transition to this state has the form

$$|\langle\Psi|f_{coh},\Phi_m\rangle|^2=\frac{2}{\pi}\int_{-\infty}^\infty dx\int_{-\infty}^\infty dx'\int Du\int Du'\,\Psi^*[x,u;\tau]\Psi[x',u';\tau]\times f_{coh}(x)f^*_{coh}(x')\Phi_m^*[u']\Phi_m[u]. \tag{3.2}$$

If we sum this expression over all the possible final states of the field, then, on account of the completeness of the set of functionals $\Phi_m[u]$, we obtain the relation

$$\sum_m|\Phi_m\rangle\langle\Phi_m|=1\ \text{ or }\ \sum_m\Phi_m^*[u']\Phi_m[u]=\delta_\infty[u(\cdot)-u'(\cdot)],$$

where δ_∞ is a delta functional. Then we obtain from (3.2) for the probability of transition to the final state $|f_{coh}\rangle$ of the oscillator the expression

$$P_\tau(f,f^*)=\frac{2}{\pi}\int_{-\infty}^\infty dx\int_{-\infty}^\infty dx'\,f_{coh}(x)f^*_{coh}(x')\int Du\,\Psi^*[x,u;\tau]\Psi[x',u;\tau]. \tag{3.3}$$

Let us consider the function

$$\Omega(x,x',\tau)=\int Du\,\Psi^*[x,u;\tau]\Psi[x',u;\tau].$$

Substituting (2.18) into this expression, we obtain

$$\Omega(x,x',\tau)=\exp\{-x^2-x'^2+2i\eta^*(\tau)x-2i\eta(\tau)x'-2\operatorname{Re}v(\tau)\}\Omega_0(\tau), \tag{3.4}$$

where

$$\Omega_0=\int Du\exp\left\{-2\int_0^\infty[u^2(\varkappa)-2\operatorname{Im}\xi(\varkappa,\tau)u(\varkappa)]d\varkappa\right\}$$
$$=\exp\left\{2\int_0^\infty[\operatorname{Im}\xi(\varkappa,\tau)]^2d\varkappa\right\}\int Du\exp\left\{-2\int_0^\infty[u(\varkappa)-\operatorname{Im}\xi(\varkappa,\tau)]^2d\varkappa\right\}. \tag{3.5}$$

The continuous integral entering here is reduced by the change of variables $u-\operatorname{Im}\xi=u'$ to the normalization integral (2.14)[9], and is therefore equal to unity. Thus,

$$\Omega_0=\exp\left\{2\int_0^\infty[\operatorname{Im}\xi(\varkappa,\tau)]^2d\varkappa\right\}. \tag{3.6}$$

Now substituting (3.4) and (3.6) into (3.3), and performing the integration over x and x′, we obtain

$$P_\tau(f,f^*)=A(\tau)\exp\{-ff^*+\eta^*(\tau)f+\eta(\tau)f^*\}. \tag{3.7}$$

The probabilities $P_n(\tau)$ of excitation of the n-th level of the oscillator are expressible in terms of the found function $P_\tau(f,f^*)$. Indeed, if the state of the oscillator is described by the vector $|\psi\rangle$, then $P_\tau(f,f^*) = |\langle\psi|f_{coh}\rangle|^2$, where

$$|f_{coh}\rangle=\exp(-|f^2|/2)\sum_{n=0}^\infty\frac{f^n}{\sqrt{n!}}|n\rangle.$$

Substituting this expression into $P_\tau(f,f^*)$, we obtain

$$P_\tau(f,f^*)=\exp(-|f^2|)\sum_{n,m=0}^\infty\frac{f^nf^{*m}}{\sqrt{n!\,m!}}\langle\psi|n\rangle\langle\psi|m\rangle^*. \tag{3.8}$$

If we write the expansion of the function $e^{ff^*}P_\tau$ in a Taylor series in f and f^*:

$$e^{ff^*}P_\tau(f,f^*)=\sum_{n,m=0}^\infty\frac{f^nf^{*m}}{n!\,m!}\left[\frac{\partial^{n+m}e^{ff^*}P_\tau(f,f^*)}{\partial f^n\,\partial f^{*m}}\right]_{f=f^*=0},$$

then comparing this expansion with (3.8), we obtain

$$\langle\psi|n\rangle\langle\psi|m\rangle^*=\frac{1}{\sqrt{n!m!}}\left[\frac{\partial^{n+m}e^{ff^*}P_\tau(f,f^*)}{\partial f^n\,\partial f^{*m}}\right]_{f=f^*=0}.$$

Setting m = n here, we obtain for the probability of transition of the oscillator from the initial vacuum state to the final state $|n\rangle$ the expression

$$P_n(\tau)=|\langle\psi|n\rangle|^2=\frac{1}{n!}\left[\left(\frac{\partial^2}{\partial f\,\partial f^*}\right)^n e^{ff^*}P_\tau(f,f^*)\right]_{f=f^*=0}. \tag{3.9}$$

Let us now find the generating function of the factorial moments[3]:

$$Q_\tau(\gamma)=\sum_{n=0}^\infty(1-\gamma)^nP_n(\tau). \tag{3.10}$$

As is well known, we can obtain from $Q_\tau(\gamma)$ the factorial moments

$$L_m\equiv\overline{n(n-1)\dots(n-m+1)}=(-1)^m\frac{d^mQ_\tau(\gamma)}{d\gamma^m}\bigg|_{\gamma=0}, \tag{3.11}$$

as well as the probabilities

$$P_m(\tau)=\frac{(-1)^m}{m!}\frac{d^mQ_\tau(\gamma)}{d\gamma^m}\bigg|_{\gamma=1} \tag{3.12}$$

Substituting (3.9) into (3.10), we obtain

$$Q_\tau(\gamma)=\sum_{n=0}^{\infty}\frac{(1-\gamma)^n}{n!}\left(\frac{\partial^2}{\partial f\,\partial f^*}\right)^n e^{ff^*}P_\tau(f,f^*)\,|_{f=f^*=0}$$
$$=\exp\left\{(1-\gamma)\frac{\partial^2}{\partial f\,\partial f^*}\right\}e^{ff^*}P_\tau(f,f^*)\,|_{f=f^*=0}.$$

Let us set $f = v + iw$, $f^* = v - iw$. Then

$$\frac{\partial^2}{\partial f\,\partial f^*}=\frac{1}{4}\left(\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial w^2}\right)$$

and

$$Q_\tau(\gamma)=\exp\left\{\frac{1-\gamma}{4}\frac{\partial^2}{\partial v^2}\right\}\exp\left\{\frac{1-\gamma}{4}\frac{\partial^2}{\partial w^2}\right\}\times\exp(v^2+w^2)P_\tau(v+iw,v-iw)\,|_{v=w=0}. \tag{3.13}$$

Using the formula

$$\exp\left(t\frac{\partial^2}{\partial x^2}\right)f(x)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x-x')^2}{4t}\right]f(x')\,dx',$$

which can be easily proved by reducing it to the heat equation, we find the general relation connecting $Q_T(\gamma)$ and $P_T(f, f^*)$:

$$Q_\tau(\gamma)=\frac{1}{\pi(1-\gamma)}\iint_{-\infty}^{\infty}\exp\left[-\frac{\gamma ff^*}{1-\gamma}\right]P_\tau(f,f^*)\,d^2f, \tag{3.14}$$

where $d^2f = d(\operatorname{Re} f)d(\operatorname{Im} f)$. Substituting (3.7) into this expression, and evaluating the integral, we obtain the final expression for the generating function:

$$Q_\tau(\gamma)=\exp\{-\gamma|\eta(\tau)|^2\}. \tag{3.15}$$

The normalization constant $A(\tau)$ is eliminated by the obvious condition $Q_T(0) = 1$.

The function (3.15) is the generating function of the Poisson distribution. It differs from the corresponding expression obtainable with the aid of perturbation theory in that the quantity $\eta(\tau)$ entering into it differs from the quantity obtainable in the perturbation-theory calculation. In particular, there does not arise the typical perturbation-theory difficulty connected with the fact that the transition probabilities eventually begin to exceed unity. The obtained result is valid for any arbitrarily large value of the time if we are able to find the function $\eta(\tau)$ for such a value. Furthermore, the above derivation of the formula (3.15) is free from the use of the additional, essentially purely probabilistic, arguments with the aid of which this formula is usually justified.

4. THE EFFECT OF THE THERMAL FLUCTUATIONS IN THE DETECTOR ON THE PHOTOCOUNTING STATISTICS

We can also consider in the framework of the above-employed model the influence of the thermal fluctuations in the photodetector on the photocounting statistics. If at the initial moment of time the detector has a temperature T, then the probability of finding the oscillator in the m-th excited state is equal to

$$w_m=\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\exp\left(-m\frac{\hbar\omega}{kT}\right). \tag{4.1}$$

Instead of solving the problem with the initial oscillator state $|m\rangle$, let us first consider the auxiliary problem when at $t = 0$ the oscillator is in a coherent state $|\zeta\rangle$ of the form (2.15). The solution of interest to us is easily expressible in terms of the solution to this auxiliary problem.

In the case when at $t = 0$ both the field and the oscillator are in coherent states, the initial conditions for the Eqs. (2.19) and (2.20) will have the following form:

$$\eta(0)=\zeta,\quad \xi(\varkappa,0)=z(\varkappa). \tag{4.2}$$

As a result of this, the probability $P_n(\tau)$ of excitation of the n-th oscillator level, which is given by the previous formula (3.9), also becomes a function of ζ and ζ^*: $P_{|\zeta\rangle\to|n\rangle} = P_{\zeta\to n}(\zeta, \zeta^*)$. Let us average the values of the n-th level excitation probabilities over the distribution (4.1):

$$\bar{P}_n=\sum_{m=0}^{\infty}w_mP_{m\to n}. \tag{4.3}$$

The quantity $\bar{P}_n$ is expressible in terms of the auxiliary function $P_{\zeta\to n}(\zeta, \zeta^*)$ with the aid of the formula

$$\bar{P}_n=\left[\exp\left(\frac{\hbar\omega}{kT}\right)-1\right]\iint_{-\infty}^{\infty}\exp\left\{-\left[\exp\left(\frac{\hbar\omega}{kT}\right)-1\right]\zeta\zeta^*\right\}P_{\zeta\to n}(\zeta,\zeta^*)\frac{d^2\zeta}{\pi}. \tag{4.4}$$

The derivation of this formula is similar to the derivation of the formula (3.14) from (3.7), except that the initial coherent state $|\zeta\rangle$ is expanded in a series in $|n\rangle$; therefore, we shall not give it here.

Averaging the formula (3.10), we obtain the analogous relation

$$\bar{Q}_\tau(\gamma)=\left[\exp\left(\frac{\hbar\omega}{kT}\right)-1\right]\iint_{-\infty}^{\infty}\exp\left\{-\left[\exp\left(\frac{\hbar\omega}{kT}\right)-1\right]\zeta\zeta^*\right\}Q_\tau(\gamma,\zeta,\zeta^*)\frac{d^2\zeta}{\pi} \tag{4.5}$$

for the generating functions.

Now notice that on account of the linearity of the Eqs. (2.19) and (2.20) their solutions depend linearly on the initial conditions. Therefore, for $\zeta \neq 0$ the new solution, which we denote by $\eta'(\tau)$, will have the form

$$\eta'(\tau)=\eta(\tau)+\zeta\eta_1(\tau),$$

where $\eta(\tau)$ is the previous solution, which depends linearly on $z(\kappa)$, while $\eta_1(\tau)$ is the new function. Then, substituting η' in place of η in (3.15), we have

$$Q_\tau(\gamma,\zeta,\zeta^*)=\exp\{-\gamma|\eta^2|-\gamma\eta\eta_1^*\zeta^*-\gamma\eta^*\eta_1\zeta-\gamma|\eta_1^2|\zeta\zeta^*\}. \tag{4.6}$$

Substituting this expression into (4.5) and performing the integration, we obtain

$$\bar{Q}_\tau(\gamma)=(1+\bar{n}_T\gamma)^{-1}\exp\{-\bar{n}_r\gamma(1+\bar{n}_T\gamma)^{-1}\}. \tag{4.7}$$

Here we have introduced the notations

$$\bar{n}_T=|\eta_1^2(\tau)|\left[\exp\left(\frac{\hbar\omega}{kT}\right)-1\right]^{-1},\quad \bar{n}_r=|\eta(\tau)|^2.$$

The first factor in (4.7)

$$Q_\tau^T(\gamma)=(1+\bar{n}_T\gamma)^{-1} \tag{4.8}$$

is the generating function of the Bose-Einstein distribution. The second factor

$$Q_\tau^r(\gamma)=\exp\{-\bar{n}_r\gamma(1+\bar{n}_T\gamma)^{-1}\} \tag{4.9}$$

is the generating function of the photocounts for a detector at a temperature T. Since (4.7) is a product Q^TQ^r of the two generating functions, the thermal photocounts are statistically independent of the counts due to the field. However, as follows from (4.9), for a "heated" detector there should be a deviation from the Poisson distribution. For example, the variance of the photocounts computed on the basis of (4.9) has the form

$$\sigma_r^2=\overline{n_r^2}-\bar{n}_r^2=\bar{n}_r(1+2\bar{n}_T), \tag{4.10}$$

whereas at $T = 0$ we should have $\sigma_r^2 = \bar{n}_r$. It is possible that the values obtained in many experiments for the second and higher distribution moments are slightly higher than the "Poisson" values precisely because of this effect.

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