Self-action of a bounded wave beam in nonlinear geometric optics (two-dimensional problem)

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The nonlinear evolution of high-intensity plane beams of electromagnetic waves is investigated within the framework of geometric optics. A broad class is defined of exact analytic solutions of the equations of nonlinear geometric optics describing the self-action of a beam bounded on entry into the nonlinear medium by a slit or an opaque strip. The self-action of beams with symmetric and asymmetric intensity distributions is discussed. The dependence of singularities which appear during the nonlinear evolution of the beam on the form of the boundary conditions in the focusing medium is investigated. The deformation of a wave packet defined by the pulse shape on entry into the nonlinear medium is considered.

1. INTRODUCTION

The aim of this paper is to construct exact analytic solutions of the equations of nonlinear geometric optics. These equations describe the self-action of a weakly inhomogeneous high-intensity wave beam when the nonlinear evolution of the beam field has a more substantial effect than diffraction. When a monochromatic beam of frequency ω propagates in a nonlinear, isotropic, and homogeneous medium, and the nonlinearity of the medium is described by a small increment in the permittivity ϵ_0 ($\epsilon = \epsilon_0 + \epsilon_2$, $|\epsilon_2| \ll \epsilon_0$), the diffraction effects are unimportant[1] so long as $(\omega a c^{-1})^2 |\epsilon_2| \gg 1$ where a is the characteristic beam width. In this description, the wave field at each point is characterized by the intensity W and the ray direction u at each point. The cases $\epsilon_2 > 0$ and $\epsilon_2 < 0$ correspond to focusing and defocusing media, respectively. Consider a beam, propagating in the z direction, which at z = 0 has its x coordinate restricted either by a slit or a strip (screen). Assuming that in the plane perpendicular to z, all the quantities characterizing the beam are functions of only x, the nonlinear geometric-optics equations can be written in the form ($\epsilon_2 > 0$)

$$\frac{\partial W}{\partial t}+u\frac{\partial W}{\partial x}+W\frac{\partial u}{\partial x}=0,$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\frac{\partial W}{\partial x}=0. \qquad (1)$$

In these expressions, $W = I/I_0$ is the dimensionless intensity, I_0 is the maximum intensity at z = 0, the deviation of a given ray from the axis is characterized by $u = \beta^{-1/2}k_\perp k_0^{-1}$, $k_0 = \omega_0 c^{-1}$, $\beta = |\epsilon_2| I_0 (2\epsilon_0)^{-1}$, $t = \beta^{1/2} z a^{-1}$, and $x \to x a^{-1}$ where a is the characteristic transverse linear dimension of the beam at z = 0. A detailed derivation of (1) is given in[2].

The boundary conditions for (1) are specified in the t = 0 plane: $W(x, 0) = W_0(x)$; $u(x, 0) = u_0(x)$. The analytic solution of (1) for a particular boundary condition was obtained by Talanov.[3] Another particular solution is given in[2]. In this paper, we shall obtain a broad class of exact analytic solutions of (1) which describe the self-interaction of wave beams in focusing and defocusing media for symmetric $[W_0(x) = W_0(-x)]$ and asymmetric $[W_0(x) \neq W_0(-x)]$ profiles of $W_0(x)$.

It is well known[1] that (1) can be reduced to a linear set of equations through the Legendre transformation. The quantities W and u are then the independent variables, and x(W, u) and t(W, u) are functions thereof. The quantities x and t are defined in terms of a function ψ which, in the case of a focusing medium, is the solution of the Laplace equation for the axisymmetric problem in the (p, v) space

$$\frac{1}{p}\frac{\partial}{\partial p}\left(p\frac{\partial\psi}{\partial p}\right)+\frac{\partial^2\psi}{\partial v^2}=0, \qquad (2)$$

where $p^2 = W$ and $u = 2v$, and

$$t=-\frac{1}{2p}\frac{\partial\psi}{\partial p},\qquad x=-\frac{v}{p}\frac{\partial\psi}{\partial p}-\frac{1}{2}\frac{\partial\psi}{\partial v}. \qquad (3)$$

For the sake of simplicity, consider the case of a plane beam $u_0(x) = 0$ and transform (2) to spheroidal coordinates through the substitutions

$$v=\varepsilon\eta,\quad p^2=(1+\varepsilon^2)(1-\eta^2). \qquad (4)$$

The Laplace equation then takes the form

$$\frac{\partial}{\partial\varepsilon}\left[(1+\varepsilon^2)\frac{\partial\psi}{\partial\varepsilon}\right]+\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\psi}{\partial\eta}\right]=0. \qquad (5)$$

The boundary conditions for (4) are specified on the $\epsilon = 0$ surface and have the form

$$\left.\frac{\partial\psi}{\partial\eta}\right|_{\varepsilon=0}=0,\qquad \left.\frac{\partial\psi}{\partial\varepsilon}\right|_{\varepsilon=0}=-2\eta x_0(W). \qquad (6)$$

In these expressions, $x_0(W)$ is the inverse of $W_0(x)$. The solution of (5) can be written in the form of a series in terms of oblate spheroidal harmonics of the first and second kind:

$$\psi=\sum_n\{Q_n(\eta)[A_nQ_n(i\varepsilon)+B_nP_n(i\varepsilon)]+P_n(\eta)[C_nQ_n(i\varepsilon)+D_nP_n(i\varepsilon)]\}. \qquad (7)$$

The coefficients in this expansion can be determined from the boundary conditions. We shall now use (6) and (7) to investigate the evolution of certain wave beams.

2. SELF-ACTION OF A BEAM BOUNDED BY A SLIT

Consider the wave beam defined by a slit ($-1 \le x \le 1$) in an opaque screen which, in contrast to[3], is characterized by a smooth intensity variation at the edges (Fig. 1):

$$W_0|_{x=1}=W_0|_{x=-1}=0,\qquad \left.\frac{\partial W_0}{\partial x}\right|_{x=1}=\left.\frac{\partial W_0}{\partial x}\right|_{x=-1}=0. \qquad (8)$$

Suppose that the boundary condition which specifies the intensity distribution in x for t = 0 is given by

$$x=\frac{1}{(a+b)}\left[b\left(\eta+\frac{1-\eta^2}{2}\ln\frac{1+\eta}{1-\eta}\right)+a\eta\right]$$
$$=\eta+\frac{1-\eta^2}{2(K-1)}\ln\frac{1+\eta}{1-\eta}\qquad K=\frac{a+2b}{b} \qquad (9)$$

where K is a parameter and $W_{\epsilon=0} = 1 - \eta^2$. The func-

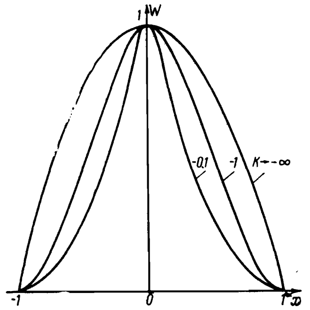


FIG. 1. Initial intensity distribution in the beam for different values of K (symmetric profile).

tion given by (9) describes the one-parameter family of symmetric beams $x(\eta) = x(-\eta)$ defined by the slit $(-1, 1)$, where $x = \pm 1$ corresponds to $\eta = \pm 1$, respectively, and on the beam axis $(x = 0)$ the intensity is a maximum $(\eta = 0, W_0 = 1)$. To ensure that (9) is a single-valued function, it is required that the equation $\partial x/\partial W = 0$ should have no roots for $-1 \le \eta \le 1$. Differentiating (9), we obtain

$$\frac{\partial x}{\partial W} = \frac{b}{2(a+b)}\left[-\frac{a+2b}{b\eta} + \ln\frac{1+\eta}{1-\eta}\right].$$

Since the function $y = \eta \ln[(1+\eta)/(1-\eta)]$ is nonnegative for $-1 \le x \le 1$, the equation $\partial x/\partial W = 0$ has no roots on this interval of η, provided

$$K = (a+2b)/b < 0. \qquad (10)$$

We shall assume henceforth that (10) is satisfied. If this is so, then by varying the value of K and using (9), we can investigate the self-action of a broad class of profiles. Substituting (9) into the boundary condition (6), we obtain the following expressions for the coefficients in (7):

$$A_1 = \frac{4ib}{5(a+b)}, \quad A_3 = \frac{8ib}{15(a+b)}, \quad C_0 = \frac{2i(a+2b)}{3(a+b)},$$
$$C_2 = -\frac{2ia}{3(a+b)}, \quad D_2 = \frac{\pi a}{3(a+b)}. \qquad (11)$$

The remaining coefficients in (7) are zero. Hence, using (3), we obtain the following formulas for the required functions:

$$t = (a+b)^{-1} f_1(\varepsilon, \eta), \quad x = (a+b)^{-1} f_2(\varepsilon, \eta); \qquad (12)$$

$$f_1(\varepsilon, \eta) = b\varepsilon\left[\eta \ln\frac{1+\eta}{1-\eta} + \frac{2}{3}\frac{3\eta^2(1+\varepsilon^2) - 2\varepsilon^2 - 3}{(1+\varepsilon^2)(1-\eta^2)}\right] + \frac{a}{2}\left[\operatorname{arctg}\frac{1}{\varepsilon} - \frac{\pi}{2} - \frac{\varepsilon}{1+\varepsilon^2}\right],$$

$$f_2(\varepsilon, \eta) = b\left[\frac{1+\varepsilon^2+\varepsilon^2\eta^2-\eta^2}{2}\ln\frac{1+\eta}{1-\eta} + \frac{\eta}{3}\frac{3(1-\eta^2)+\varepsilon^4(1+3\eta^2)}{(1+\varepsilon^2)(1-\eta^2)}\right] + \frac{a\eta}{1+\varepsilon^2}.$$

In these expressions, ϵ and η are related to W and u by (4).

The solution given by (12) describes the evolution of a smooth initial beam profile, which leads to the appearance of singularities. The physical significance of these singularities is connected with the crossing of the beam rays in the nonlinear medium. We shall now consider singularities which appear both on the axis and elsewhere.

An axial singularity is defined by the condition

$$x = 0, \quad \partial x/\partial W|_t = 0. \qquad (13)$$

Substituting (12) into these conditions, we obtain the coordinates of the ray crossing point $(\eta = 0, \epsilon = \epsilon_k)$. Hence, we find the coordinates of this point on the (x, z) plane and the value of the intensity:

$$\varepsilon_k^2 = -\frac{3}{4} + \left[\left(\frac{3}{4}\right)^2 - \frac{3K}{4}\right]^{1/2}, \quad W_k = 1 + \varepsilon_k^2. \qquad (14)$$

The parameter $K < 0$ is defined in (10). The intensity determined by (14) is finite; as b decreases, the intensity increases and the parabolic Talanov solution[3] is obtained as a special case $(b \to 0, K \to -\infty, W_k \to \infty)$.

The evolution of the beam may lead to another singularity connected with the existence of a point of inflection on the initial profile (9). Near this point, the intensity gradient $\partial W/\partial x$ is maximum, and in the course of the evolution of the beam the gradient may become infinite at some point. This singularity is analogous to the spillover of a simple wave in hydrodynamics. The conditions for the appearance of this singularity are[4]

$$\partial x/\partial W|_t = 0, \quad \partial^2 x/\partial W^2|_t = 0. \qquad (15)$$

Substituting (12) into these expressions, we obtain a very unwieldy set of algebraic equations for the coordinates η and ϵ of the spillover point. Here, we shall confine our attention to the case where this singularity appears in the region of high intensities $(\epsilon^2 \gg 1)$ and not too far from the axis $(\eta^2 \ll 1)$. In this case, (14) yields

$$3\eta^2(26\varepsilon^8 + 15K\varepsilon^4 - 18K^2) - 8(\varepsilon^4 + {}^3/_4K)^2 = 0, \qquad (16)$$
$$2(\varepsilon^4 + {}^3/_4K)(190\varepsilon^8 + 45K\varepsilon^4 - 54K^2) + \eta^2\varepsilon^4[1858\varepsilon^8 + 453K\varepsilon^4 + 270K^2] = 0. \qquad (17)$$

It is clear from (15) that the inequality $\eta^2 > 0$ is possible only for values of ϵ^4 within the range $(-1.17K > \epsilon^4 > 0, K < 0)$. In this interval, the set of equations defined by (16) and (17) has a unique solution: $\eta_s^2 = 1.1 \times 10^{-3}$; $\epsilon^2 = \epsilon_s^2 = (-2 \times 3^{-1}K)^{-1/2}$. The singularity appears on the (x, z) plane at points with coordinates

$$x_s = \frac{4b\eta_s\varepsilon_s^2}{3(a+b)}; \quad t_s = -\frac{a\pi}{4(a+b)}\left[1 + \frac{16\varepsilon_s}{3\pi(K-2)}\right]. \qquad (18)$$

In this case, the evolution of the initial profile results in an intensity distribution near the beam maximum, which takes the form of a thin intensity spike with a characteristic size $s = 2x_s$ which for $|K| \gg 1$ becomes $s = |\eta|\,|K|^{-1/2} \ll 1$. Since $t_{\eta=0} = t(W)$ is a monotonically increasing function of W, comparison of (18) and (14) readily shows that for high intensities $(\epsilon^2 \gg 1)$ the spillover is the first to appear $(t_s < t_f)$ if $|K| < 235$, whilst for $|K| > 235$ the axial singularity appears first. This is readily understood in a qualitative way if we recall that for large values of $|K|$ the initial profile is not very different from the parabolic profile which has a tendency to self-focusing on the axis. For smaller values of $|K|$ $(|K| < 235)$ the effect of the "edges" of the profile $W_0(x)$ leads to a qualitatively different result of beam self-action, namely, to an increase in the steepness of the wave front and the appearance of the ray spillover outside the beam axis.

3. ANGULAR SCANNING OF AN ASYMMETRIC BEAM

We now consider the self-interaction of an asymmetric beam $W_0(x) \ne W_0(-x)$ which satisfies the conditions given by (8). This beam is not only deformed during self-action in the nonlinear medium but is also displaced in a direction perpendicular to the direction of propagation (this is angular scanning). Let us again consider the family of initial profiles

$$x|_{t=0} = R\left[\frac{b(1-\eta^2)}{2}\ln\frac{1+\eta}{1-\eta} + \eta(a+b) + d\eta^2\right],$$
$$R = [d + (a+b)\operatorname{sign}\eta]^{-1}. \qquad (19)$$

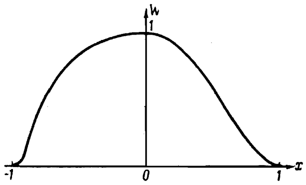


FIG. 2. Asymmetric initial intensity profile.

In these expressions sign $\eta = 1$ for $\eta > 0$ and sign $\eta = -1$ for $\eta < 0$. To ensure that the dependence of W on x in (19) is single-valued, the equation $\partial x/\partial W = 0$ should have no roots in the interval $-1 \le \eta \le 1$. This leads to the condition

$$K+2M(K-1)\eta<0, \quad M=\frac{d}{a+b}\neq 1, \quad \frac{1}{R}\neq 0, \qquad (20)$$

which will be assumed to be satisfied henceforth. Expressing the coefficients in (19) in terms of K and M, we can readily see that, in contrast to the symmetric profile (9), the asymmetric profile given by (19) is described by a two-parameter family of initial conditions (K and M are the parameters). Figure 2 shows this profile for $K = -9$ and $M = 0.4$. Using the boundary conditions (6), we obtain the coefficients in (7):

$$A_1=\frac{4ibR}{5}, \quad A_3=\frac{8ibR}{15}, \quad C_0=\frac{2iR(q+2b)}{3}, \quad C_2=-\frac{2iaR}{3},$$
$$D_2=\frac{\pi R(a+2b)}{3}, \quad D_1=\frac{6idR}{5}, \quad D_3=-\frac{8i}{15}dR. \qquad (21)$$

The remaining coefficients in (7) are, in this case, all zero.

We now use (3) to find the required functions t and x:

$$t=R[f_1(\varepsilon,\eta)-2d\varepsilon\eta], \quad x=R[f_2(\varepsilon,\eta)+d(\eta^2-\varepsilon^2-\varepsilon^2\eta^2)], \qquad (22)$$

where $f_1(\epsilon, \eta)$ and $f_2(\epsilon, \eta)$ are defined in (12). It is clear directly from the solution given by (22) that, in the course of the beam volution, the point corresponding to the intensity maximum ($\eta = 0$) is displaced away from the beam axis so that the coordinate of this point becomes $x_{max} = -Rd\epsilon^2$. At the same time, the direction in which the beam maximum propagates is different from the z direction (this is angular scanning). The evolution of the solution leads to an increase in the steepness of the front and to the formation of singularities of the form described by (15).

4. SELF-ACTION OF A BEAM BOUNDED BY AN OPAQUE STRIP

Consider the self-action of a beam which is bounded on entry into the nonlinear medium ($t = 0$ plane) by an opaque strip ($-1 \le x \le 1$). This is the opposite situation to the case of a slit ($-1 \le x \le 1$) in an opaque screen (Secs. 2 and 3). When this "internal" problem is considered, it is convenient to use the toroidal coordinates (α, γ) to solve the Laplace equation given by (2) with the aid of the formula

$$p=\frac{\operatorname{sh}\alpha}{\operatorname{ch}\alpha-\cos\gamma}, \quad v=\frac{\sin\gamma}{\operatorname{ch}\alpha-\cos\gamma}. \qquad (23)$$

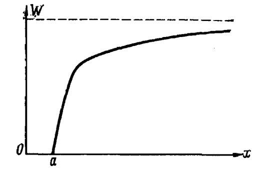
In terms of these coordinates, the Laplace equation is

$$\frac{\partial}{\partial\alpha}\left[\frac{1}{\operatorname{sh}\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\operatorname{sh}\alpha}{\operatorname{ch}\alpha-\cos\gamma}\frac{\partial\psi}{\partial\alpha}\right)\right]+\frac{\partial}{\partial\gamma}\left[\frac{1}{\operatorname{ch}\alpha-\cos\gamma}\frac{\partial\psi}{\partial\gamma}\right]=0. \qquad (24)$$

The coordinate α varies from zero to ∞, and the coordinate γ from π to zero. The boundary conditions in (24) are specified on the $\gamma = \pi$ ($v = 0$) surface, and have the form

$$\left.\frac{\partial\psi}{\partial\alpha}\right|_{\gamma=\pi}=0; \quad \left.\frac{\partial\psi}{\partial\gamma}\right|_{\gamma=\pi}=\operatorname{ch}^{-2}\frac{\alpha}{2}\,x_0(\alpha). \qquad (25)$$

FIG. 3. Profile of the beam bounded by a strip.



In these expressions, $x_0(\alpha)$ is the inverse of $W_0(x)$. At the same time, on the $\gamma = \pi$ surface, the intensity is $W = p^2 = \tanh^2(\alpha/2)$. The solution of (24) can be written in the form of the sum of toroidal harmonics of the first and second kind:

$$\psi=\sqrt{2(\operatorname{ch}\alpha-\cos\gamma)}\sum_n\{Q_{n-1/2}(\operatorname{ch}\alpha)(A_n\cos n\gamma+B_n\sin n\gamma)$$
$$+P_{n-1/2}(\operatorname{ch}\alpha)[C_n\cos n\gamma+D_n\sin n\gamma]\}. \qquad (26)$$

The coefficients in (26) are determined from the boundary conditions (25).

It is shown in[1] that (2) has a self-similar solution of the form

$$\psi=p^\lambda F_\lambda\left(\frac{\tau}{p}\right), \quad \tau=\frac{v}{p}. \qquad (27)$$

For integral values $\lambda = n$, the solutions given by (27) are Legendre functions; linear combinations of such solutions are used in Sec. 2 and 3. For half-integral values $\lambda = n - 1/2$, the solutions are expressed in terms of the toroidal harmonics (26) which, in turn, are expressed in terms of the complete elliptic integrals of the first and second kind. The modulus of such integrals is connected with the beam intensity, so that these integrals can be used to construct a family of analytic solutions of the nonlinear equations of geometric optics. For the simplest of these ($n = 1$), the boundary condition is of the form

$$x|_{\gamma=\pi}=P_{1/2}(\operatorname{ch}\alpha)\operatorname{ch}^3\frac{\alpha}{2}. \qquad (28)$$

where we have used (25) and (26). The profile $W = W_0(x)$ corresponding to (28) is shown in Fig. 3.

As can be seen, the intensity increases monotonically from $W = 0$ for $|x| \le 1$ to $W \to 1$ for $|x| \to \infty$. In the function given by (26), all the coefficients are then zero except for $D_1 = -1/2$ for $x \ge 1$ and $D_1 = 1/2$ for $x \le -1$. Hence, using (3), we have

$$t=-\frac{\sqrt{2}}{2}R^{3/2}\sin\gamma\left[\frac{3}{2}\cos\gamma P_{1/2}(\operatorname{ch}\alpha)-T\frac{\partial P_{1/2}(\operatorname{ch}\alpha)}{\partial\alpha}\right], \qquad (29)$$

$$x=\frac{\sqrt{2}}{4}R^{1/2}\left\{P_{1/2}(\operatorname{ch}\alpha)\left[\operatorname{ch}\alpha\frac{3\sin^2\gamma-2}{2}+\cos\gamma(1+3\sin^2\gamma)\right]\right.$$
$$\left.+\sin^2\gamma\left[\operatorname{sh}\alpha-2T\frac{\partial P_{1/2}(\operatorname{ch}\alpha)}{\partial\alpha}\right]\right\},$$
$$R=\operatorname{ch}\alpha-\cos\gamma, \quad T=\operatorname{sh}^{-1}\alpha[1-\operatorname{ch}\alpha\cos\gamma]. \qquad (30)$$

In calculations involving (29) and (30), it is convenient to use the representation of the function $P_{1/2}(\operatorname{ch}\alpha)$ in terms of the complete elliptic integral of the first (E) and second (K) kind:

$$P_{1/2}(\operatorname{ch}\alpha)=\frac{2}{\pi\operatorname{ch}{}^1/_2\alpha}\left[2\operatorname{ch}^2\frac{\alpha}{2}\mathbf{E}\left(\operatorname{th}\frac{\alpha}{2}\right)-\mathbf{K}\left(\operatorname{th}\frac{\alpha}{2}\right)\right]. \qquad (31)$$

Using the asymptotic property

$$\lim_{\alpha\to 0}P_{1/2}(\operatorname{ch}\alpha)=1+{}^3/_{16}\alpha^2, \qquad (32)$$

it is readily verified that, for low intensities $W = \tanh^2(\alpha/2) \approx \alpha^2/4 \ll 1$, the initial condition (28) has the form $\Delta x = x - 1 = 9W/4$. In this special case, the solution of (29) and (30) becomes identical with the well-known result for the evolution of a "linear" profile in a nonlinear medium. In contrast to this, the formulas given by (29) and (30) describe the intensity profile defined for all values in the range $0 \le W \le 1$.

The evolution of the solution given by (29) and (30) leads to an increase in the steepness of the front of the profile and to the appearance of a singularity whose position is given by (15).

5. SELF-ACTION OF A BEAM IN A DEFOCUSING MEDIUM

The equations of nonlinear geometric optics reduce to the Laplace equation (2) in the case of a focusing medium ($\epsilon_2 > 0$). In a defocusing medium ($\epsilon_2 < 0$), the last equation in (1) has a different sign in front of the derivative $\partial W/\partial x$. If we then substitute[4]

$$t=\partial\psi/\partial x, \quad x=2v\partial\psi/\partial W-{}^1/_2\partial\psi/\partial v$$

we find that, instead of (1), we have

$$-\frac{1}{p}\frac{\partial}{\partial p}\left(p\frac{\partial\psi}{\partial p}\right)-\frac{\partial^2\psi}{\partial v^2}=0. \tag{33}$$

To investigate the bounded wave beams with the aid of this equation, it is convenient to use the substitution $v \to -iv$, $\epsilon \to i\epsilon$. Substituting

$$p^2=(1-\varepsilon^2)(1-\eta^2), \quad v=\varepsilon\eta, \tag{34}$$

we find that the function ψ given by (33) satisfies (5) if we replace ϵ by $i\epsilon$ in this last equation. If we consider, for example, self-interaction in a defocusing medium in the case of a profile bounded by a slit, and if we take the boundary conditions in the form (19), we find that the expansion coefficients in (7) are given by

$$A_3=-{}^8/_{15}bR,\ A_1=-{}^4/_5bR,\ C_2={}^2/_3aR,\ C_0=-{}^2/_3R(a+2b), \\ D_1=-{}^6/_5Rd,\ D_3={}^8/_{15}Rd,\ D_2={}^1/_3\pi R(a+2b). \tag{35}$$

The functions t and x which describe the evolution of the beam now take the form

$$t=R\left\{b\varepsilon\left[-\eta\ln\frac{1+\eta}{1-\eta}+\frac{2}{3}\frac{3(1-\eta^2)+3\eta^2\varepsilon^2-2\varepsilon^2}{(1-\varepsilon^2)(1-\eta^2)}\right]\right. \\ \left.+\frac{a}{4}\ln\frac{1+\varepsilon}{1-\varepsilon}+\frac{a}{2}\frac{\varepsilon}{1-\varepsilon^2}+2d\varepsilon\eta\right\}, \tag{36}$$

$$x=R\left\{\frac{b}{2}\left[1-\eta^2-\varepsilon^2-\varepsilon^2\eta^2\right]\ln\frac{1+\eta}{1-\eta}+\frac{b\eta}{3}\frac{3(1-\eta^2)+\varepsilon^4(1+3\eta^2)}{(1-\varepsilon^2)(1-\eta^2)}\right. \\ \left.+\frac{a\eta}{1-\varepsilon^2}+d(\varepsilon^2\eta^2+\eta^2-\varepsilon^3)\right\}. \tag{37}$$

When $d = 0$, these solutions describe the self-action of a symmetric profile. It is then clear that (13) is not satisfied for any values of ϵ, i.e., the evolution of this family of symmetric profiles does not lead to the appearance of an axial singularity. At the same time, the appearance of the spillover represented by (15) is possible both for $d = 0$ and $d \neq 0$.

6. SELF-CONTRACTION OF A WAVE PACKET IN A NONLINEAR MEDIUM

It is well known that there is a space-time analogy between the self-focusing of a two-dimensional beam and the self-action of a plane wave packet. In a medium with a quadratic nonlinearity ($\epsilon = \epsilon_0 + \epsilon_2|E|^2$), the self-action of the packet, which is connected with the dependence of the group velocity on the amplitude E, is described by

$$2ik\frac{\partial E}{\partial z}+b^2\frac{\partial^2 E}{\partial \xi^2}-\frac{k^2\varepsilon_2|E|^2E}{2\varepsilon_0}=0 \tag{38}$$

In this expression, $b_2 = -kv'_\omega v_0^{-2}$, $v'_\omega = \partial v/\partial\omega$, $\xi = l - zv_0^{-1}$, v_0 is the group velocity in the linear approximation, t is the time, z is the direction of propagation of the packet, $k_0 = \omega_0 c^{-1}\sqrt{\epsilon}$, ω is the wave frequency, and c is the velocity of light.

When the frequency dispersion of the group velocity is small and the beam intensity is substantial, then provided the inequality $(v_0\xi)^2|\epsilon_2|\,|E|^2\epsilon_0^{-1} \gg k_0^{-1}v'_\omega$ is satisfied, we can use the geometric-optics approximation and substitute $E = A\exp(ik_0 s)$. It is now convenient to transform to dimensionless variables: $q = b^2c^2T^{-1}\xi$, $\tau = b^3c^2T_Z^{-1}[\epsilon_2|E_0|^2(2\epsilon_0)^{-1}]^{1/2}$ in (38), where 2T is the pulse length at entry into the nonlinear medium ($z = 0$) and E_0 is the maximum amplitude of the pulse at $z = 0$. To be specific, we consider the case of the defocusing medium ($\epsilon_2 < 0$), so that the equations of nonlinear geometric optics can be written in the form

$$\frac{\partial h}{\partial\tau}+h\frac{\partial h}{\partial q}-\frac{\partial W}{\partial q}=0, \quad \frac{\partial W}{\partial\tau}+\frac{\partial}{\partial q}(hW)=0. \tag{39}$$

In these expressions, W is the dimensionless intensity and the function h is related to the frequency modulation Ω of the beam, which develop in the nonlinear medium:

$$h=b\Omega/k\sqrt{\beta}, \tag{40}$$

where $\Omega = \partial s/\partial t$, $\Omega \ll \omega_0$.

The set of equations given by (39) is analogous to (1) and to equation (5) which describes the spatial self-action of a beam. Therefore, to describe the nonlinear evolution of a packet, we can use the previously found exact solutions (12) and (22). These solutions show the possibility of a controlled deformation of a packet, which depends on the initial shape of the pulse. Such solutions are also of interest for other problems which lead to the equations of nonlinear geometric optics, for example, the equations describing the flow of plasma across a magnetic field,[5] the problem of the dynamics of transparent plasma in a high-frequency high-intensity wave,[6] and so on.

The boundary conditions for (39) are determined by the shape of the envelope and the modulation of the pulse on entry into the nonlinear medium ($z = 0$, $\epsilon = 0$, $v = 0$, $\epsilon = 0$). For the sake of simplicity, consider a pulse which is not frequency-modulated at $z = 0$ ($\Omega_{z=0} = 0$). The boundary conditions for (3) can then be written in the form

$$\left.\frac{\partial\psi}{\partial\eta}\right|_{\varepsilon=0}=0, \quad \left.\frac{\partial\psi}{\partial\varepsilon}\right|_{\varepsilon=0}=-2\eta q=-2\eta b^2c^2\frac{t_0(W)}{T}. \tag{41}$$

In these expressions, $t_0(W)$ is the inverse of $W_0(t)$ at $z = 0$. Consider, for example, the family of pulses defined, by analogy with (19), by the function

$$\frac{t_0(W)}{T}=R\left[\frac{b(1-\eta^2)}{2}\ln\frac{1+\eta}{1-\eta}+\eta(a+b)+d\eta^2\right], \\ R=[d+(a+b)\,\mathrm{sign}\,\eta]^{-1}. \tag{42}$$

The values of t/T vary for $\epsilon = 0$ from $t/T = 1$ ($W = 0$, $\eta = 1$, leading front of the pulse) to $t/T = -1$ ($W = 0$, $\eta = -1$, rear front).

Assuming that the single-valuedness condition (20) is satisfied, and substituting $x \to q$, $t \to \tau$, we have from (22) the following expression which describes the deformation of the packet:

$$\frac{z}{Tv_0}=R(v_0b)^{-1}\left[\frac{|\varepsilon_2||E_0|^2}{2\varepsilon_0}\right]^{-1/2}[f_1(\varepsilon,\eta)-2d\varepsilon\eta], \\ \frac{t}{T}=\frac{z}{Tv_0}+R[f_2(\varepsilon,\eta)-d(\eta^2-\varepsilon^2-\varepsilon^2\eta^2)]. \tag{43}$$

By varying the values of the coefficients a, b, and d in (42), we can investigate the self-action of a broad class of pulses described by the solution (43). Thus, for example, when $b = 0$ this solution describes the deformation of a pulse with "unsmoothed" fronts: $\partial W/\partial z|_{W\to 0} \neq 0$. There is then an increase in the steepness of the rear pulse front (defocusing medium), so that at some point $z = z_0$ we have $\partial W/\partial z|_t \to \infty$. At this point, $W = 0$, $\eta = -1$, and the coordinate ϵ is found from the condition $\partial z/\partial W|_t = 0$. Substituting the solution (43) into this condition, we find that the coordinate ϵ satisfies the equation

$$\frac{2a}{d}\varepsilon^2+1+\varepsilon^2+2(1+\varepsilon^2)^3=0. \tag{44}$$

When $a/d < 0$, $|a|d|| > 3.5$, we find, for example, that when $\epsilon^2 \ll 1$, we have $\epsilon = [-3d/2a]^{1/2}$. Once the singularity appears, the equations given by (39) are no longer valid. When $b \neq 0$, the deformation of the "smoothed" pulse $\partial W/\partial z|_{W=0} = 0$ can lead to the appearance of a singularity not at the edge but "inside" the pulse. The coordinate and time of appearance of this singularity is obtained by substituting (43) into the conditions $\partial z/\partial W|_t = 0$, $\partial^2 z/\partial W^2|_t = 0$.

It is clear from (43) that, in the case of a nonlinear evolution of the pulse, the maximum-intensity point ($W = W_{max}$; $\eta = 0$) is displaced toward one of the edges of the pulse. The size and direction of this displacement, and the value of W_{max}, are determined by the shape of the pulse (42) at entry into the nonlinear medium. Therefore, the medium plays the role of a kind of delay line whose properties depend on the initial pulse shape.

In this paper, we consider the evolution of a beam prior to the appearance of singular points ($t \leq t_f$, t_s). Regions of ray crossing appear after the singular points. The equations given by (1) are then invalid. In the limit of large values of t, when wave effects are appreciable, the asymptotic solution of the problem of diffraction of a plane wave by a strip in a nonlinear defocusing medium is given in[7].

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