Conductivity of type-II superconductors near the critical field H_{c2} for arbitrary mean free paths

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An expression is derived for the conductivity of a superconductor near the critical field H_{c2} for arbitrary concentrations of paramagnetic impurities. The limiting case of a short mean free path and also the case of low temperatures with a long mean free path are investigated.

1. INTRODUCTION

In investigating the conductivity of superconductors in the mixed state, the treatment is usually confined to superconductors with a rather high concentration of impurities, when the electron mean free path l is much smaller than the pair size ξ.[1,2] This is associated with the fact that the majority of type-II superconductors are alloys with relatively short mean free paths. It is also significant that the problem of calculating the conductivity is complicated even for very dirty superconductors, which are described with the aid of differential equations. For an arbitrary mean free path the superconductor is described by a system of integral equations, and the problem of calculating the conductivity becomes even more complicated. Nevertheless, the study of the properties of superconductors for arbitrary mean free paths is of considerable interest. In addition to very dirty superconducting alloys, type-II superconductors also exist with very long mean free paths of the electrons. The study of type-II superconductors with arbitrary mean free paths may help us to better understand the phenomena which occur in superconductors and, in particular, it may help to establish the domain in which the approximation of a short mean free path is valid.

Below we shall determine the conductivity of type-II superconductors in the mixed state near the critical field H_{c2} for arbitrary mean free path and arbitrary temperature.

2. THE CONDUCTIVITY NEAR THE CRITICAL FIELD H_{c2}

It was shown in the article by Larkin and the author[3] that the transverse conductivity tensor $\hat{\sigma}$ is expressed in terms of the matrix elements of the operator K by the formula

$$\hat{\sigma}=B^{-2}\tau_y\tilde{K}\tau_y, \tag{1}$$

where B is the magnetic induction, τ_y is the Pauli matrix,

$$e_\alpha^{(1)}\tilde{K}_{\alpha\beta}e_\beta^{(2)}=\left\langle(\mathbf{e}^{(1)}\partial_+\Delta^*,\ \mathbf{e}^{(1)}\partial_-\Delta,\ [\mathbf{He}^{(1)}])\hat{K}\begin{pmatrix}\mathbf{e}^{(2)}\partial_-\Delta\\ \mathbf{e}^{(2)}\partial_+\Delta^*\\ [\mathbf{He}^{(2)}]\end{pmatrix}\right\rangle, \tag{2*}$$

$\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are unit vectors in the plane perpendicular to the magnetic field, and $\hat{\partial}_\pm=\partial/\partial\mathbf{r}+2ie\mathbf{A}$ (a vector operator). The operator $\hat{K}$ is found by linearizing the system consisting of Gor'kov's equations for the order parameter and Maxwell's equations for the vector potential with respect to small, slowly varying corrections to the order parameter $\hat{\Delta}$ and to the vector potential A:

$$\left(\hat{L}+\hat{K}\frac{\partial}{\partial t}\right)\begin{pmatrix}\delta\Delta_1\\ \delta\Delta_2\\ \delta\mathbf{A}\end{pmatrix}=0. \tag{3}$$

The operator $\hat{L}$ in formula (3) is equal to the second variational derivative of the free energy with respect to Δ and $\mathbf{A}$.

The system of equations for the linear response $\hat{G}_1$ has the form[4]

$$\begin{gathered}\mathbf{v}\frac{\partial\hat{G}_1}{\partial\mathbf{r}}+\omega^+\tau_z\hat{G}_1-\omega\hat{G}_1\tau_z+[-ie\mathbf{vA}\tau_z-i\hat{\Delta},\hat{G}_1]+in\Sigma_{\mathbf{pp}}(\omega_+)\hat{G}_1\\ -in\hat{G}_1\Sigma_{\mathbf{pp}}(\omega)+(-ie\mathbf{vA}_1\tau_z-i\hat{\Delta}^{(1)}+ie\varphi+in\hat{\Sigma}^{(1)}_{\mathbf{pp}})\hat{G}(\omega)\\ -\hat{G}(\omega_+)(-ie\mathbf{vA}_1\tau_z-i\hat{\Delta}^{(1)}+ie\varphi+in\hat{\Sigma}^{(1)}_{\mathbf{pp}})=0,\\ \hat{G}(\omega_+)\hat{G}_1+\hat{G}_1\hat{G}(\omega)=0,\end{gathered} \tag{4}$$

where $[\ldots,\ldots]$ denotes the commutator, $\omega_+=\omega+\omega_0$, φ and $\mathbf{A}_1$ are the amplitudes of the scalar and vector potentials of the variable field, G is the Green's function in the absence of the variable field, and $\hat{G}_1$ is the correction to the Green's function which arises in the presence of the variable field.

The function $\hat{G}(\omega)$ satisfies the following system of equations:[5,6]

$$\mathbf{v}\frac{\partial\hat{G}}{\partial\mathbf{r}}+\hat{\omega}\hat{G}-\hat{G}\hat{\omega}=0, \tag{5}$$

$$\begin{gathered}\operatorname{Sp}\hat{G}=0,\quad \hat{G}^2=1;\\ \hat{\omega}=\omega\tau_z-ie\mathbf{vA}\tau_z-i\hat{\Delta}+in\hat{\Sigma}_{\mathbf{pp}}(\omega),\quad \hat{\Delta}=\begin{pmatrix}0&\Delta\\ -\Delta^*&0\end{pmatrix},\\ \hat{\Sigma}_{\mathbf{pp}'}=\chi_{\mathbf{pp}'}-\frac{i\nu}{4}\int d\Omega_{\mathbf{p}_1}\chi_{\mathbf{pp}_1}\hat{G}_{\mathbf{p}_1}\hat{\Sigma}_{\mathbf{p}_1\mathbf{p}'}-\frac{i}{2n\tau_s}\tau_z\int\frac{d\Omega_{\mathbf{p}'}}{4\pi}\hat{G}_{\mathbf{p}}\tau_z,\end{gathered} \tag{6}$$

$\nu=mp/2\pi^2$ is the density of states at the Fermi surface, τ_s is the time of flight with spin flip, n is the impurity concentration, and the quantity $\chi_{\mathbf{pp}'}$ is related to the scattering amplitude $f_{\mathbf{pp}'}$ by the equation

$$-\frac{2\pi}{m}f_{\mathbf{pp}'}=\chi_{\mathbf{pp}'}+\frac{ip}{4\pi}\int\chi_{\mathbf{pp}_1}f_{\mathbf{p}_1\mathbf{p}'}\,d\Omega_{\mathbf{p}_1}. \tag{7}$$

The correction $\hat{\Sigma}^{(1)}$ to the self-energy part is given by

$$\hat{\Sigma}^{(1)}_{\mathbf{pp}}=-\frac{i\nu}{4}\int\hat{\Sigma}_{\mathbf{pp}_1}(\omega_+)\hat{G}^{(1)}_{\mathbf{p}_1}\hat{\Sigma}_{\mathbf{p}_1\mathbf{p}}(\omega)\,d\Omega_{\mathbf{p}_1}-\frac{i}{2n\tau_s}\tau_z\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\hat{G}^{(1)}_{\mathbf{p}}(\omega)\tau_z. \tag{8}$$

The current density $\mathbf{j}_1$ and the order parameter $\hat{\Delta}^{(1)}$ are expressed in terms of the function $\hat{G}^{(1)}$ according to the formulas

$$\begin{gathered}\mathbf{j}_1=-\frac{iep}{\pi}T\sum_\omega\int\frac{d\Omega_{\mathbf{p}}}{4\pi}(\mathbf{p}\hat{G}_1)_{11},\quad \hat{\Delta}^{(1)}=\begin{pmatrix}0\,;&\Delta_1\\ -\Delta_2;&0\end{pmatrix};\\ \Delta_1=i|\lambda|\nu\pi T\sum_\omega\int\frac{d\Omega_{\mathbf{p}}}{4\pi}(\hat{G}_1)_{12},\quad \Delta_2=-i|\lambda|\nu\pi T\sum_\omega\int\frac{d\Omega_{\mathbf{p}}}{4\pi}(\hat{G}_1)_{21}.\end{gathered} \tag{9}$$

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The system of Eqs. (4)–(6) and (8) can be solved near the critical field H_{c2} by making an expansion in powers of the order parameter Δ. Writing down the zero-order Green's function $\hat{G}$ in the form

$$\hat{G}(\omega)=\begin{pmatrix} \alpha_p(\omega); & -i\beta_p(\omega) \\ i\tilde{\beta}_p(\omega); & -\alpha_p(\omega) \end{pmatrix}, \tag{10}$$

we obtain from the system of Eqs. (5) and (6) the following expressions for the functions $\alpha_p(\omega)$ and $\beta_p(\omega)$:

$$\begin{gathered} \alpha_p(\omega)=\operatorname{sign}\omega[1-{}^1\!/_2\beta_p(\omega)\tilde{\beta}_p(\omega)], \\ \beta_p(\omega)=2\operatorname{sign}\omega\hat{L}_p^{-1}(\omega)\Delta, \quad \tilde{\beta}_p(\omega)=2\operatorname{sign}\omega\hat{M}_p^{-1}(\omega)\Delta^*, \end{gathered} \tag{11}$$

where the operators $\hat{L}_p(\omega)$ and $\hat{M}_p(\omega)$ are given by

$$\begin{aligned} \hat{L}_p(\omega)=\operatorname{sign}\omega\Big\{&\operatorname{sign}\omega\mathbf{v}\partial_-+2|\omega|+\tau^{-1}+\tau_s^{-1} \\ &-nv\int d\Omega_{p_1}\sigma_{pp_1}+\tau_s^{-1}\int\frac{d\Omega_p}{4\pi}\Big\}, \end{aligned} \tag{12}$$

$$\hat{M}_p(\omega)=\operatorname{sign}\omega\left\{-\operatorname{sign}\omega\mathbf{v}\partial_++2|\omega|+\tau^{-1}+\tau_s^{-1}-nv\int d\Omega_{p_1}\sigma_{pp_1}+\tau_s^{-1}\int\frac{d\Omega_{p_1}}{4\pi}\right\},$$

where $\mathbf{v}$ is the velocity at the Fermi surface, $\sigma_{pp'}$ is the scattering cross section, and $\tau = (nv\sigma)^{-1}$ is the electron time of free flight.

Let us represent the Green's function $\hat{G}_1$ in the form

$$\hat{G}_1=\begin{pmatrix} g_1; & f_1 \\ -f_2; & g_2 \end{pmatrix}. \tag{13}$$

The equations for the functions $g_{1,2}$ and $f_{1,2}$ substantially depend on the frequency ranges in which these functions are considered. In the frequency region sign ω = sign ω_+ it turns out to be convenient to write down the equations for the functions $f_{1,2}$ and the functions $g_{1,2}$ are found from the normalization condition (4). In the region sign ω = – sign ω_+ the functions $f_{1,2}$ are expressed in terms of the functions $g_{1,2}$, and the latter are determined by using the system of Eqs. (4). First let us consider the frequency range in which

$$\operatorname{sign}\omega=\operatorname{sign}\omega_+. \tag{14}$$

In the frequency range (14) we obtain the following result from the system of Eqs. (4):

$$\begin{gathered} f_1=(\omega_0+\hat{L}_p(\omega))^{-1}[e\mathbf{vA}_1(\beta_p(\omega)+\beta_p(\omega_+))-2i\operatorname{sign}\omega\Delta_1], \\ f_2=(\omega_0+\hat{M}_p(\omega))^{-1}[e\mathbf{vA}_1(\tilde{\beta}_p(\omega)+\tilde{\beta}_p(\omega_+))-2i\operatorname{sign}\omega\Delta_2], \\ g_1=-{}^1\!/_2 i\operatorname{sign}\omega(\beta_p(\omega_+)f_2+\tilde{\beta}_p(\omega)f_1). \end{gathered} \tag{15}$$

In order to construct the operator $\hat{K}$ it is also necessary to find expressions for the functions $f_{1,2}$ and $g_{1,2}$ in the frequency range

$$-\operatorname{sign}\omega=\operatorname{sign}\omega_+, \quad \operatorname{sign}\omega_+>0. \tag{16}$$

In the zero-order approximation with respect to Δ, we obtain the following results for the functions $g_{1,2}$:

$$g_1=-2ie\hat{N}_p^{-1}\mathbf{vA}_1, \quad g_2=2ie\hat{O}_p^{-1}\mathbf{vA}_1, \tag{17}$$

$$\begin{aligned} \hat{N}_p&=\mathbf{v}\frac{\partial}{\partial\mathbf{r}}+\omega_0+\tau^{-1}+\tau_s^{-1}-\tau_s^{-1}\int\frac{d\Omega_p}{4\pi}-nv\int d\Omega_{p_1}\sigma_{pp_1}, \\ \hat{O}_p&=-\mathbf{v}\frac{\partial}{\partial\mathbf{r}}+\omega_0+\tau^{-1}+\tau_s^{-1}-\tau_s^{-1}\int\frac{d\Omega_p}{4\pi}-nv\int d\Omega_{p_1}\sigma_{pp_1}. \end{aligned} \tag{18}$$

From the normalization condition (4) and formulas (11) and (17) we find the following expressions for the functions $f_{1,2}$:

$$\begin{aligned} f_1&=-2e[(\hat{L}_p^{-1}(\omega_+)\Delta)\hat{O}_p^{-1}+(\hat{L}_p^{-1}(\omega)\Delta)\hat{N}_p^{-1}]\mathbf{vA}_1, \\ f_2&=-2e[(M_p^{-1}(\omega_+)\Delta^*)\hat{N}_p^{-1}+(\hat{M}_p^{-1}(\omega)\Delta^*)\hat{O}_p^{-1}]\mathbf{vA}_1. \end{aligned} \tag{19}$$

By using formulas (17) and (19) one can obtain, as a result of simple but rather lengthy calculations, from the system of Eqs. (4) an expression for the function g_1 correct to terms of second order in Δ. This expression is essential in order to construct the K_{33} element of the operator $\hat{K}$. Expressions (15) and (19) are sufficient for the construction of all remaining elements:

$$\begin{gathered} K_{11}=-\frac{\nu}{2}\int d\Omega_pT\sum_{\omega>0}\frac{\partial}{\partial\omega}\hat{L}_p^{-1}(\omega), \quad K_{22}=-\frac{\nu}{2}\int d\Omega_pT\sum_{\omega>0}\frac{\partial}{\partial\omega}\hat{M}_p^{-1}(\omega), \\ K_{12}=K_{21}=0, \\ K_{13}=-\frac{ie}{m}\nu T\sum_{\omega>0}\frac{\partial}{\partial\omega}\int d\Omega_p[\hat{L}_p^{-1}(\omega)(\hat{L}_p^{-1}(\omega)\Delta)\mathbf{p}+(\hat{L}_p^{-1}(\omega)\Delta)\hat{O}_p^{-1}\mathbf{p}], \\ K_{31}=-\frac{ie}{m}\nu T\sum_{\omega>0}\frac{\partial}{\partial\omega}\int d\Omega_p\mathbf{p}[(\hat{M}_p^{-1}(\omega)\Delta^*)\hat{L}_p^{-1}(\omega)+\hat{O}_p^{-1}(\hat{M}_p^{-1}(\omega)\Delta^*)], \\ K_{23}=-\frac{ie}{m}\nu T\sum_{\omega>0}\frac{\partial}{\partial\omega}\int d\Omega_p[\hat{M}_p^{-1}(\omega)(\hat{M}_p^{-1}(\omega)\Delta^*)\mathbf{p}+(\hat{M}_p^{-1}(\omega)\Delta^*)\hat{N}_p^{-1}\mathbf{p}], \\ K_{32}=-\frac{ie}{m}\nu T\sum_{\omega>0}\frac{\partial}{\partial\omega}\int d\Omega_p\,\mathbf{p}[(\hat{L}_p^{-1}(\omega)\Delta)\hat{M}_p^{-1}(\omega)+\hat{N}_p^{-1}(\hat{L}_p^{-1}(\omega)\Delta)]. \end{gathered} \tag{20}$$

The element K_{33} is rather unwieldy for the case of arbitrary scattering by impurities; we shall not explicitly write it down, but at once present the expression for the conductivity associated with isotropic scattering by impurities. For triangular and square lattices the conductivity does not depend on the direction in the plane perpendicular to the magnetic field. In our approximation the operator $\hat{K}$ is Hermitian and the Hall angle is equal to zero. Taking this into account, we find the following expression for the conductivity tensor:

$$\begin{aligned} &\hat{\sigma}_{\alpha\beta}=\delta_{\alpha\beta}\sigma; \\ &\sigma=\frac{e^2p^2v}{3\pi^2(\tau^{-1}+\tau_s^{-1})}-\frac{4\pi ve}{H}\langle|\Delta|^2\rangle T\sum_{\omega>0}\frac{\partial}{\partial\omega}D_3(\omega) \\ &-\frac{8e^2p\langle|\Delta|^2\rangle}{\pi(\tau^{-1}+\tau_s^{-1})}T\sum_{\omega>0}\frac{\partial}{\partial\omega}S_4(\omega)+\frac{8e^2p\langle|\Delta|^2\rangle}{\pi}T\sum_{\omega>0}\frac{\partial}{\partial\omega}Q^{(1)}(\omega) \\ &+\frac{ie^2p\langle|\Delta|^2\rangle}{4\pi^2mT(\tau^{-1}+\tau_s^{-1})}\int_{-i\infty}^{i\infty}\frac{d\omega}{\cos^2(\omega/2T)}\Big\{(S_1(\omega)+S_1(\omega_+)) \\ &+\frac{1}{\tau^{-1}+\tau_s^{-1}}\Big[(\tau^{-1}-\tau_s^{-1})(D_1(\omega)+D_1(\omega_+))(S_2(\omega)+S_2(\omega_+)) \\ &-\frac{4}{3}\pi nvp^2(f^2S_3(\omega_+)+f^{*2}S_3(\omega))+4\pi eHnv(S_4(\omega)+S_4(\omega_+)) \\ &\times(f^2S_4(\omega)+f^{*2}S_4(\omega_+))-eH\tau_s^{-1}(S_4(\omega)+S_4(\omega_+))^2-\frac{p^2}{3\tau_s}(S_3(\omega)+S_3(\omega_+)) \\ &+\frac{2ip^3}{3\tau}(fD_1^2(\omega_+)-f^*D_1^2(\omega))\Big]\Big\}+\frac{8e^2v\langle|\Delta|^2\rangle}{\pi}T\sum_{\omega>0}\frac{\partial}{\partial\omega}S_5(\omega). \end{aligned} \tag{21}$$

Here and later on the angular brackets ⟨...⟩ denote averages over a cell, f is the amplitude for impurity scattering, and the arguments ω_+ and ω denote continuation with $\omega > 0$ and $\omega < 0$, respectively. The quantities S_i and D_i are defined in the following manner:

$$\begin{gathered} \int\frac{d\Omega_p}{4\pi}\hat{L}_p^{-1}(\omega)\Delta=D_1(\omega)\Delta, \quad \int\frac{d\Omega_p}{4\pi}\hat{L}_p^{-1}(\omega)\partial_-\Delta=D_3(\omega)\partial_-\Delta, \\ \int\frac{d\Omega_p}{4\pi}\mathbf{p}\hat{M}_p^{-1}(\omega)\Delta^*=S_4(\omega)\partial_+\Delta^*, \quad \int\frac{\partial\Omega_p}{4\pi}\mathbf{p}\hat{L}_p^{-1}(\omega)\Delta=-S_4(\omega)\partial_-\Delta, \\ S_3(\omega)=\frac{1}{\langle|\Delta|^2\rangle}\int\frac{d\Omega_p}{4\pi}\langle(M_p^{-1}(\omega)\Delta^*)(\hat{L}_p^{-1}(\omega)\Delta)\rangle=-\frac{1}{2}\frac{\partial}{\partial\omega}D_1(\omega), \\ \mathbf{ab}\,S_2(\omega)=\frac{1}{\langle|\Delta|^2\rangle}\int\frac{d\Omega_p}{4\pi}\langle\Delta(\mathbf{ap})(\mathbf{bp})\hat{M}_p^{-1}(\omega)\Delta^*\rangle, \\ \mathbf{ab}\,S_1(\omega)=\frac{1}{\langle|\Delta|^2\rangle}\int\frac{d\Omega_p}{4\pi}\langle(\mathbf{ap})(\mathbf{bp})(\hat{M}_p^{-1}(\omega)\Delta^*)(\hat{L}_p^{-1}(\omega)\Delta)\rangle, \\ \mathbf{ab}\,S_5(\omega)=\frac{1}{2\langle|\Delta|^2\rangle}\int\frac{d\Omega_p}{4\pi}\langle(\mathbf{ap})\{(\hat{M}_p^{-1}(\omega)\Delta^*)\hat{L}_p^{-1}(\omega)(\mathbf{pb})\hat{L}_p^{-1}(\omega)\Delta \\ +(\hat{L}_p^{-1}(\omega)\Delta)\hat{M}_p^{-1}(\omega)(\mathbf{pb})\hat{M}_p^{-1}(\omega)\Delta^*\}\rangle, \\ \int\frac{d\Omega_p}{4\pi}\hat{L}_p^{-1}(\omega)(\hat{L}_p^{-1}(\omega)\Delta)\mathbf{p}=Q^{(1)}(\omega)\partial_-\Delta, \end{gathered} \tag{22}$$

where **a** and **b** are vectors in the plane, which is perpendicular to the magnetic field.

In connection with the derivation of formulas (22) it is essential that near the critical field H_{c2} the order parameter Δ is an eigenfunction of the operator ∂_-^2:[7]

$$-\partial_-^2\Delta=2eH_{c2}\Delta. \tag{23}$$

The second term in formula (21) is related to the elements K_{11} and K_{22}; the third and fourth terms arise from the elements K_{13}, K_{31}, K_{23}, and K_{32}. All of the remaining terms in formula (21) are associated with the element K_{33}.

3. CALCULATION OF THE QUANTITIES S_i AND D_i

Let us demonstrate the method, which can be used to find all of the quantities D_i and S_i, by calculating D_1 as an example. The first formula in (22) follows from formula (23) and from the form of the operator $\hat{L}_p^{-1}$. This relation is fulfilled for any function Δ which satisfies Eq. (23). Let us choose it in the form

$$\Delta\to\exp(-eHx^2). \tag{24}$$

Here and below we are using the gauge in which the vector potential is given by

$$\mathbf{A}=Hx(0,\ 1,\ 0). \tag{25}$$

Let us introduce the function ψ_p, which satisfies the equation

$$\hat{L}_{p\,\mathrm{sign}\,\omega}\psi_p=\exp(-eHx^2). \tag{26}$$

Using the definition (22) of the function $D_1(\omega)$ and the explicit form (12) of the operator $\hat{L}_p(\omega)$, we reduce Eq. (26) to the form

$$\left[v_x\frac{\partial}{\partial x}-2ieHxv_y+\alpha\right]\psi_p=\mathrm{sign}\,\omega[1+\mathrm{sign}\,\omega(\tau^{-1}-\tau_s^{-1})D_1(\omega)]\exp(-eHx^2),$$
$$\alpha=2|\omega|+\tau^{-1}+\tau_s^{-1}. \tag{27}$$

The solution of Eq. (27) has the form

$$\psi_p=\mathrm{sign}\,\omega[1+\mathrm{sign}\,\omega(\tau^{-1}-\tau_s^{-1})D_1(\omega)]\int_{-\infty}^{\infty}dx_1\mathscr{K}_p(x,x_1)\exp(-eHx_1^2), \tag{28}$$

where the kernel is given by

$$\mathscr{K}_p(x,x_1)=\begin{cases}\theta(x-x_1)v_x^{-1}\exp\left[-\dfrac{\alpha(x-x_1)}{v_x}+\dfrac{ieHv_y(x^2-x_1^2)}{v_x}\right], & v_x>0,\\ \theta(x_1-x)|v_x|^{-1}\exp\left[-\dfrac{\alpha(x-x_1)}{v_x}+\dfrac{ieHv_y(x^2-x_1^2)}{v_x}\right], & v_x<0.\end{cases} \tag{29}$$

The function $D_1(\omega)$ can be expressed in terms of the kernel $\mathscr{K}_p(x, x_1)$ according to the formula

$$D_1(\omega)=[\mathrm{sign}\,\omega+(\tau^{-1}-\tau_s^{-1})D_1(\omega)]$$
$$\times\int\frac{d\Omega_p}{4\pi}\int_{-\infty}^{\infty}\mathscr{K}_p(x,x_1)\exp[-eH(x_1^2-x^2)]dx_1. \tag{30}$$

In calculating the function $D_1(\omega)$ it is sufficient to consider the asymptotic behavior in Eq. (30) as $x\to\infty$. In this connection it turns out to be necessary to evaluate the quantity

$$\int\frac{d\Omega_p}{4\pi}\mathscr{K}_p(x,x_1)$$

only for large values of both the sum and the difference of the arguments

$$\int\frac{d\Omega_p}{4\pi}\mathscr{K}_p(x,x_1)\to\frac{1}{2v}\left(\frac{\pi}{2eH}\right)^{1/2}\frac{1}{|x^2-x_1^2|^{1/2}}\exp(-eH|x^2-x_1^2|)$$
$$\times\left\{1-\Phi\left(\left(\frac{\alpha^2}{2eHv^2}\left|\frac{x-x_1}{x+x_1}\right|\right)^{1/2}\right)\right\}, \tag{31}$$

where $\Phi(x)$ denotes the probability integral.

Using formulas (30) and (31), we obtain the following expression for the function $D_1(\omega)$:

$$D_1(\omega)=\mathrm{sign}\,\omega\frac{J(\omega)}{1-(\tau^{-1}-\tau_s^{-1})J(\omega)}, \tag{32}$$

$$J(\omega)=\frac{1}{v(2eH)^{1/2}}\int_0^\infty\frac{dy}{y^{1/2}}\mathrm{arctg}\left(\frac{v(2eHy)^{1/2}}{\alpha}\right)e^{-y}. \tag{33}$$

From formulas (22), (32), (33) and the equation for Δ we find

$$\Delta=\frac{|\lambda|mp}{2\pi}T\sum_\omega\int\frac{d\Omega_p}{4\pi}\beta_p(\omega), \tag{34}$$

and we obtain the following expression for the critical field H_{c2}:[7]

$$\ln\frac{T_c}{T}=2\pi T\sum_{\omega>0}[\omega^{-1}-2D_1(\omega)]. \tag{35}$$

Formula (35) goes over into the expression derived by Gor'kov[8] in the limit $\tau\to\infty$ and $\tau_s\to\infty$.

For superconductors containing a rather high concentration of impurities, when the condition $l\ll\xi$ is satisfied, it follows from formulas (32) and (33) that

$$D_1(\omega)={}^1/_2\,\mathrm{sign}\,\omega[|\omega|+\tau_s^{-1}+eH_{c2}D]^{-1},\quad D=vl_{tr}/3. \tag{36}$$

By substituting the value of D_1 into formula (35) we obtain the well known expression[9-11] for the critical field

$$\ln\frac{T_c}{T}=\psi\left(\frac{1}{2}+\frac{\tau_s^{-1}+eH_{c2}vl_{tr}/3}{2\pi T}\right)-\psi\left(\frac{1}{2}\right), \tag{37}$$

where $\psi(x)$ is the psi-function.

The method used to calculate the function $D_1(\omega)$ also enables us to determine all remaining functions: $D_3(\omega)$, $S_i(\omega)$, and $Q^{(1)}(\omega)$. Omitting the intermediate steps, we immediately present the answer:

$$D_3(\omega)=-\mathrm{sign}\,\omega\cdot\frac{\alpha}{2eHv^2}(\mathrm{Ei}(-z)e^z)\Big/\left[1+\frac{\alpha(\tau^{-1}-\tau_s^{-1})}{2eHv^2}\mathrm{Ei}(-z)e^z\right],$$

$$S_1(\omega)=\frac{p^2}{2eHv^2}[1+\mathrm{sign}\,\omega(\tau^{-1}-\tau_s^{-1})D_1(\omega)]^2[1-\alpha J(\omega)],$$

$$S_2(\omega)=\frac{\alpha p^2}{4eHv^2}\mathrm{sign}\,\omega[1+\mathrm{sign}\,\omega(\tau^{-1}-\tau_s^{-1})D_1(\omega)]$$
$$\times\left\{\frac{\alpha^2+eHv^2}{\alpha}J(\omega)-\left[1+\frac{1}{2}\mathrm{Ei}(-z)e^z\right]\right\},$$

$$S_4(\omega)=\frac{p}{2eHv}[1+\mathrm{sign}\,\omega(\tau^{-1}-\tau_s^{-1})D_1(\omega)][1-\alpha J(\omega)], \tag{38}$$

$$S_5(\omega)=\mathrm{sign}\,\omega\frac{\alpha p^2}{(2eHv^2)^2}\Big[(1+\mathrm{sign}\,\omega(\tau^{-1}-\tau_s^{-1})D_1(\omega))$$
$$+\frac{2eH}{m}(\tau^{-1}-\tau_s^{-1})Q^{(1)}(\omega)\Big]\left[\frac{eHv^2}{\alpha}J(\omega)+\frac{1}{2}\mathrm{Ei}(-z)e^z\right][1-(\tau^{-1}-\tau_s^{-1})J(\omega)]^{-1},$$

$$Q^{(1)}_{(\omega)}=Q\left[1+\frac{\alpha}{2eHv^2}(\tau^{-1}-\tau_s^{-1})\mathrm{Ei}(-z)e^z\right]^{-1},$$

$$Q=-\frac{\alpha m}{2e^2H^2v^2}[1+\mathrm{sign}\,\omega(\tau^{-1}-\tau_s^{-1})D_1(\omega)]\left[\frac{eHv^2}{\alpha}J(\omega)+\frac{1}{2}\mathrm{Ei}(-z)e^z\right]$$
$$z=\alpha^2/2eHv^2.$$

Formulas (21), (22), and (38) determine the conductivity of a superconductor near the critical field H_{c2} for arbitrary mean free path and arbitrary temperature.

It is necessary to relate the quantity $\langle|\Delta|^2\rangle$ in formula (21) for the conductivity to an experimentally measurable quantity—namely, the value of the magnetic moment. Using formulas (10) and (11) we obtain the following expression for the current:

$$\mathbf{j}=\frac{2iep}{\pi}T\sum_\omega\int\frac{d\Omega_p}{4\pi}\mathbf{p}\,\mathrm{sign}\,\omega(\hat{L}_p^{-1}(\omega)\Delta)(\hat{M}_p^{-1}(\omega)\Delta^*). \tag{39}$$

Using the method described above, we find the following result for the right hand side of Eq. (39):

$$\int \frac{d\Omega_{\mathbf{p}}}{4\pi} \mathbf{p} \operatorname{sign}\omega (\hat{L}_{\mathbf{p}}^{-1}(\omega)\Delta)(\hat{M}_{\mathbf{p}}^{-1}(\omega)\Delta^{*}) = \tilde{\chi}(\Delta\partial_{+}\Delta^{*} - \Delta^{*}\partial_{-}\Delta), \quad (40)$$

$$\tilde{\chi} = \frac{\alpha p}{4e^2H^2v^3}[1+\operatorname{sign}\omega(\tau^{-1}-\tau_s^{-1})D_1(\omega)]\left\{\frac{eHv^2}{\alpha}J(\omega) + \frac{1}{2}\operatorname{Ei}\left(-\frac{\alpha^2}{2eHv^2}\right)\exp\left(\frac{\alpha^2}{2eHv^2}\right)\right\}[1-(\tau^{-1}-\tau_s^{-1})J(\omega)]^{-1}. \quad (41)$$

Substituting expression (40) into formula (39), we obtain

$$\mathbf{j} = \frac{2iep}{\pi}[\Delta\partial_{+}\Delta^{*} - \Delta^{*}\partial_{-}\Delta]T\sum_{\omega}\tilde{\chi}(\omega). \quad (42)$$

Using expression (42) for the current, we find that the magnetic moment is given by

$$4\pi M = -8ep\langle|\Delta|^2\rangle T\sum_{\omega}\tilde{\chi}(\omega). \quad (43)$$

4. LIMITING CASES

Let us investigate the general expression (21) for the conductivity in two special cases.

A. Low temperature $T \ll T_c$ and long mean free path $l \gg \xi$. In this limiting case the major contribution to the correction to the conductivity arises from the third and fifth terms in formula (21). The expressions for the functions S_i and D_i are determined from formulas (22), (32), and (38) and are given by

$$D_1(\omega) = \operatorname{sign}\omega\frac{\pi^{3/2}}{2(2eHv^2)^{1/2}}; \quad D_3(\omega) = \operatorname{sign}\omega\frac{\alpha}{eHv^2}\ln\frac{(2eHv^2/\gamma)^{1/2}}{\alpha},$$
$$S_1(\omega) = \frac{p^2}{2eHv^2}; \quad S_2(\omega) = \operatorname{sign}\omega\frac{p^2\pi^{3/2}}{8(2eHv^2)^{1/2}},$$
$$S_3(\omega) = \frac{1}{eHv^2}\ln\frac{(eHv^2/2\gamma)^{1/2}}{|\omega|+(\tau^{-1}+\tau_s^{-1})/2}, \quad S_4(\omega) = \frac{p}{2eHv}, \quad (44)$$
$$S_5(\omega) = \operatorname{sign}\omega\frac{p^2\pi^{3/2}}{4(2eHv^2)^{3/2}}, \quad Q^{(1)} = Q = -\frac{\pi^{3/2}pv}{2(2eHv^2)^{3/2}},$$
$$|\omega|^2 \ll eHv^2.$$

Substituting the values of the functions S_i and D_i into formula (21), we obtain

$$\sigma = \frac{e^2p^2v}{3\pi^2(\tau^{-1}+\tau_s^{-1})}\left\{1 + \frac{3\langle|\Delta|^2\rangle}{eHv^2(\tau^{-1}+\tau_s^{-1})}\left[2\tau_s^{-1} + \frac{2}{3}\left(\frac{1}{\tau}+\frac{1}{\tau_s}\right)\times\left(\ln\frac{(2\gamma eHv^2)^{1/2}}{\pi T} + \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2}+\frac{\tau^{-1}+\tau_s^{-1}}{4\pi T}\right)\right)\right.\right. \quad (45)$$
$$\left.\left.+\frac{\sigma_t p^2}{2\pi\tau}\left(1+\frac{\pi^3}{12}-\frac{2}{3}\left(\ln\frac{(2\gamma eHv^2)^{1/2}}{\pi T}+\psi\left(\frac{1}{2}\right)-\psi\left(\frac{1}{2}+\frac{\tau^{-1}+\tau_s^{-1}}{4\pi T}\right)\right)\right)\right]\right\},$$

where $\sigma_t = 4\pi|f|^2$ is the total scattering cross section.

The last term in formula (45) appears only in an exact calculation of the amplitude for electron scattering by impurities. This term contains an extra factor $\sigma_t p^2/2\pi$ in comparison with the second term inside the square brackets. The latter term is small in the Born approximation, when $\sigma_t p^2/2\pi \ll 1$.

We also present expressions for the magnetic moment and critical field in the same limiting case; these expressions follow from formulas (43) and (35):

$$eH_{c2}v^2 = (2.718\pi T_c)^2/2\gamma, \quad 4\pi M = -p^2\langle|\Delta|^2\rangle/\pi Hv. \quad (46)$$

B. Superconductors with a short mean free path ($l \ll \xi_0$). From formulas (22), (32), and (38) we find the following expressions for the coefficients D_i and S_i in the limiting case $l \ll \xi_0$:

$$D_1(\omega) = \frac{\operatorname{sign}\omega}{2(|\omega|+\tau_s^{-1}+\lambda)}, \quad D_3(\omega) = \frac{\operatorname{sign}\omega}{2(|\omega|+\tau_s^{-1}+3\lambda)}, \quad S_1(\omega) = \frac{p^2D_1^2(\omega)}{3},$$
$$S_2(\omega) = \frac{p^2D_1(\omega)}{3}, \quad S_3(\omega) = D_1^2(\omega), \quad S_4(\omega) = \frac{\operatorname{sign}\omega\, pv\tau D_1(\omega)}{3}, \quad (47)$$
$$Q^{(1)}(\omega) = -\operatorname{sign}\omega\frac{pv\tau D_1(\omega)}{3[|\omega|+\tau_s^{-1}+3\lambda]}, \quad S_5(\omega) = \frac{p^2\tau D_1(\omega)}{6[|\omega|+\tau_s^{-1}+3\lambda]},$$
$$\lambda = eH_{c2}vl_{tr}/3.$$

Substituting these values of the coefficients into formula (21), we find

$$\sigma = \frac{e^2p^2l}{3\pi^2}\left\{1 + \frac{\langle|\Delta|^2\rangle}{4\pi T\lambda}\psi'\left(\frac{1}{2}+\frac{\tau_s^{-1}+\lambda}{2\pi T}\right) + \frac{\langle|\Delta|^2\rangle}{4\pi T}\left[\frac{1}{\lambda+\tau_s^{-1}}\psi'\left(\frac{1}{2}+\frac{\lambda+\tau_s^{-1}}{2\pi T}\right) + \frac{1}{2\pi T}\psi''\left(\frac{1}{2}+\frac{\lambda+\tau_s^{-1}}{2\pi T}\right)\right]\right\}. \quad (48)$$

As $\tau_s \to \infty$ formula (48) goes over into the corresponding expression given in Thompson's article.[2]

It follows from formula (45) that the correction to the conductivity intrinsically depends on the length of the mean free path. For long mean free paths, the major contribution to the conductivity comes from the anomalous terms. At low temperatures an additional logarithmic divergence appears in the conductivity. The exact calculation of the amplitude for impurity scattering leads to the appearance of non-Born terms in the conductivity when the mean free paths are long. The non-Born terms vanish in the limit of high impurity concentrations, and the well known expression[2] is obtained for the conductivity in the limit $\tau_s \to \infty$. Comparison of formulas (45) and (48) shows that at low temperatures this situation occurs only for $l < \xi_0/3$.

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*[He] ≡ H × e.

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