On the gravitational field of a massless particle

N. A. Voronov and I. Yu. Kobzarev

Institute for Experimental and Theoretical Physics
(Submitted October 10, 1973)
Zh. Eksp. Teor. Fiz. 66, 1179–1186 (April 1974)

A formula in terms of retarded potentials is derived for the gravitational field of an extended wave packet of light produced at a finite instant of time. Limiting cases of this formula are compared with the results of other papers, as well as with the result of a direct integration of the retarded potentials for a point particle, carried out by means of distribution-theory methods.

1. The problem of gravitational fields of massless particles has been discussed by several authors[1-3]. Aichelburg and Sexl[1] and Andreev[2] have considered the field of point particles created at infinity. According to relativistic quantum mechanics a zero-mass particle (in the sequel we shall call it a photon, for the sake of brevity) cannot be localized with an accuracy exceeding $\lambdabar$[4] ($\lambdabar$ is the wavelength divided by 2π), therefore the concept of point-photon is an approximate way of describing a wave packet of finite dimensions Δ. Bonnor[3] has found an exact solution for the gravitational field of a wave packet of arbitrary shape, also produced at infinity. However, the very concept of photon existing at $t=-\infty$ is an idealization; in reality one should consider a photon produced at a finite instant of time, $-T$. In the present paper we determine in the linear approximation the gravitational field of a light packet of finite size Δ produced at the time $-T$. The solution of the linearized Einstein equation was taken in the form of retarded potentials. As $T\to\infty$ the equations obtained here, after a gauge transformation, go over into the corresponding expressions of Bonnor's paper[3]. Letting Δ go to zero we then obtain a formula which coincides with the results of[1,2].

In the recent paper by Andreev[2] it is asserted that for sources moving with the speed of light the solution in the form of retarded potentials is invalid, in view of the lack of E(2)-invariance[1)]. We note however, that for an extended packet as well as for a point photon produced at a finite distance this condition should not be true, since the sources do not exhibit this invariance. This condition is valid only for a point photon produced at infinity. In integrating the retarded potentials in the latter case one runs into definite computational difficulties which can be overcome by making use of distribution theory. The retarded solution obtained in this manner exhibits E(2)-invariance.

We note that in the case of infinite T the retarded solution contains singular terms which, however, can be transformed away by a gauge transformation.

2. We write the equations of the weak gravitational field in the standard gauge[6]

$$\Box\psi_{ik}=16\pi kT_{ik},\quad \partial\psi_i^k/\partial x^k=0, \tag{1}$$

where $\psi_{ik}=h_{ik}-\frac{1}{2}\delta_{ik}h$ and T_{ik} is the energy-momentum tensor of the source. The solution of the equations (1) in the form of retarded potentials is

$$\psi_{ik}=-8k\int d^4x'\delta[(x-x')^2]\Theta(t-t')T_{ik}(x'), \tag{2}$$

where $x_i\equiv(t,\mathbf{r})$ is the event of the observer; equivalently

$$\psi_{ik}=-4k\int d^3r'\frac{T_{ik}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}. \tag{3}$$

We consider a wave packet of length Δ uniformly spread along the z axis with linear density $1/\Delta$. Let the coordinates of its endpoints be $z_1(t)=t$ and $z_2(t)=t-\Delta$. Then

$$T_{ik}=\frac{p_ip_k}{\varepsilon}\delta(\rho)\frac{1}{\Delta}[\Theta(t-z)-\Theta(t-z-\Delta)], \tag{4}$$

where $p_i=(\epsilon,0,0,-\epsilon)$ is the photon four-momentum, ϵ is its energy and $\rho=(x,y)$. In order to make the problem more definite we shall consider that the packet appears at the point $-L$. At the instant $t=-T=-L$ the anterior front appears and at $t=-L+\Delta$ the posterior front appears, so that the formation of the packet is completed. In the sequel the packet propagates according to Eq. (4). Formally the introduction of a finite point of formation of the packet leads to the appearance of the factor $\Theta(z+L)$ in T_{ik} (4). But this violates the conservation law

$$T^k_{i,k}=0, \tag{5}$$

which follows from (1). One may, however, consider a more general problem in which the total energy-momentum tensor is conserved and within the framework of which there appears the question of finding the gravitational field of a wave packet produced at a finite distance. One of the possible versions of such a problem is treated in Appendix A.

Let us now calculate the field of the wave packet. Substituting the expression (4) into (3) we have $\psi_{ik}=h_{ik}$ and

$$\psi_{ik}=-\frac{4k}{\varepsilon}p_ip_k\Phi,$$

where

$$\Phi=0\quad\text{for}\quad -L+\sqrt{(z+L)^2+\rho^2}>t,$$

$$\Phi=\frac{1}{\Delta}[\ln(t-z)-\ln(\sqrt{(z+L)^2+\rho^2}-z-L)]$$

$$\text{for } -L+\sqrt{(z+L)^2+\rho^2}<t<-L+\Delta+\sqrt{(z+L)^2+\rho^2},$$

$$\Phi=\frac{1}{\Delta}[\ln(t-z)-\ln(t-z-\Delta)]$$

$$\text{for } -L+\Delta+\sqrt{(z+L)^2+\rho^2}<t$$

or

$$\Phi=\frac{1}{\Delta}[\ln(t-z)-\ln(\sqrt{(z+L)^2+\rho^2}-z-L)][\Theta(t+L-\sqrt{(z+L)^2+\rho^2}) -\Theta(t+L-\Delta-\sqrt{(z+L)^2+\rho^2})]+\frac{1}{\Delta}[\ln(t-z) -\ln(t-z-\Delta)]\Theta(t+L-\Delta-\sqrt{(z+L)^2+\rho^2}). \tag{6}$$

Let us explain this result. For a given observer with coordinates (t,ρ,z) we call initial point of radiation z_e the site on the trajectory from which comes the excitation from the anterior front of the wave packet (we note that here we do not deal with real radiation). The coordinate z_e is determined by

$$z_e=\frac{t^2-z^2-\rho^2}{2(t-z)},\quad z_e<t.$$

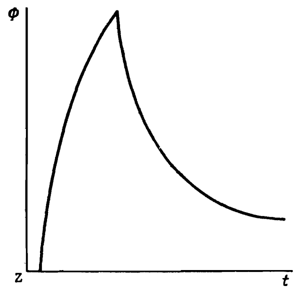

FIG. 1

For $\rho \neq 0$ the point z_e tends to $-\infty$ for $t \to z$, however, for $z < -L$ the packet did never exist, so $z_e \geq -L$. From here and from the natural condition $t \geq -L$ it follows that the signal appears at a given point of observation at time $t = -L + ((z + L)^2 + \rho^2)^{1/2}$. The other cases are considered similarly. The solution ψ_{ik} determined this way is finite and everywhere continuous, with the exception of the z axis.

As will be shown in Appendix A, for $z + L \gg \Delta$ and $\rho \ll z + L$ one may neglect the influence of concrete processes accompanying the production of the packet. Under these conditions the solution (6) will have the form

$$\Phi=\frac{1}{\Delta}\left[\ln(t-z)-\ln\frac{\rho^2}{2(z+L)}\right]\left[\Theta\left(t-z-\frac{\rho^2}{2(z+L)}\right)-\Theta\left(t-z-\Delta-\frac{\rho^2}{2(z+L)}\right)\right]+\frac{1}{\Delta}[\ln(t-z)-\ln(t-z-\Delta)]\Theta\left(t-z-\Delta-\frac{\rho^2}{2(z+L)}\right). \tag{7}$$

We describe the behavior of (7) at an arbitrary observation point (ρ, z), $\rho \neq 0$ as a function of time (cf. Fig. 1). The signal appears at a given point at the instant $t = z + \rho^2/2(z + L)$ and further increases from zero, attaining a maximum for $t = z + \Delta + \rho^2/2(z + L)$. At that point (7) takes on the value $\ln[1 + 2\Delta(z + L)/\rho^2$. For $t > z + \Delta + \rho^2/2(z + L)$ the field (7) starts decreasing and for $t \gg z + \Delta + \rho^2/2(z + L)$ it tends to zero as $1/(t - z)$.

3. We now consider different limiting cases of the expression (7). For finite Δ and $L \to \infty$ we obtain

$$\Phi=\frac{1}{\Delta}[\Theta(t-z)-\Theta(t-z-\Delta)][\ln(t-z)-\lim_{L\to\infty}\ln\rho^2/2L]+\frac{1}{\Delta}[\ln(t-z)-\ln(t-z-\Delta)]\Theta(t-z-\Delta). \tag{8}$$

This equation contains singular terms, but they depend only on $(t - z)$ and therefore can be removed by means of a gauge transformation. Indeed, the part of ψ_{ik} which depends only on $t - z$, or for arbitrary direction of p_i

$$\psi_{ik}=p_ip_kF(x_jp^j), \tag{9}$$

does not contribute to the Riemann tensor calculated in the approximation linear in h_{ik}. This means that the field (9) is not physical and can be transformed away by means of a gauge transformation

$$h'_{ik}=\psi'_{ik}=\psi_{ik}+\frac{\partial\xi_i}{\partial x^k}+\frac{\partial\xi_k}{\partial x^i},\quad \xi_i=-\frac{1}{2}p_i\int^{x_jp^j}F(y)\,dy.$$

After the gauge transformation, (8) takes the form

$$\Phi'=-\frac{1}{\Delta}[\Theta(t-z)-\Theta(t-z-\Delta)]\ln\rho^2. \tag{8'}$$

This equation coincides with the exact solution determined in[3]. Letting Δ go to zero we obtain a solution corresponding to a point particle[1-3]:

$$\Phi=-\delta(t-z)\ln\rho^2.$$

If at the point $z = L$ one installs an absorbing wall and lets the length Δ of the packet go to infinity, leaving its linear density finite and constant, then after the anterior front passes we are in the situation which was previously considered by Tolman, Ehrenfest, and Podolsky[7], who have computed the gravitational field of a stationary beam of light emitted at the point $z = -L$ and absorbed at the point $z = L$ of the z axis.

4. We now consider a point photon moving along the z axis: $x = 0$, $y = 0$, $z = t$; then

$$T_{ik}=\frac{p_ip_k}{\varepsilon}\delta(t-z)\delta(\rho). \tag{10}$$

Substituting (10) into (2) and carrying out the transformations we obtain

$$\psi_{ik}=-\frac{4k}{\varepsilon}p_ip_k\int_{-(t-z)}^{\infty}d\tau\,\delta[(t-z)\tau-\rho^2]=-\frac{4k}{\varepsilon}p_ip_k\lim_{L\to\infty}\int_{-(t-z)}^{2L}d\tau\,\delta[(t-z)\tau-\rho^2]. \tag{11}$$

We have introduced the limiting process in order to achieve a junction with the solution obtained in Sec. 2. The generally accepted rules of operation with delta functions do not allow us to determine the field ψ_{ik} in the $z = t$ plane: we find that $\psi_{ik} = 0$ for $z \geq t$, $\psi_{ik} \sim 1/(t - z)$ for $t > z$ and ψ_{ik} is undetermined for $z = t$.

However, the integral (11) can be calculated for $\rho \to 0$ if one uses the generalized function methods developed by Gel'fand and Shilov[8]. Without dwelling on the derivation (cf. Appendix B) we give here the formula for $\delta(xy - c)$:

$$\delta(xy-c)=-2\ln c\,\delta(x,y)+\frac{\delta(x)}{|y|_R}+\frac{\delta(y)}{|x|_R}+o(c), \tag{12}$$

where $1/|x|_R$ and $1/|y|_R$ are regularizations[2)] of the functions $1/|x|$ and $1/|y|$ and $o(c)$ are the remaining terms which vanish for $c \to 0$. A regularization of the function $1/|x|$ is the function defined by the following linear functional:

$$\int_{-\infty}^{+\infty}\frac{dx}{|x|_R}\varphi(x)=\int_0^{\infty}\frac{\varphi(x)+\varphi(-x)-2\varphi(0)\Theta(1-x)}{x}dx. \tag{13}$$

Here $\varphi(x)$ is a test function. We note that the one inside the Θ-function is essential. If $\varphi(x)$ is such that $\varphi(0) = 0$ (13) coincides with the integral for the usual function $1/|x|$. Therefore everywhere except at the origin $1/|x|_R$ coincides with $1/|x|$. In the expression for $\delta(xy - c)$ obtained in[8] the leading term of the expansion with respect to c coincides with (12), the terms which do not depend on c being erroneous.

Making use of (12) we obtain an expression for the integral (11) in the following form:

$$-2\ln c\,\delta(t-z)\int_{-(t-z)}^{\infty}d\tau\,\delta(\tau)+\frac{1}{|t-z|_R}\int_{-(t-z)}^{\infty}d\tau\,\delta(\tau)+\delta(t-z)\lim_{L\to\infty}\int_{-(t-z)}^{2L}\frac{d\tau}{|\tau|_R}.$$

In the first term the presence of $\delta(t - z)$ allows us to change the lower limit to zero. We further note that

$$\int_0^{\infty}d\tau\,\delta(\tau)={}^1/_2$$

(this is obtained most simply by considering $\delta(\tau)$ as a weak limit of a smeared sequence near the origin). In the second term for $t < z$ the domain of integration does not comprise the point $\tau = 0$, therefore it vanishes. For $t > z$

$$\int_{-(t-z)}^{\infty} d\tau\delta(\tau)=1, \quad \frac{1}{|t-z|_R}=\frac{1}{t-z}.$$

In the third term we retain the upper limit, since for $L = \infty$ the integral diverges, which is related to the fact that formally integrals over generalized functions have to be taken with an integrable weight $\varphi(x)$. For finite L this integral equals

$$\int_{-(t-z)}^{2L} \frac{d\tau}{|\tau|_R}=-|\ln(t-z)|+\ln 2L$$

Thus, we have obtained $\psi_{ik} = -4k\epsilon^{-1}p_i p_k \Phi$,

$$\Phi=-\delta(t-z)\ln\rho^2+\frac{\Theta(t-z)}{t-z}+\delta(t-z)\lim(-|\ln(t-z)|+\ln 2L). \quad (14)$$

In the third term the coefficient in front of $\delta(t-z)$ is undetermined since $\ln(t-z) \to -\infty$ for $t \to z$. This indeterminacy is removed as

$$\lim_{L\to\infty}(\ln\frac{a^2}{2L}+\ln 2L)=\ln a^2,$$

where a is an arbitrary constant. Formally (14) coincides with the limit of Φ in((8) as $L \to \infty$ and $\Delta \to 0$.

In conclusion we note that in[1] the field of a point photon was obtained from the exact expression for the field of a point particle with mass by means of a Lorentz transformation and the limiting procedure $v \to 1$ and $m \to 0$. However, the field of a particle with mass $m \neq 0$ does not go over into the retarded solution for $m = 0$, since it does not vanish for $t < z$. We note that a similar situation occurs for the electromagnetic field of a massless particle[9].

The authors express their gratitude to M. I. Graev for a useful discussion.

APPENDIX A

We consider the decay of a point massive particle at rest at the origin into two packets moving along the z-axis in opposite directions. The tensor T_{ik} for this problem is

$$\begin{aligned}T_{ik}=&\frac{p_ip_k}{\varepsilon}\delta(\mathbf{r})\left\{[1-\Theta(t)]-\frac{1}{\Delta}[\Theta(t)-\Theta(t-\Delta)](t-\Delta)\right\}\\&+\frac{p_{1i}p_{1k}}{\varepsilon}\delta(\rho)\Theta(z)\frac{1}{\Delta}[\Theta(t-z)-\Theta(t-z-\Delta)]\\&+\frac{p_{2i}p_{2k}}{\varepsilon}\delta(\rho)[1-\Theta(z)]\frac{1}{\Delta}[\Theta(t+z)-\Theta(t+z-\Delta)],\end{aligned} \quad (1A)$$

where $p_i = (2^{1/2}\epsilon, 0, 0, 0)$, $p_{1i} = (\epsilon, 0, 0, -\epsilon)$, $p_{2i} = (\epsilon, 0, 0, \epsilon)$. It is easy to verify that (1A) satisfies the condition (5). The first term in (1A) describes a body of variable mass at rest at the origin. The second and third terms are wave packets moving in opposite directions away from $z = 0$. By a shift $t \to t + L$ and $z \to z + L$ the second term becomes Eq. (4). Integrating (3) with T_{ik} from (1A), we obtain the following expression for the field ψ_{ik}:

$$\begin{aligned}\psi_{ik}=&-\frac{p_ip_k}{\varepsilon}\frac{4k}{r}\left\{[1-\Theta(t-r)]-\frac{1}{\Delta}[\Theta(t-r)-\Theta(t-r-\Delta)](t-r-\Delta)\right\}\\&-\frac{p_{1i}p_{1k}}{\varepsilon}\frac{4k}{\Delta}\{[\ln(t-z)-\ln(r-z)][\Theta(t-r)-\Theta(t-r-\Delta)]\\&+[\ln(t-z)-\ln(t-z-\Delta)]\Theta(t-r-\Delta)\}-\\&-\frac{p_{2i}p_{2k}}{\varepsilon}\frac{4k}{\Delta}\{[\ln(t+z)-\ln(r+z)][\Theta(t-r)-\Theta(t-r-\Delta)]\\&+[\ln(t+z)-\ln(t+z-\Delta)]\Theta(t-r-\Delta)\}.\end{aligned} \quad (2A)$$

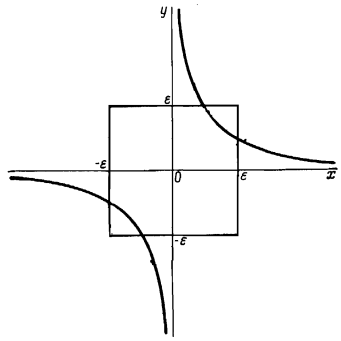

FIG. 2

At $z \gg \Delta$ and $\rho \ll z$ the first and third terms are small compared with the second and therefore in that region it makes sense to talk about the gravitational field of a wave packet produced at a finite distance.

APPENDIX B

Equations (12) and (13) can be obtained either directly by means of the methods of[8] or by the method described below. We consider the integral

$$\int\delta(xy-c)f(x,y)\,dx\,dy, \quad c>0,$$

where $f(x, y)$ is an arbitrary test function. Integrating with respect to y, we obtain

$$\int_{-\infty}^{+\infty} dx\frac{f(x,c/x)}{|x|}. \quad (1B)$$

We introduce an auxiliary function $f_\epsilon(x, y)$ of the form

$$f_\varepsilon(x,y)=\begin{cases}f(0,0) & \text{for } |x|,\ |y|\leqslant\varepsilon\\ f(x,y) & \text{for } |x|,\ |y|>\varepsilon\end{cases};$$

with ϵ chosen such that $c < \epsilon^2$ (cf. Fig. 2). For $c \to 0$, $\epsilon \to c^{1/2}$ the function f_ϵ converges to $f(x, y)$. Therefore, by this method we obtain in the expansion of $\delta(xy - c)$ only the singular term and the terms which do not depend on c. The integral over the region $|x|$, $|y| \leq \epsilon$ can be calculated explicitly

$$\int_{|x|,|y|\leqslant\varepsilon} f_\varepsilon(x,y)\delta(xy-c)\,dx\,dy=2f(0,0)(2\ln\varepsilon-\ln c). \quad (2B)$$

The part of the integral (1B) corresponding to integration over the second region decomposes into four terms

$$\int_{\varepsilon}^{\infty}\frac{f(x,c/x)}{x}dx+\int_{0}^{c/\varepsilon}\frac{f(x,c/x)}{x}dx+\int_{-\infty}^{-\varepsilon}\frac{f(x,c/x)}{|x|}dx+\int_{-c/\varepsilon}^{0}\frac{f(x,c/x)}{|x|}dx. \quad (3B)$$

In the first term $x > \epsilon$; since $c < \epsilon^2$ we obtain from here that $c/x < \epsilon$. For $c \to 0$ the value of ϵ can be chosen sufficiently small so that $f(x, c/x)$ can be expanded in a power series. As a result we obtain

$$\int_{\varepsilon}^{\infty}\frac{f(x,0)}{x}dx.$$

The next term of the expansion is of order of c. We make the change of variable $c/x = y$ in the second term. The result is

$$\int_{0}^{c/\varepsilon}\frac{f(x,c/x)}{x}dx=\int_{\varepsilon}^{\infty}\frac{f(c/y,y)}{y}dy=\int_{\varepsilon}^{\infty}\frac{f(0,y)}{y}dy+o(c).$$

The other terms are treated similarly. If in (2B) we replace $\ln\epsilon$ by

$$-\int_{\varepsilon}^{1}\frac{dx}{x},$$

we obtain, adding (2B) and (3B)

$$-2\ln cf(0,0)+\int_\epsilon^1 \frac{f(x,0)+f(-x,0)-2f(0,0)}{x}dx+\int_1^\infty \frac{f(x,0)+f(-x,0)}{x}dx$$
$$+\int_\epsilon^1 \frac{f(0,y)+f(0,-y)-2f(0,0)}{y}dy+\int_1^\infty \frac{f(0,y)+f(0,-y)}{y}dy.$$

Finally, we replace the integration limit ϵ by zero, so that the error is of order

$$\int_0^\epsilon \frac{f(x,0)+f(-x,0)-2f(0,0)}{x}dx\sim\epsilon^2\sim c \qquad (4B)$$

and vanishes for c, $\epsilon \to 0$. Taking this into account (4B) is equivalent to Eqs. (12), (13).

[1)]E(2) is the subgroup of the Lorentz group ("little group") which leaves an isotropic vector invariant.

[2)]We use the terminology introduced in [8].

[1]P. C. Aichelburg and R. U. Sexl, Gen. Rel. Grav. 2, 303 (1971).

[2]A. F. Andreev, ZhETF Pis. Red. 17, 424 (1973) [JETP Lett. 17, 303 (1973)].

[3]W. B. Bonnor, Commun. Math. Phys. 13, 163 (1969).

[4]V. B. Berestetskiĭ, E. M. Lifshitz and L. P. Pitaevskiĭ, Relyativistskaya kvantovaya teoriya (Relativistic Quantum Theory) p. I, Nauka 1968, p. 15 [Engl. Transl. Addison-Wesley-Pergamon].

[5]I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, Predstavleniya gruppy vrashcheniĭ i guppy Lorentza (Representations of the Rotation Group and of the Lorentz Group), Fizmatgiz 1958, p. 336 [Engl. Transl. Pergamon].

[6]L. D. Landau and E. M. Lifshitz, Teoriya polya (Classical Field Theory), 4-th ed., Nauka, 1967, p. 418 [Engl. Transl. Addison-Wesley-Pergamon].

[7]R. Tolman, P. Ehrenfest, and B. Podolsky, Phys. Rev. 37, 602 (1931).

[8]I. M. Gel'fand and G. E. Shilov, Obobshennye funktsii i deĭstviya nad nimi (Generalized functions and operations on them) Fizmatgiz, 1959, p. 408 [Engl. Transl. Generalized Functions vol I, Academic Press, 1961].

[9]V. G. Vaks, Zh. Eksp. Teor. Fiz. 40, 792 (1961) [Sov. Phys. JETP 13, 556 (1961).

Translated by M. E. Mayer
121