The interaction of short gravitational waves with electromagnetic waves in arbitrary external electromagnetic fields

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(Submitted October 10, 1973)
Zh. Eksp. Teor. Fiz. **66**, 1187–1199 (April 1974)

We show that the propagation of short electromagnetic and gravitational waves occurs as a process of successive mutual transformation of these waves into one another, for external electromagnetic fields with the principal axes transported parallel along the rays carrying the waves. In this case the polarization plane of the waves along the rays does not rotate. An analysis is carried out of wave propagation in a Nordstrøm–Reissner field. In the general case the polarization plane of initially linearly polarized short waves does rotate, but the total intensity of the waves along an isotropic tube along the ray is conserved. The propagation of short waves in a Kerr–Newman field is described. We show that the rotation angle of the polarization plane relative to an isotropic tetrad (which is parallel transported along the rays) will be small compared to the rotation of the tetrad itself (owing to rotation of the black hole) for large impact parameters of the rays. However, both angles will be comparable for near-critical impact parameters of rays not situated in the equatorial plane and for finite charges of the black hole.

Owing to the nonlinearity of the Einstein-Maxwell equations gravitational fields in vacuum always interact with one another. Of particular interest is the interaction of fields having the character of waves. Wave properties are exhibited by solutions having different scales of variation in different regions of space-time; in this case a region with smooth variation of the solution is naturally termed background and regions with "abrupt" variations of the solution will be called waves. If the background is homogeneous, the propagation of the waves does not depend on their degree of inhomogeneity. This is true of most of the known exact solutions with singled-out algebraic properties. An isolated act of transformation of electromagnetic waves into gravitational waves and vice versa, in the presence of a static transverse magnetic or electric field has been considered in[1-4]. In the general case only solutions which vary abruptly compared to the background can make their way along null-geodesics through the inhomogeneities of the background[1)]. In this case the regions of abrupt change of the solution will be concentrated along the characteristic isotropic cones with vertices in the initial inhomogeneity regions. An example of such solutions are the rapidly oscillating paths of gravitational waves or discontinuities of order ≥ 1 of gravitational fields[7,8,5].

For such solutions (of the type of a traveling wave) in the absence of background electromagnetic fields, propagation according to the laws of geometric optics along null-geodesics in the curved background space is a characteristic feature[7,8]. From the "transport equations" for the wave amplitudes it follows that the intensity (brightness) of the radiation is inversely proportional to the area element of the wave front, area element which is determined by the intersection of the wave fronts with the same rays. The polarization of the waves is covariantly conserved along rays that carry the wave and which "generate" an isotropic wave front surface in 4-space. When a small gravitational wave amplitude becomes comparable with the wavelength (in units where the vacuum velocity of light has been set equal to one), then trains of short gravitational waves start to bend the underlying background and it becomes necessary to consider the process of wave propagation together with the problem of determining the background field.[8,9]

We show below that in the propagation of interacting short gravitational and electromagnetic waves in arbitrary external electromagnetic fields there appear essentially new effects. Any of these waves causes the appearance of the other and the propagation occurs with mutual modulation of the wave amplitudes.[2)] For sources of external electromagnetic fields concentrated in compact regions, the waves experience a finite number of acts of mutual transformation.

The total intensity of the gravitational and electromagnetic waves obeys a continuity equation with an isotropic velocity field along the characteristic rays that carry the wave. In the quasiclassical approximation developed below one may talk of a gas of photons and gravitons transforming into one another. The total distribution function of these particles will obey a Liouville equation[9]. The Einstein equations for the background will involve the energy-momentum tensor of the photons and gravitons as the mean-square of the "noise" of the waves (Section 1). Therefore the reaction of the waves on the background manifests itself in exactly the same manner as in the absence of external electromagnetic fields[8,9].

Press and Thorne[11] have qualitatively transferred the results of Gertsenshteĭn[1] and Vladimirov[2] to waves in the field of charged black holes. Section 2 contains a study of the propagation of short waves in a Nordstrøm-Reissner field; in particular, we calculate the amplitude of the appearing gravitational component in the reflected wave as a function of the impact parameter of the incident of the electromagnetic wave, as well as of the charge and mass of the black hole.

For the special case of constant transverse external fields[1-4,10] the curious effect of rotation of the polarization plane of an initially linearly polarized wave relative to a tetrad which is parallel-transported along a ray is absent.

In Sec. 3 we describe the propagation of short waves in the field of a rotating black hole (the Kerr-Newman solution[12]), where such an effect occurs. In the general case the intensity of the electromagnetic wave is related to the rotation of its polarization plane.

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1. DERIVATION OF THE EQUATIONS FOR THE AMPLITUDES OF INTERACTING WAVES AND THE NONLINEAR REACTION OF THE WAVES ON THE BACKGROUND

We shall search for an asymptotic solution of the Einstein-Maxwell equations in vacuum in the form of formal expansions of the metric g'_{ij} and of the electromagnetic field bivector (2-form) F'_{ij}

$$g_{ij}'=g_{ij}^{(0)}+\sum_{l=1}^{\infty}\omega^{-l}[A_{ij}^{(l)}\cos(\omega sl)+B_{ij}^{(l)}\sin(l\omega s)], \quad (1.1)$$

$$F_{ij}'=F_{ij}^{(0)}+\omega\sum_{l=1}^{\infty}\omega^{-l}[C_{ij}^{(l)}\cos(l\omega s)+D_{ij}^{(l)}\sin(l\omega s)]. \quad (1.2)$$

Here the functions $A_{ij}^{(1)}$, $B_{ij}^{(1)}$, $C_{ij}^{(1)}$, and $D_{ij}^{(1)}$ are expansions in the reciprocal powers of ω starting from zero; the background fields $g_{ij}^{(0)}$ and $F_{ij}^{(0)}$ are expansions in the squares of reciprocal powers of ω.

We denote the leading terms in the expansions $g_{ij}^{(0)}$, $F_{ij}^{(0)}$, $A_{ij}^{(1)}$, $B_{ij}^{(1)}$, $C_{ij}^{(1)}$, $D_{ij}^{(1)}$ respectively by g_{ij}, F_{ij}, $h_{ij(1)}$, $h_{ij(2)}$, $f_{ij(1)}$, $f_{ij(2)}$. Substituting the expansions (1.1), (1.2) into the system of Einstein-Maxwell equations, we verify its self-consistency, in view of the compatibility of the system of equations obtained by equating the coefficients of the same harmonics and the same reciprocal powers of ω. Equating to zero the terms of order ω of the first harmonic, we obtain an eikonal equation for the function s: $g^{ij}s_{,i}s_{,j} = 0$ and the following algebraic restrictions on $h_{ij(A)}$, $f_{ij(A)}$ (A = 1, 2):

$$s_{,i}h^{i}_{j(A)}-s_{,j}h^{k}_{k(A)}/2=0; \quad s_{,i}f^{ij}_{(A)}=0. \quad (1.3)$$

For simplicity we restrict ourselves in the sequel, without contradiction, to the case $f_{ij(1)} = h_{ij(2)}$, which in the absence of interaction corresponds to linearly polarized waves, and we omit the additional index A. The wave fronts s = const define a family of rays $l_i \equiv s_{,i}$ tangent to the congruence of null-geodesics without rotation in the background space. For our purpose the formalism of optical frames of Newman-Penrose[13] is particularly convenient. We shall assume that in the region filled by the rays the field of null-frames is obtained by parallel transport along the rays l_i from the field onto arbitrary hypersurfaces, which are intersected only once by each ray. The isotropic basis vectors l, n, m, m* of the frames satisfy by definition the relations $l_i m^i = n_i m^i = 0$, $l_i n^i = -m_i m^{*i} = 1$. Let the background electromagnetic field have in the indicated field of tetrads the components

$$F_{ij}=\varphi_0 U_{ij}+\varphi_1 M_{ij}+\varphi_2 V_{ij}+\text{c.c.} \quad (1.4)$$

Here

$$V_{ij}=2l_{[i}m_{j]}, \quad M_{ij}=2(n_{[i}l_{j]}+m_{[i}m^*_{j]}), \quad U_{ij}=2m^*_{[i}n_{j]}$$

form a basis in the complex 3-space of self-dual bivectors $\tilde{F}_{ik} = F_{ik} + i\epsilon_{iklm}F^{lm}/2$.

The function φ_0 characterizes the radiation of the external field along rays which carry the short waves, since the square of its absolute value is proportional to the flux of energy-momentum of the external field along the direction l_i:

$$|\varphi_0|^2=4\pi T_{ij}l^i l^j,$$

where T_{ij} is the energy momentum tensor of the external field. All the peculiarities of behavior of the short waves described below are related to the component φ_0.

As shown in the Appendix, the variation of the argument of the component φ_0 along the rays l_i is expressed in terms of the Ricci rotation coefficients of the principal axes of the bivector describing the external electromagnetic field[3]. It follows from Eqs. (1.3) that the coefficients in front of the first harmonic of the perturbations in the basis l, m, m*, n have the form

$$h_{ij}=Pm_i m_j+P^* m_i^* m_j^*+2l_{(i}A_{j)}; \quad f_{ij}=fV_{ij}+\text{c.c.} \quad (1.5)$$

Here the functions P and f are the absolute values of the amplitudes of the gravitational and electromagnetic waves, and their argument characterizes the polarization of the corresponding waves. The term $l_{(i}A_{j)}$ is related to the selection of the coordinate grid and can be made to vanish by means of the coordinate transformation $x^{i'} = x^i - \sin(\omega s) A^i \omega^{-1}$.

Equating to zero the totality of nonoscillating terms with zero power of ω in the Einstein equations $R'_{ij} = \kappa T'_{ij}$ (here T'_{ij} is the energy-momentum tensor of the electromagnetic field) into which the expansions (1.1) and (1.2) have been substituted, we obtain

$$R_{ij}=\varkappa T_{ij}+s_{,i}s_{,j}\left(\frac{|P|^2}{4}+\frac{G}{c^4}|f|^2\right), \quad (1.6)$$

where G is the gravitational constant. Equation (1.6) describes the curving of the background along which the short waves travel. In order to close the system (1.6) it is necessary to obtain equations for the complex amplitudes P and f. For this purpose we equate to zero in the Einstein equations the coefficients of the first harmonic and zeroth power of ω and then take its tetrad component (m*, m*) (i.e., we carry out a contraction of the equations so obtained with $m^{*i}m^{*j}$). Using

$$(T_{ij}'-T_{ij})m^{*i}m^{*j}\approx(4\pi)^{-1}f\varphi_0^*,$$

we obtain easily

$$l^k\nabla_k P+\frac{1}{2}(\nabla_k l^k)P=\frac{2G}{c^4}f\varphi_0^*. \quad (1.7)$$

In the same way, equating to zero the coefficients of the first harmonic and the zeroth power of ω in the Maxwell equations $\nabla_i\tilde{F}^{ij} = 0$ we take the tetrad component m*, obtaining

$$l^k\nabla_k f+{}^1/_2 f\nabla_k l^k+{}^1/_2\varphi_0 P=0. \quad (1.8)$$

From (1.7) and (1.8) it is easy to derive the continuity equation for the total brightness (energy) of the electromagnetic and gravitational waves:

$$\nabla_i[l^i(c^4|P^2|+4G|f|^2)]=0. \quad (1.9)$$

In the isotropic-geodesic system constructed on the rays l_i:

$$dS^2=2ds\,d\alpha+\hat{g}_0 ds^2+\hat{g}_{\mu\nu}(d\xi^\mu+\hat{g}^\mu ds)(d\xi^\nu+\hat{g}^\nu ds); \quad \mu,\nu=1,2$$

(α is the affine parameter along the rays l_i), it follows from Eq. (1.9) that

$$c^4|P^2|+4G|f|^2=(\hat{g})^{-1/2}\sigma(s,\xi^1,\xi^2).$$

Therefore along the rays the following quantity is conserved

$$(c^4|P^2|+4G|f|^2)\sqrt{\hat{g}}\,d\xi^1 d\xi^2.$$

But $\sqrt{\hat{g}}\,d\xi^1 d\xi^2$ has the meaning of the elementary area of the wave-front surface subtended by the same rays.

Therefore the total intensity of the waves is inversely proportional to the elementary area of the wave front. The continuity equation (1.9) for the total intensity implies that mutually related rays of electromagnetic and gravitational waves act on the background in the same manner as "pure" gravitational waves in the absence of external electromagnetic fields [8,9]. As was shown in [9], in this case one can introduce for the null-particles a distribution function that is subject to a Liouville equation.

The largest intensity (brightness) is attained by the waves in focal points (on caustics), where $\hat{g} = 0$ (focal points appear unavoidably as a consequence of the Einstein equations for any normal congruence of null-geodesics with nonzero convergence [15], the so-called Landau-Raychaudhury effect). At focal points the geometrical optics approximation becomes useless, since it leads to fictitious singularities for the wave amplitudes. The leading terms of the asymptotic behavior on the caustics for large ω can be obtained in the following manner. The algebraic conditions (1.3) for the leading terms of the expansions (from which we will now not separate the rapidly oscillating factor $e^{i\omega s}$) will be considered valid as before. In the approximation of geometric optics the operator $l^i\nabla_i + \frac{1}{2}\nabla_i l^i$ which occurs in the left-hand sides of (1.7) and (1.8) is equivalent to the D'Alembertian, however the latter is meaningful also on the caustics, where the former loses its meaning. Therefore, near the caustics the leading terms of the asymptotics for large ω will be determined by means of expansions in fractional powers of the wavelength $1/\omega$ from the equations

$$\Box P = \frac{4G}{c^4}\omega f \varphi_0^*, \quad \Box f = \varphi_0 \omega P. \tag{1.10}$$

The character of interaction of the waves, according to (1.10), near simple focal points as well as near multiple foci (unstable closed light rays) is described on the example of waves propagating in the Nordstrøm-Reissner field in Section 2.

2. PROPAGATION OF SHORT WAVES IN THE NORDSTRØM–REISSNER FIELD

On account of Eqs. (1.7), (1.8) a change of the phase of φ_0 induces a self-consistent change of phase of the functions f and P, i.e., a rotation of the polarization plane of the electromagnetic and gravitational waves. The condition that the argument of φ_0 be constant along each null-geodesic imposes restrictions on the structure of space-time (cf. the Appendix).

In Eqs. (1.7) and (1.8) we make the substitution

$$f = \left(\frac{4G}{c^4}\right)^{1/2} \mathscr{F}(\hat{g})^{-1/4}, \quad P = \mathscr{P}(\hat{g})^{-1/4}, \quad \varphi = \varphi_0 \left(\frac{G}{c^4}\right)^{1/2};$$

we then obtain

$$\frac{d}{d\alpha}\mathscr{P} = \varphi^* \mathscr{F}; \quad \frac{d}{d\alpha}\mathscr{F} = -\varphi \mathscr{P}, \tag{2.1}$$

where α is the affine parameter along the ray.

In those cases when the principal axes of the external electromagnetic field do not rotate along the rays that carry the waves, one can use the rotation $m' = e^{i\chi} m$ to make the function φ_0 real everywhere along the ray without violating the parallel-transport property of m along the ray. In this case the solution of the system (2.1) is of the form

$$\mathscr{F} = A\cos\left(\int \varphi\, d\alpha + \gamma\right), \quad \mathscr{P} = A\sin\left(\int \varphi\, d\alpha + \gamma\right). \tag{2.2}$$

(The complex number A and the phase shift γ are constant along a fixed ray.) From the form of the solutions (2.2) it follows that the amplitudes of the electromagnetic and gravitational wave turn out to be sinusoidally modulated with a frequency which can be determined from the equation $2\pi = \int \varphi d\alpha$. Before reaching the region with a strong electromagnetic field $|\tilde{\varphi}| \sim 1$, the gravitational and electromagnetic wave from a single source propagate independently, having identical wavefronts, but in general different polarizations. Then in the region $|\varphi| \sim 1$ these waves suffer only partial mutual transformations, with the polarization plane of each suffering a rotation relative to a tetrad which is parallel-transported along the rays. Formally this corresponds to complex phases in the expressions (2.2). The case of real γ corresponds to the fact that either the polarization plane of the gravitational and electromagnetic short waves coincides before entering the region with $|\varphi| \sim 1$, or initially only one of these waves was incident. In this case the solutions (2.2) characterize the complete mutual transformation over a length of the period $\int \varphi d\alpha = 2\pi$. If φ_0 is real along the rays that carry the wave, the behavior of the short waves at the focal points is described in a particularly simple way. In this case the equations (1.10) decompose into two independent equations of second order for the linear combinations $f \pm (c^4/4G)^{1/2} P \equiv \chi_\pm$:

$$\Box \chi_\pm \pm 2\omega\varphi\chi_\pm = 0. \tag{2.3}$$

In the case of a charged black hole without rotation (the Nordstrøm-Reissner solution) the only nonvanishing component of the electromagnetic field is $F_{0r} = e/r^2$, where e is the charge of the black hole. The Nordstrøm-Reissner metric takes the form

$$dS^2 = Adt^2 + A^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$A = 1 - r_g/r + Ge^2/c^4r^2.$$

It is easy to verify that in this case the component φ_0 of the electromagnetic field is real for any geodesic. Therefore

$$\varphi^2 = 4\pi Gc^{-4} T_{ij} l^i l^j = Gc^{-4} e^2 A r^{-6} [(s_{,\theta})^2 + (s_{,\varphi})^2/\sin^2\theta]. \tag{2.4}$$

The eikonal equation in the Nordstrøm-Reissner metric admits a complete integral of the form

$$s = t \pm R(r) + \psi(\theta) + N\varphi, \quad R(r) = \int A^{-1}(1 - \lambda^2 A/r^2)^{1/2} dr,$$
$$\psi(\theta) = \int (\lambda^2 - N^2/\sin^2\theta)^{1/2} d\theta.$$

From (2.4) it follows that

$$\varphi = \sqrt{G}\lambda e/c^2 r^3. \tag{2.4'}$$

The affine parameter α is conveniently replaced by the radial coordinate r by means of the equation $g^{ij}s_{,j} = dx^i/d\alpha$, hence

$$d\alpha = dr(1 - A\lambda^2/r^2)^{-1/2}.$$

Therefore the period of the modulating sinusoid can be determined as a function of the radius from the equation

$$2\pi = \int \varphi\, d\alpha = e\lambda\sqrt{G}c^{-2}\int dr\, r^{-3}(1 - \lambda^2 A/r^2)^{-1/2}. \tag{2.5}$$

It follows from this equation that the mutual transformation effect depends on the impact parameter of the ray. The waves captured by the black hole have impact parameters such that the equation $1 - A\lambda^2/r^2 = 0$ has no real roots. The corresponding values of λ satisfy the inequality $\lambda < \lambda_{cr}$:

$$\lambda_{cr}^2 = r_g^2[x^2 + 5/2 + \sqrt{1+8x^2} + (8x^2)^{-1}(\sqrt{1+8x^2} - 1)],$$

$$x^2 = 1 - Ge^2/M^2$$

(for e = 0 we have $\lambda_{cr} = (27)^{1/2} r_g/2$ for $eG^{1/2} = M$ we have $\lambda_{cr} = 2r_g$).

At $\lambda > \lambda_{cr}$ the short waves are reflected ideally from the first turning point r_1 which they encounter, where r_1 is a root of the equation $1 - A\lambda^2/r^2 = 0$. If an uncaptured electromagnetic wave is incident with amplitude B on the black hole from $+\infty$, after suffering several acts of mutual transformation into gravitational waves in the field of the black hole this wave will go off to infinity in the form of an electromagnetic wave and a produced gravitational wave with the amplitudes given respectively by

$$f = B\cos\left(2e\lambda G^{1/2}c^{-2}\int_{r_1}^{\infty} dr\, r^{-3}(1-\lambda^2 A/r^2)^{-1/2}\right), \qquad (2.6)$$

$$P = 2c^{-2}BG^{1/2}\sin\left(2e\lambda G^{1/2}c^{-2}\int_{r_1}^{\infty} dr\, r^{-3}(1-\lambda^2 A/r^2)^{-1/2}\right). \qquad (2.7)$$

The expressions (2.6) and (2.7) diverge when $1 - \lambda^2 A/r^2 = 0$ has a multiple root, i.e., when $\lambda = \lambda_{cr}$. The corresponding value of the impact parameter belongs to a ray which winds itself onto an unstable limit cycle: a closed circular orbit of particles of mass zero, with radius

$$r_{cr} = r_g(3+\sqrt{1+8x^2})2^{-2}$$

(for e = 0 the radius $r_{cr} = 3r_g/2$, and $r_{cr} = r_g$ for $eG^{1/2} = M$).

In the neighborhood of the limit cycle the geometrical optics approximation becomes invalid, there occurs a penetration of the short waves through the potential barrier with comparable reflection and transmission coefficients[16].

Equations (2.3) for the Nordstrøm-Reissner field become, after expanding in spherical harmonics $Y_{lm}(\theta, \varphi)$ and a Fourier transform with respect to time,

$$A\frac{\partial}{\partial r}\left(A\frac{\partial}{\partial r}r'\chi_\pm\right) + r\chi_\pm\left[\omega^2 - A\left(\frac{(l+1)l}{r^2} \mp \left(\frac{G}{c^4}\right)^{1/2}\frac{e\lambda\omega}{r^3}\right)\right] = 0. \qquad (2.8)$$

The ratio $((l+1)l)^{1/2}/\omega$ has the meaning of an impact parameter λ. Near the turning points r_1, Eqs. (2.8) reduce to the Airy equation

$$\frac{r_1^4}{\lambda^4}\frac{d^2}{dr^2}\chi_\pm + \chi_\pm\omega^2\left[(r-r_1)\frac{4r_1^2 - 2\lambda^2 r_1 + r_g\lambda^2}{r_1^4} \pm \left(\frac{G}{c^4}\right)^{1/2}\frac{e\lambda}{r_1^3\omega}\right] = 0.$$

Therefore on the simple caustics the amplitude of the wave increases $\omega^{1/6}$ times compared to the ordinary points, where the Airy functions go over into the WKB solution of the appropriate equation of (2.8) by means of the stationary-phase method (the general case of the behavior of waves near simple caustics, in particular the field discontinuities, is described by means of solutions of the Tricomi equation, solutions which for periodic waves go over into Airy functions[17,5]; cf. also[14], Sec. 59).

For rays with a near-critical impact parameter $|\lambda - \lambda_{cr}| \sim O(1/\omega)$ near the closed ray $|r - r_{cr}| \sim O(1/\omega^{1/2})$, Eqs. (2.8) reduce to the parabolic cylinder equation:

$$\frac{r_{cr}^4}{\lambda_{cr}^4}\frac{d^2}{dr^2}\chi_\pm + \omega^2\chi_\pm\left[(r-r_{cr})^2\frac{6r_{cr}^2 - \lambda_{cr}^2}{r_{cr}^4} + \frac{2}{\lambda_{cr}}(\lambda_{cr}-\lambda) \pm \left(\frac{G}{c^4}\right)^{1/2}\frac{e\lambda_{cr}}{\omega r_{cr}^3}\right] = 0.$$

The parabolic-cylinder functions can be expressed in terms of confluent hypergeometric functions, which allow one to prove (cf.[18]) that the amplitudes of the incident, reflected, and transmitted waves are in the ratio

$$1:[1+\exp(-\pi a_\pm)]^{-1/2}:[1+\exp(\pi a_\pm)]^{-1/2} = 1:|R_\pm|:|T_\pm|,$$

$$a_\pm = 2\omega\lambda_{cr}(6r_{cr}^2 - \lambda_{cr}^2)^{-1/2}\left(\lambda - \lambda_{cr} \pm \sqrt{\frac{G}{c^4}}\frac{e}{\lambda_{cr}r_{cr}\omega}\right). \qquad (2.9)$$

Near the limit cycle $r = r_{cr}$ the wave amplitude increases by a factor of $\omega^{1/4}$ compared to the regular points where geometric optics is valid. This property is characteristic for wave amplitudes near the "return wedges" of the caustic surfaces, also in the general case.

Assume that an electromagnetic wave with impact parameters of the rays close to the critical value is incident on the black hole:

$$f = Ar^{-1}\exp[i\omega(t+r)].$$

The gravitational wave which appears during reflection will take, according to (2.9), the form

$$P = \frac{A\sqrt{G}}{2ric^2}\exp[i\omega(t-r)]\left\{|R_+|\exp\left[2i\int_{\mathrm{Re}\,r_1}^{\infty}\left(\sqrt{\Phi_+} - \omega\frac{\partial s}{\partial r^*}\right)dr^* + \frac{\gamma_+}{2}\right]\right.$$
$$\left. - |R_-|\exp\left[2i\int_{\mathrm{Re}\,r_1}^{\infty}\left(\sqrt{\Phi_-} - \omega\frac{\partial s}{\partial r^*}\right)dr^* + \frac{\gamma_-}{2}\right]\right\}. \qquad (2.10)$$

Here

$$r_1[\mathrm{Re}\,r_1 > {}^1\!/_2 r_g(1+\sqrt{1-Ge^2/M^2})]$$

is the root of the equation

$$1 - \frac{A\lambda^2}{r^2} = 0, \quad \Phi_\pm = \omega^2\frac{\partial s}{\partial r^*} \pm \left(\frac{G}{c^4}\right)^{1/2}\frac{e\lambda\omega}{r^3}A,$$

$$\gamma_\pm = \arg\Gamma\left(\frac{1}{2} + \frac{ia_\pm}{2}\right) + \frac{a_\pm}{2} - \frac{a_\pm}{2}\ln\frac{|a_\pm|}{2}.$$

If there is a finite difference between λ and λ_{cr} and $\lambda > \lambda_{cr}$ the "height" of the barriers for the uncaptured short waves becomes impenetrable, so that $a_\pm$ becomes of order ω. In this case

$$\gamma_\pm \to 0, \quad \exp(-\pi a_\pm) \to 0,$$

$$\sqrt{\Phi_\pm} - \omega\frac{\partial s}{\partial r^*} \approx \left(\frac{G}{c^4}\right)^{1/2}\frac{e\lambda A}{r^3}\left(\frac{\partial s}{\partial r^*}\right)^{-1/2},$$

and we again arrive at Eq. (2.6).

For $|\lambda - \lambda_{cr}| \sim O(1/\omega)$ the total intensity of the electromagnetic and the produced gravitational waves will make up a finite part of the initial intensity

$$I_{out} = A^2{}^1\!/_2(|R_+|^2 + |R_-|^2).$$

The number of acts of mutual transformation of waves is comparable to one only in the case of large charges $e \sim G^{-1/2}M$ and small λ: $\lambda \sim r_g$ (M is the mass of the black hole), which follows from (2.6) and (2.10).

In the same manner as in the case of the uncharged black hole[16] in the region $|r - r_{cr}| \sim O(1/\omega)$ a peculiar halo (an "aureole") is created on account of the strong scattering of the incident waves, so that the black hole becomes a "source" of secondary radiation. Such a situation can be realized in a double system, one component of which is a black hole and the other, a source of high-energy electromagnetic radiation[16].

3. PROPAGATION OF SHORT WAVES IN THE KERR-NEWMAN METRIC

Here we consider the general case when the argument of the tetrad component φ_0 varies along the rays which carry the wave. We represent φ in the form $|\varphi|e^{i\nu}$ and introduce in place of the affine parameter α the variable m: $dm = |\varphi|d\alpha$. Eliminating $\mathscr{P}$ from Eqs. (2.1) we obtain

$$\frac{d^2\mathscr{F}}{dm^2}+\mathscr{F}-i\frac{d\nu}{dm}\frac{d\mathscr{F}}{dm}=0. \tag{3.1}$$

Equation (3.1) describes the change in amplitude of the electromagnetic wave self-consistently with the change of its polarization plane. We note that in the absence of the electromagnetic background the polarization vector of the rapidly oscillating electromagnetic wave is parallel-transported along the propagation ray of the wave: the structure of the wave is described by the equation $F_{ij} = f_{ij}e^{i\omega s}$, where s = const describes the wavefront, and the amplitudes f_{ij} satisfy the relations

$$f_{ij}s_{,k}g^{jk}=0,\quad 2s_{,j}g^{ij}\nabla_i f_{kl}+f_{kl}\Box s=0.$$

In the special case of an uncaptured ray in the gravitational field of a rotating body, a tetrad which is parallel-transported along this ray will turn out at the "exit" of the ray from the gravitational field to be rotated relative to its position at the "entrance" (one can assess the rotation of the tetrad by means of a parallel transport to pseudoeuclidean infinity). This fact was noted by Skrotskiĭ[18] (cf. also[19]). According to (3.1) the polarization plane in the presence of the electromagnetic background will rotate relative to a parallel-transported tetrad, so that the effect noticed by us has an essentially different nature than in[18].

In the sequel we restrict our attention to the investigation of Eq. (3.1) for the case of a charged rotating black hole. The Kerr-Newman solution has the metric

$$dS^2=\left(1-\frac{r_g r-Gc^{-4}e^2}{\Sigma}\right)dt^2+2\Sigma^{-1}a\sin^2\theta\,(r_g r-Gc^{-4}e^2)\,dt\,d\varphi$$
$$-\frac{dr^2\Sigma}{\Delta}-\Sigma\,d\theta^2-d\varphi^2\,\Sigma^{-1}\sin^2\theta\,((r^2+a^2)^2-\Delta a^2\sin^2\theta); \tag{3.2}$$
$$\Sigma=r^2+a^2\cos^2\theta,\quad \Delta=r^2+a^2-r_g r+e^2Gc^{-4}.$$

The eikonal equation in this metric admits a complete integral of the form

$$s=t\pm R(r)\pm\psi(\theta)+N\varphi,$$

where

$$R(r)=\int[(r^2+a^2)^2+2aN(r_g r-Ge^2c^{-4})+a^2N^2-\lambda^2\Delta]\Delta^{-1}\,dr,$$
$$\psi(\theta)=\int[\lambda^2-N^2/\sin^2\theta-a^2\sin^2\theta]^{1/2}\,d\theta. \tag{3.3}$$

In the metric (3.2) the principal axes of the bivector of the electromagnetic field are given by the components

$$\{l^i\}=\left(\frac{r^2+a^2}{\Delta},\,1,\,0,\,\frac{a}{\Delta}\right),\quad \{n^i\}=\Sigma^{-1}2^{-1}(r^2+a^2,-\Delta,0,a),$$
$$\{m^i\}=[\sqrt{2}(r+ia\cos\theta)]^{-1/2}(ia\sin\theta,\,0,\,1,\,i/\sin\theta). \tag{3.4}$$

(the distinction from the "uncharged" Kerr model consists only in a different expression for Δ, see (3.2), cf.[20]). In the general case (cf. Appendix) the expression $d\nu/d\alpha$ is given by[4)]

$$\frac{d\nu}{d\alpha}=3\,\mathrm{Im}\left[s_{,(m)}\pi-s_{,(m^*)}\tau+s_{,(n)}\rho-s_{,(l)}\mu-\frac{s_{,(n)}s_{,(m^*)}}{s_{,(l)}}k\right.$$
$$\left.+\frac{s_{,(l)}s_{,(m)}}{s_{,(n)}}\nu+\frac{s^2_{,(m^*)}}{s_{,(l)}}\sigma-\frac{s^2_{,(m)}}{s_{,(n)}}\lambda\right]. \tag{3.5}$$

Making use of the expressions (3.3) and (3.4) we find

$$s_{,(m)}=s_{,i}m^i=[\sqrt{2}(r+ia\cos\theta)]^{-1}[\psi(\theta)+i(a\sin\theta+N/\sin\theta)],$$
$$s_{,(n)}=s_{,i}n^i=(2\Sigma)^{-1}\left(r^2+a^2+aN-\Delta\frac{dR}{dr}\right), \tag{3.6}$$
$$s_{,(l)}=s_{,i}l^i=\Delta^{-1}\left(r^2+a^2+\Delta\frac{dR}{dr}+Na\right).$$

For the Kerr-Newman field which has Petrov type D[21], the rotation coefficients k, ν, λ, σ of the tetrad vanish. Therefore Eq. (3.5) takes the form

$$\frac{d\nu}{d\alpha}=\frac{3}{2i}[s_{,(m)}(\pi+\tau^*)-s_{,(m^*)}(\pi^*+\tau)+s_{,(n)}(\rho-\rho^*)-s_{,(l)}(\mu-\mu^*)]. \tag{3.7}$$

Using the definitions of π, τ, ρ, μ, we have

$$\pi+\tau^*=m_i^*(n^k\partial_k l^i-l^k\partial_k n^i)=\sqrt{2}ira\cos\theta\Sigma^{-1}(r-ia\cos\theta)^{-1},$$
$$\rho-\rho^*=m^i m^{*k}(\partial_k l_i-\partial_i l_k)=-2ia\cos\theta\Sigma^{-1}, \tag{3.8}$$
$$\mu-\mu^*=m^i m^{*k}(\partial_i n_k-\partial_k n_i)=-ia\cos\theta\Delta\Sigma^{-2}.$$

We now use the equations

$$\frac{d\theta}{d\alpha}=-\Sigma^{-1}\frac{d\psi}{d\theta},\quad \frac{dr}{d\alpha}=-\Delta\Sigma^{-1}\frac{dR}{dr}$$

and substitute into (3.7) the expressions (3.6), (3.8). We then obtain

$$\nu=-3\,\mathrm{arctg}\,(r/a\cos\theta). \tag{3.9}$$

For a charged black hole with rotation in the tetrad (3.4) the background electromagnetic field has the form

$$F_{ij}=2^{-1/2}e(r-ia\cos\theta)^{-2}M_{ij}+\mathrm{c.c.}$$

Using the transformation formulas for the tetrad components of the electromagnetic field[20], we obtain

$$|\varphi_0|=\Sigma^{-3/2}e\sqrt{\lambda^2+2aN}.$$

Finally, making use of the expression for the argument of φ_0, (3.9) we obtain for the tetrad component φ_0 the elegant expression

$$\varphi_0=e\sqrt{\lambda^2+2aN}(r-ia\cos\theta)^{-3}. \tag{3.10}$$

At a = 0, Eqs. (2.4′) and (3.10) coincide and the variable m is related to the radius r in the following manner:

$$m=eG^{1/2}c^{-2}\sqrt{\lambda^2+2aN}\int\Sigma^{-1/2}\Delta^{-1}(dR/dr)^{-1}\,dr. \tag{3.11}$$

Therefore the function $d\nu/dm$ which occurs in (3.1) is given implicitly in terms of m by means of the formulas (3.11) and (3.12):

$$\frac{d\nu}{dm}=3a\Sigma^{-1/2}e^{-1}(\lambda^2+2aN)^{-1/2}\left(\Delta\frac{dR}{dr}\cos\theta+r\sin\theta\frac{d\psi}{d\theta}\right)\frac{c^2}{\sqrt{G}}. \tag{3.12}$$

As in the case of a nonrotating charged black hole the largest number of acts of mutual transformation and rotation of the polarization plane by 180° is suffered by waves along rays which wind onto a limit cycle, the radius of which is a multiple root of the equation

$$(r^2+a^2)^2+2aN(r_g r-Ge^2c^{-4})+a^2N^2-\lambda^2\Delta=0.$$

This root yields the radius of a closed trajectory of null-particles. For trajectories situated in the equatorial plane $\theta = \pi/2$ there is no rotation of the polarization plane. In this case the period of the modulating frequency can be determined from Eq. (3.11) by setting in it $\theta = \pi/2$. The mutual transformation of waves will not occur in the equatorial plane for rays with N = −a.

A second interesting case is formed by the trajectories which wind themselves onto the cone $\theta = \arcsin(|N|a^{-1})$ (for $|N| < a$ and $\lambda^2 = 2aN$). In this case, for trajectories with $N < 0$, there is no mutual transformation of waves, since $\varphi_0 = 0$.

The general solution of (3.1) admits the following series expansion for large r

$$\mathscr{F}=C_1\left\{1-\frac{1}{2r^4}\left(\frac{G^{1/2}e\sqrt{\lambda^2+2aN}}{2c^2}\right)^2+\frac{2ia\cos\theta_0}{5r^5}\left(\frac{G^{1/2}e\sqrt{\lambda^2+2aN}}{2c^2}\right)^2+\ldots\right\}$$
$$+C_2\left\{\frac{1}{r^2}-\frac{2ia\cos\theta_0}{r^3}+\frac{\lambda^2-2a^2-10a^2\cos^2\theta_0}{r^4}-\frac{1}{r^5}\left(\frac{2aN+\lambda^2}{5}\right.\right.$$
$$\left.\left.-i\frac{4a\cos\theta_0}{5}(3a^2\cos^2\theta_0+2a^2-\lambda^2)\right)+\ldots\right\}.$$

Therefore, if a ray approaches a charged rotating body to the maximally close distance $R \gg r_g$, the polarization plane of the wave rotates relative to the tetrad (parallel-transported along the ray) by an angle δ:

$$\delta\sim a\cos\theta_0 Ge^2c^{-4}R^{-3}.$$

In conclusion we note that for rays with large impact parameter λ or in the case of a small charge of the black hole, the rotation of the polarization plane of the wave due to the interaction will be much smaller than the rotation of the tetrad which is parallel-transported along the ray; the ratio of the respective small rotation angles for $\lambda \gg \lambda_{cr}$ is of the order $e^2M^{-1}c^{-2}R^{-1}$. Both rotation effects of the polarization plane become comparable for $\lambda \sim \lambda_{cr}$ and $e^2 \sim GM^2$, i.e., when the mutual transformation of the waves is not small (cf. Sec. 2).

The author thanks A. A. Starobinskiĭ for remarks which required the inclusion of some additional research in this paper.

APPENDIX

Here we derive Eq. (3.5).

As is well known, the six-parameter group of Lorentz rotations which preserves the orthogonality relations of l, m, m*, n decomposes into the product of three Abelian subgroups[22,23]. Let l, m, m*, n denote the tetrad in which the bivector of the external field takes on the canonical form in the nondegenerate case: $F_{ij} = \varphi_1 M_{ij}$ + c.c. We first rotate the vector l in such a manner that it becomes tangent to the given congruence of null-geodesics without rotation:

$$n'=n,\quad l'=l+am^*+a^*m+|a^2|n,\quad m'=m+an.$$

We then stretch the vector l' so that it becomes self-parallel—transportable:

$$n''=\Lambda^{-1}n',\quad l''=\Lambda l',\quad m''=m'.$$

Then along the geodesics a and Λ will satisfy the usual equations

$$\begin{aligned} \tilde{D}(a\Lambda)&=\Lambda m_i\tilde{D}l^i+\Lambda|a^2|m_i\tilde{D}n^i+a\Lambda m_i\tilde{D}m^{*i},\\ \tilde{D}(\Lambda)&=-[\Lambda n_i\tilde{D}l^i+a\Lambda n_i\tilde{D}m^{*i}+\Lambda a^*n_i\tilde{D}m^i]. \end{aligned} \qquad (A.1)$$

In these equations the operator D denotes the covariant derivative along the geodesic l''.

We now rotate the vector n'' in such a manner that it becomes covariantly constant along the geodesic l'': $n''' = n'' + |b|^2 l'' + bm''^* + b^*m''$, ... (the condition on b is determined from the equation $Dn''' = 0$). Finally, we rotate the vector m''' conserving the direction of the vectors $l''' = l''$ and n''':

$$\tilde{m}=e^{i\theta}m''',\quad \tilde{n}=n''',\quad \tilde{l}=l'''.$$

From the condition that m be covariantly constant along the given null-geodesics we derive a condition on θ:

$$i\tilde{D}\theta=m_i^{*'''}\tilde{D}m'''^i=m_i^*\tilde{D}m^i+a^*n_i\tilde{D}m^i+am_i^*\tilde{D}n^i. \qquad (A.2)$$

Thus the tetrad l, m, m*, n, which was discussed in Secs. 1—3 has been constructed. In this case the tetrad component $\tilde{\varphi}_0$ of the electromagnetic field has the form

$$\tilde{\varphi}_0=2\Lambda ae^{i\theta}\varphi_1. \qquad (A.3)$$

It follows from the Maxwell equations $\nabla_i(\varphi_1 M^{ij}) = 0$ that

$$\tilde{D}(\ln\varphi_1)=2\Lambda(m_i\delta^*l^i+|a|^2m_i^*\delta n^i-a^*l_i\Delta m^i-an_iDm^{*i}). \qquad (A.4)$$

With the help of Eqs. (A.1)—(A.4) we obtain the law of variation of φ_0 along the congruence of null-geodesics:

$$\tilde{D}(\ln\tilde{\varphi}_0)=\Lambda a^{-1}(k+3|a|^2\tau+3a\rho+a^*\sigma)-a^*(3\pi+|a^2|\nu+3\mu a^*+a\lambda). \qquad (A.5)$$

We now use the condition of absence of rotation of the null-geodesics. Then $l_i = \partial_i s$. Therefore $\Lambda = s_{,i}n^i$, $\Lambda|a^i| = s_{,(l)}$, $\Lambda a = s_{,(m)}$. Separating the imaginary part in Eq. (A.5) we obtain Eq. (3.5), as required.

A necessary condition for the absence of rotation of the polarization planes of originally linearly polarized short waves consists in the equations (the Petrov type D)

$$\rho=\rho^*,\quad \mu=\mu^*,\quad \pi+\tau^*=0,\quad k=\sigma=\nu=\lambda.$$

If the electromagnetic field is degenerate then we have for some tetrad $F_{ij} = U_{ij}$ + c.c. Computations analogous to the ones above yield

$$\tilde{D}(\ln\tilde{\varphi}_0)=2[\varepsilon s_{,(n)}+(\gamma-\nu)s_{,(l)}-s_{,(m^*)}\beta+s_{,(m)}(\pi-\alpha)$$
$$+(s_{,(m)}/s_{,(n)})(s_{,(l)}\nu-s_{,(m)}\lambda)].$$

---

1) The wave properties of the solution are determined by its differential but not algebraic structure: the algebraic degeneracy manifests itself as a property of only the principal terms of the asymptotic expansions of solutions of the traveling-wave type [5] or of the type of waves far from the "island" source (the Sachs splitting theorem [6]).

2) Through the kindness of L. P. Grishchuk, I have recently become acquainted with a preprint by Zel'dovich [10], which deals with periodic mutual transformation of electromagnetic and gravitational waves in a constant transverse magnetic field against the background of flat space; the causes of violation of the mutual conversion are shown to be the presence of plasma and pair production in magnetic fields.

3) A distinction must be made between two cases: 1) both invariants are equal to zero: $\tilde{F}_{ik}\tilde{F}^{ik} = 0$ (the so called pure-radiation state); 2) at least one of the invariants differs from zero. In either case, there is a certain reference frame in which the electromagnetic-field bivector assumes a canonical form: in the former case $\tilde{F}_{ij} = U_{ij}$, in the latter $\tilde{F}_{ij} = CM_{ij}$, where $C^2 = 2(E^2 - H^2 + 2i\mathbf{E}\cdot\mathbf{H})$ (cf. [14]).

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Translated by Meinhard E. Mayer
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