The stability of the polarization of colliding beams

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The depolarizing effect of a colliding beam on the polarization of particles in storage rings, due to the stochastic mixing of the particle trajectories in an external field (owing, for example, to quantum fluctuations in the synchrotron radiation), is studied. In particular, a limitation on the maximum number of colliding-beam particles that do not destroy the radiative polarization of the forward beam is obtained for an electron (positron) beam.

1. As is well known, electrons (positrons) get polarized under the action of the synchrotron radiation when they move for long periods in storage rings[1-6]. The direction n of the equilibrium polarization (i.e., the direction of the axis of precession of the spin of a particle moving in the storage ring along a closed equilibrium trajectory) in a nonuniform field varies along the particle orbit, repeating itself every other revolution[1][7]. The degree of equilibrium polarization and the time required for its establishment depend on the proximity to spin resonances. The role of the depolarizing effects becomes more and more important as we approach resonances. It is usually possible in storage rings to choose the particle energy such that the "dangerous" resonances are sufficiently far away. The electrons will then undergo self-polarization to a degree of polarization close to unity.

The situation changes in storage rings with colliding beams. Because of the strong nonlinearity of the collective field of the colliding bunches, the spin resonances decrease sufficiently slowly in strength as their number increases. The frequencies of the orbital motion become dependent on the amplitudes of the betatron and synchrotron oscillations of the particle. It is evident that resonant harmonics always exist under these conditions, and the depolarizing influence of the quantum fluctuations in the synchrotron radiation should be investigated in greater detail. In order for a region of stability of the radiative polarization to exist, the resonance density, which is proportional to the number of colliding-beam particles, should not be too high. It follows from the results of the investigation that we can, by properly choosing the parameters of the motion of the beams, obtain polarized electrons with a number of colliding-beam particles corresponding to the maximum radiant emittance of the storage device.

In a preceding paper[6] we have, by solving the equation for the polarization density, obtained formulas allowing the determination of the establishment time for, and the degree of, the equilibrium polarization under steady-state conditions for the motion of a beam in a storage ring with an arbitrary field without restrictions on the proximity of the spin resonances. In particular, the strength of the depolarizing action of the resonant harmonics that is connected with the quantum jumps in the momentum during the radiation emission is characterized by the inverse time τ_d^{-1}:

$$\tau_d^{-1} = \pi \sum_k \langle |w_k|^2 \delta(\nu - \nu_k) \rangle, \qquad (1)$$

where ν is the frequency of spin precession about the direction of the precession axis n, $\nu_k = k_0 + k_\alpha \nu_\alpha$ is an integral combination of the frequencies of the orbital

motion[2], the ν_α's are the frequencies of the betatron and synchrotron oscillations (k_0 and the k_α's are whole numbers), the w_k's are the resonating harmonics' powers due to the deviation of the particle from the equilibrium orbit in the inhomogeneous field, the angle brackets denote averaging over the distribution and equilibrium motion of the particles in the beam, and $\delta(\nu - \nu_k)$ is a delta function. The quantities w_k, ν, and ν_k are functions of the amplitudes (the action variables) of the synchrotron and betatron oscillations.

The degree ζ of equilibrium polarization depends on the relation between the strengths of the depolarizing and polarizing mechanisms of the action of the synchrotron radiation. If τ_p and ζ_0 are the time and degree of equilibrium polarization for $w_k = 0$, then there gets established during the time $\tau_p \tau_d / (\tau_p + \tau_d)$ an equilibrium polarization of degree[3]

$$\zeta = \zeta_0 \tau_d / (\tau_p + \tau_d).$$

The formula (1) is the starting point in the present paper, and it can be obtained on the basis of simple arguments, which clarify the physical meaning. The quantum fluctuations in the particle momentum during the emission of radiation give rise to a stochastic straying of the detuning $\epsilon_k \equiv \nu - \nu_k$ as a result of the mixing of the particle trajectories in the inhomogeneous field. It then becomes possible for the resonances to be transmitted with velocity $\dot\epsilon_k$ equal to

$$|\dot\epsilon_k| \sim \langle \epsilon_k^2 \rangle^{1/2} / \tau_r,$$

where τ_r is the radiative trajectory-mixing time (τ_r^{-1} is of the order of the decrement of the radiative damping: $\tau_r^{-1} \sim \gamma^3 e^2 m^{-1} |\dot{\mathbf{v}}|$, $\gamma = (1 - v^2)^{-1/2}$ is the relativistic factor, and e, m, and $\dot{\mathbf{v}}$ are the electron charge, mass, and acceleration respectively). Under conditions of rapidity of transmission ($|w_k|^2 \ll |\dot\epsilon_k|$), the change in the component of the particle-spin vector s along the direction n is equal to[8,9]

$$\Delta s_n = (s^2 - s_n^2)^{1/2} (2\pi |w_k|^2 / |\dot\epsilon_k|)^{1/2} \cos(\Phi_k + \pi/4),$$

where w_k, $\dot\epsilon_k$, and Φ_k are the values of the power, the transmission velocity, and the phase of the spin precession at the instant when $\epsilon_k = 0$.

Let $f(I_\alpha)$ be the stationary distribution function of the particles over the action variables I_α of the orbital motion. The number of passages of a resonance per unit time is equal to $|\dot\epsilon_k| f \delta(\nu - \nu_k) d\Gamma$. Then we obtain for the mean rate $\dot\zeta$ of change of ζ the expression

$$\dot\zeta = \overline{\dot{s}_n} = \int dI_\alpha f(I_\alpha) |\dot\epsilon_k| \delta(\nu - \nu_k) \frac{1}{2} \overline{\frac{\partial}{\partial s_n} (\Delta s_n)^2} = -\pi \langle |w_k|^2 \delta(\nu - \nu_k) \rangle \zeta,$$

which, after summing over the resonances, agrees with (1). Notice that since in deriving (1) we did not use the

radiative character of the particle-trajectory mixing, formula (1) describes the depolarizing influence in the resonance of any agency that randomizes the orbital motion.

2. Let us consider the simplest case when the motion is such that the direction n of the equilibrium polarization is almost constantly along the orbit and deviates little from the direction of the driving magnetic field. In the linear approximation in the deviations from the equilibrium plane orbit, the expression for w_k can be reduced to the form[10]

$$w_k = \langle \alpha z R^{-1} \exp(-i\Psi_k) \rangle_I, \qquad (2)$$

where R is the radius of the storage ring (the length of the equilibrium orbit divided by 2π), z is the vertical deviation from the equilibrium plane orbit for which $n = e_z = $ const, Ψ_k is an integral combination of the phases ψ_α of the orbital motion ($\dot{\Psi}_k = \nu_k$), the symbol $\langle ... \rangle_I$ denotes averaging over the phases of the motion, and $\alpha(\theta)$ is a periodic function of the generalized azimuth θ of the particle motion:

$$\alpha = -\left[\nu^3 K^2 + i(\nu^2 - 1)\frac{dK}{d\theta} \right] \exp\left[-i\nu \int_0^\theta (K-1)d\theta \right].$$

Here $K(\theta)$ is the curvature of the orbit in units of $\frac{2\pi}{R}$ $R^{-1}(\int_0^{2\pi} Kd\theta = 2\pi)$ and $\nu = \gamma G$, G being the ratio of the anomalous magnetic moment of the electron to the normal moment. In an azimuthally symmetric storage ring we have $\alpha = -\nu^3 = $ const.

As is well known, the beam's truly vertical dimension, which is due to the quantum fluctuations in the radiation, is determined by the relation between the vertical oscillations and the radial and phase oscillations. Therefore, in describing the perturbing influence of a colliding bunch on the vertical deviations, it is, generally speaking, necessary to take into account, besides the direct action of the z component F_z of the force of the colliding bunches, that action of the radial component F_x that is due to the z-x coupling. We shall neglect the action of the force F_x, assuming that the steady-state amplitudes of the z-oscillations are given: allowance for F_x is not essential to the elucidation of the basic characteristics of the effect of the colliding bunches on the polarization and greatly complicates all the formulas. The equation describing the vertical deviation then has the form

$$d^2z/d\theta^2 + g_z z = F_z(z, x, \theta). \qquad (3)$$

Let us consider the case of a head-on encounter between electrons and positrons, when the equilibrium orbits of the bunches in the interaction region coincide. Then the effective potential V can be written in the form[11]

$$F_z = -\frac{\partial V}{\partial z}, \quad V = -2R\frac{Nr_e}{\gamma}u(x, z)g(2\theta),$$

$$u(x, z) = -\int dx' \, dz' \sigma(x', z') \ln[(x-x')^2 + (z-z')^2], \qquad (4)$$

where N is the total number of particles of the colliding beam and r_e is the classical electron radius. The functions σ and g describe the density distribution of the counterbunches about the equilibrium orbit, and are normalized in the following fashion:

$$\int \sigma \, dx \, dz = 1, \quad \int_0^{2\pi} g(\theta) d(\theta) = 1.$$

Let us choose the distributions of σ and g in the form

$$\sigma = \frac{1}{\pi x_0 z_0} \exp\left(-\frac{x^2}{x_0^2} - \frac{z^2}{z_0^2} \right),$$

$$g(\theta) = \frac{2}{p}\sum_{k=0}^{p/2-1} \delta\left(\theta - \frac{4\pi}{p}k \right), \qquad (5)$$

where $2x_0$ and $2z_0$ are the radial and vertical dimensions of the colliding bunches at the places of encounter ($x_0 \gtrsim z_0$) and p/2 is the number of bunches (p is the number of places of encounter in the storage ring). Let us assume that the magnetic system of the storage ring has a period of $\theta_0 = 2\pi/p$.

Let us begin with the study of one-dimensional resonances due to the colliding beam:

$$\nu \approx k_0 p + k_z \nu_z \quad (|k_z| \neq 0, 1).$$

The maximum number k_z^{max} starting from which the harmonics $|k_z| > k_z^{max}$ can be neglected is found from the equation[4]

$$(\tau_d)_{min} = \tau_p = \left(\frac{5\sqrt{3}}{8}\gamma^5 \frac{\lambda r_e}{R_M^2} \right)^{-1},$$

where λ is the Compton wavelength, $R_M = |\dot{\nu}|_{max}^{-1}$ is the radius of curvature of the trajectory at the sections with magnetic fields, and $(\tau_d)_{min}$ is the minimum value of τ_d. The harmonic w_k and the particle vertical-oscillation frequency shift $\Delta\nu_z$ due to the colliding beam determine through the formula (1) the quantity τ_d.

For a Gaussian distribution over the amplitudes a_z of the vertical oscillations:

$$f(a_z^2) = \langle a_z^2 \rangle^{-1} \exp(-a_z^2 \langle a_z^2 \rangle^{-1})$$

($2\langle a_z^2 \rangle^{1/2}$ and a_z are respectively the vertical dimension of the beam and the amplitude at the places of encounter); the time τ_d is a minimum when

$$a_z^2 = |k_z| \frac{\langle a_z^2 \rangle z_0}{(z_0^2 + 2\langle a_z^2 \rangle)^{1/2}} \quad (|k_z| \gg 1).$$

Then[5] (see the Appendix)

$$\frac{1}{(\tau_d)_{min}} = \frac{4}{\pi}\frac{Nr_e}{\gamma R}|A|^2 \frac{B}{|k_z|} \left[\frac{\langle a_z^2 \rangle}{\langle a_z^2 \rangle + z_0^2 + z_0(z_0^2 + 2\langle a_z^2 \rangle)^{1/2}} \right]^{|k_z|}. \qquad (6)$$

Here

$$A = \frac{1}{2i}\left\{ \frac{1}{1 - \exp[-2\pi i(\nu - \nu_z)]}\int_0^{2\pi}\alpha f_z e^{-i\nu\theta}\,d\theta \right.$$

$$\left. - \frac{1}{1 - \exp[-2\pi i(\nu + \nu_z)]}\int_0^{2\pi}\alpha f_z^* e^{-i\nu\theta}\,d\theta \right\};$$

$f_z(\theta)$ is the Floquet solution to the equations describing the beam's vertical oscillations, the solution being defined in such a way that its phase is equal to zero when $\theta = 0$ ($f_z(0) = |f_z|_0$);

$$B = \frac{z_0^2}{x_0^2}\frac{\langle a_z^2 \rangle}{z_0^2 + 2\langle a_z^2 \rangle}$$

when

$$a_z^2 = |k_z|\frac{\langle a_z^2 \rangle z_0}{(z_0^2 + 2\langle a_z^2 \rangle)^{1/2}} \gg x_0^2,$$

$$B = \left(\frac{\pi}{|k_z|} \right)^{1/2}\frac{z_0}{x_0}\frac{(\langle a_z^2 \rangle/z_0^2)^{1/2}(1 + 2\langle a_z^2 \rangle/z_0^2)^{1/4}}{1 + \langle a_z^2 \rangle/z_0^2}$$

when $z_0^2 \ll a_z^2 \ll x_0^2$.

In an azimuthally symmetric storage device with weak focusing $f_z = \exp(i\nu_z\theta)\nu_z^{-1/2}$, the quantity $A = \nu^3/(\nu^2 - \nu_z^2)$. The equation for the maximum number k_z^{max} has the form

$$\frac{\tau_p}{(\tau_d)_{min}} = \frac{8}{5\sqrt{3}}\frac{4}{\pi}\frac{NR_M^2}{\gamma^5\lambda R}|A|^2\frac{B}{k_z^{max}}.$$

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$$x\left[\frac{\langle a_z^2\rangle}{\langle a_z^2\rangle+z_0^2+z_0(z_0^2+2\langle a_z^2\rangle)^{1/2}}\right]^{k_z^{max}}=1. \qquad (7)$$

When conditions are sufficiently close to the resonances with $|k_z| < k_z^{max}$, the beam is depolarized. The "dangerous" frequency interval $\Delta\nu$ occupied by the resonance $\nu \approx k_0 p + k_z \nu_z$ is a function of the number k_z:

$$|\Delta\nu| \lesssim |k_z|\Delta\nu_z^{max};$$

here $\Delta\nu_z^{max}$ is the maximum shift introduced into the frequency ν_z by the colliding bunch (see the Appendix):

$$\Delta\nu_z^{max}=\frac{2Nr_eR|f_z|_0^2}{\pi\gamma z_0(x_0+z_0)}.$$

In order for ν-value regions free from "dangerous" intervals to exist, it is clearly sufficient that

$$\sum_{k_z=-k_z^{max}}^{k_z^{max}}|\Delta\nu| \lesssim (k_z^{max})^2\,\Delta\nu_z^{max} < p. \qquad (8)$$

The condition (8), together with Eq. (7), imposes limitations on the maximum possible number N^{max} of particles in the colliding beam that does not destroy the polarization, limitations which are roughly the same as the limitations necessary for a stable realization of the beam encounters (the nonlinear resonances $\nu_z \approx pn/m$ are dangerous in the orbital motion). The conditions limiting the maximum possible number N of particles (the radiant emittance) for which the encounters are still stable can be written down on the basis of the well-developed theory of motion near nonlinear resonances[11-13]. Such coincidence is not accidental, and is explained by the following: the decrement τ_p^{-1} of the radiative polarization is $(\gamma^2 \lambdabar/R)^{-1}$ times smaller than the decrement τ_r^{-1} of the orbital motion $(\tau_p/\tau_r \sim (\gamma^2\lambdabar/R)^{-1})$. But $|w_k|^2$, the square of the power of the spin resonances, is smaller than the corresponding square of the power of the orbital resonances by the same factor.

As is evident from (7) and (8), N^{max} strongly depends on the ratio $\langle a_z^2\rangle/z_0^2$, increasing as this ratio decreases. A decrease in the modulus of the Floquet solutions (which are β functions) at the places of encounter leads, as is well known, to a gain in the radiant emittance of the storage device (owing to increases in the beam densities at the places of encounter). To increase the radiant emittance of a storage device with polarized beams, it is also advantageous to decrease $|f_z|_0 \sim |f_x|_0$ (the degree of equilibrium polarization is less sensitive to changes in $|f_z|_0 \sim |f_x|_0$; to wit, $\tau_p/\tau_d \sim N|A|^2)$. In fact, notwithstanding the increase in the force $\sim |f_z|_0^{-1}$ of action of the colliding bunch, the betatron-frequency shift and the harmonics of the vertical deviation, which determine the power $|w_k|$ of the spin resonances, are less sensitive to changes in the β functions at the places of encounter.

The inclusion of resonances with radial and synchrotron particle oscillations leads to an increase in the number of "operating" resonances. The condition (8) is then replaced by (usually, $\Delta\nu_z^{max} \gtrsim \Delta\nu_x^{max} \gg \Delta\nu_c^{max}$) the condition[6)]

$${}^1\!/{}_3(k_z^{max})^2 k_x^{max} k_c^{max}\Delta\nu_z^{max} < p, \qquad (9)$$

which imposes a stronger limitation on N^{max} (k_x^{max} and k_c^{max} are the maximum numbers of the "operating" resonances with radial and synchrotron oscillations). The equations for k_x^{max} and k_c^{max} can be obtained in similar fashion. In allowing for the slow synchrotron oscillations, we must, generally speaking, take into consideration, besides the modulation of the betatron har-

monics w_{k_z}, k_x, k_0 of the perturbation propagating perpendicularly to n, the synchrotron modulation of $w \cdot n$ and the betatron frequencies ν_x and ν_z.

The formula (1) of course allows us to carry out a more detailed investigation with allowance for all the real factors determining the properties of the orbital motion. However, because of the above-noted complex dependence of w_k and ϵ_k on the amplitudes of the oscillations, it is more expedient to carry out such investigations for specific storage devices. The main results, however, remain valid independent of the structure of the electromagnetic field acting on the colliding beams inside the storage ring.

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APPENDIX

The expression for the vertical component $F_z(z, \theta)$ of the force exerted by the colliding bunches at $x = 0$ is obtained from (4) (σ is given in the form (5)):

$$F_z=-\frac{4Nr_eR}{\gamma z_0}\int_0^1 dt\,\frac{z\exp(-z^2t/z_0^2)}{[z_0^2(1-t)+x_0^2t]^{1/2}}.$$

The frequency shift $\Delta\nu_z$ induced by the colliding beam is computed from the formula[11-13]:

$$\Delta\nu_z=-\frac{|f_z|_0^2}{a_z}\frac{1}{(2\pi)^2}\int_0^{2\pi}d\theta\int_0^{2\pi}d\psi_z\cos\psi_z F_z(a_z\cos\psi_z,\theta)$$

$$=\frac{2Nr_eR}{\pi\gamma}|f_z|_0^2\begin{cases}1/a_z^2 & a_z\gg x_0\\ 2/\sqrt\pi\,x_0 a_z, & z_0\ll a_z\ll x_0.\\ 1/z_0(x_0+z_0), & a_z=0\end{cases}$$

The expression for w_k can be reduced with the aid of (2) and (3) to the form

$$w_k=-\frac{Nr_e}{\pi\gamma}A|f_z|_0\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial u}{\partial z}\exp(-ik_z\psi_z)\,d\psi_z.$$

For even k_z, $w_k = 0$, while for odd k_z

$$|w_k|=\left|\frac{Nr_ef_z(0)}{\pi\gamma z_0}A\int_0^{a_z^2/2z_0^2}dt\,\frac{2z_0e^{-t}}{[a_z^2+2(x_0^2-z_0^2)t]^{1/2}}[I_{(k_z-1)/2}(t)-I_{(k_z+1)/2}(t)]\right|, \qquad (A.1)$$

where $I_m(t)$ is the Bessel function of imaginary argument. The asymptotic form of the functions $I_m(t)$ for $m \gg 1$ is known[14]:

$$I_m(t)=\frac{\exp[(m^2+t^2)^{1/2}]}{\sqrt{2\pi}(m^2+t^2)^{1/4}}\left[\left(1+\frac{m^2}{t^2}\right)^{1/2}-\frac{m}{t}\right]^m.$$

The dominant contribution in the integral over t in (A.1) lies, for $|k_z| \gg 1$, in the vicinity of $t = a_z^2/2z_0^2$. Thus, the final expression for $|w_k|$ has the form

$$|w_k|=\frac{4Nr_e|f_z|_0}{\pi\sqrt{2\pi}\,\gamma x_0}|A|\frac{a_z}{z_0}\left[\left(1+\frac{k_z^2z_0^4}{a_z^4}\right)^{1/2}+1\right]^{1/2}$$

$$\times\exp\left\{\frac{1}{2}\left[\left(k_z^2+\frac{a_z^4}{z_0^4}\right)^{1/2}-\frac{a_z^2}{z_0^2}\right]\right\}$$

$$\times\left(k_z^2+\frac{a_z^4}{z_0^4}\right)^{-1/4}\left[\frac{|k_z|z_0^2}{a_z^2}+\left(1+\frac{k_z^2z_0^4}{a_z^4}\right)^{1/2}\right]^{-|k_z|/2}$$

For a Gaussian amplitude (a_z) distribution, we obtain the following expression for τ_d^{-1}:

$$\tau_d^{-1}=\pi\frac{|w_k|^2\exp(-a_z^2/\langle a_z^2\rangle)}{|k_z|\langle a_z^2\rangle|\partial\Delta\nu_z/\partial a_z^2|} \qquad (A.2)$$

(here a_z is the value of the amplitude for which $\epsilon_k = 0$). Investigation of (A.2) for its extremal values shows that the minimum value of τ_d is determined by the formula (6).

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[1]In motion across a unidirectional magnetic field, n is obviously directed along the field, and does not depend on the azimuth of the particle motion.

[2]We measure all the frequencies in units of the equilibrium frequency of revolution of the particles of the storage element.

[3]The formulas for τ_p and ζ_0 are given in [5].

[4]The formula for τ_p can be found in [1-6].

[5]Since F_z is an odd function of z, only resonances with odd numbers k_z are possible. Resonances of even parity arise when the coupling between the vertical and radial oscillations is taken into account.

[6]The condition (9) alters if the frequency of the synchrotron oscillations is small compared to the spread $\Delta\epsilon_k$ of the frequencies of the motion. Then instead of one distinct resonance in the band $\Delta\epsilon_k$, there appears a series of synchrotron resonances of roughly the same power, over which the appropriate expressions must clearly be summed.

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