Nonlinear interaction between differently polarized waves traveling in opposite directions in a ring laser

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The nonlinear polarization effects in a gas laser with a ring resonator possessing polarization anisotropy and located in zero magnetic field are investigated on the basis of the semiclassical vector theory. It is shown that because their polarization vectors are different and the active medium possesses nonlinear anisotropy, the opposing waves are generated at different frequencies and intensities. The difference between the frequencies of the opposing waves increases from zero in proportion to the intensity of generation. Expressions for the frequencies and intensities of generation of the waves are derived as functions of the parameters of the resonator and the active medium.

INTRODUCTION

The polarization characteristics of ring resonators have recently been intensively studied (see the bibliography in[1,2]). It has been shown that there exist in each direction two running-wave-polarization eigenstates with different eigenvalues, which determine the frequencies and losses. In the absence of a magnetic field, the eigenvalues for the opposing waves are equal. The eigenvectors belonging to one and the same eigenvalue are different for the opposite directions when the losses in the radiation from the resonator are taken into account. Thus, in a real resonator without a magnetic field, the frequencies of the opposing waves are equal, while their polarizations are different.

Upon the application of a magnetic field to the active medium[3,4] or to the resonator elements, in which the polarizations of the opposing waves are nonlinear, there occurs, owing to the magneto-optical effects, a splitting of the opposing-wave frequencies. The magnitude of the difference frequency is proportional to the magnetic field, but does not depend on the generation intensity.

In the present paper we show that in the absence of a magnetic field there arises as a result of the nonlinear interaction with the active medium a polarization-induced difference between the frequencies of the opposing waves that is proportional to the generation intensity. A qualitative explanation of this new effect consists in the following. In the case of an elliptically polarized running wave, the active medium, which is isotropic in the absence of an electromagnetic field, becomes anisotropic because of the nonlinear interaction between the circular components of the field[5] via the common σ_+- and σ_--transition sublevels. The polarization anisotropies in the saturation and nonlinear dispersion of the medium lead to the deformation of the polarization state of the field and to a change in the nonlinear shift of the frequency of generation of the wave. Since the opposing-wave polarizations, which are determined by the properties of the resonator, are, as a rule, different in the region where the active medium is located, the saturation and nonlinear dispersion of the medium turn out to be different for them. This determines the differences between the intensities, between the deformations of the polarizations, and between the nonlinear generation-frequency shifts of the opposing waves. The last effect gives rise to a change in the difference frequency[1)].

The problem is solved under the assumption that there obtains a single-mode regime in which there is generated in each direction a monochromatic running wave with a definite polarization state possessing the least losses. It is assumed that the resonator possesses a sufficiently large amplitude or phase anisotropy, so that the perturbation introduced by the active medium has little effect on the polarization state.

The polarization-induced frequency difference can be measured directly if its value exceeds the width of the opposing-wave-frequency synchronization region. In the opposite case, it can be distinguished when the sign of the frequency "support" produced by rotation or by a nonreciprocal Faraday element changes.

1. THE RING RESONATOR WITH AN ARBITRARY POLARIZATION ANISOTROPY

1. The polarization properties of a resonator are conveniently described with the aid of the Jones matrix method[6], in which to each polarization element corresponds a square matrix of rank two. The matrix $\hat{P}$ of a series of elements is obtained by multiplying the matrices of the individual elements in the order in which they act on the traveling-wave vector: $\hat{P} = \hat{A}_N \ldots \hat{A}_2\hat{A}_1$. If by chance the indicated series of elements forms a closed ring resonator, then we can find the eigenvectors $\mathbf{q}$ and eigenvalues λ, which determine the frequencies and losses in the resonator:

$$\hat{P}\mathbf{q} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\mathbf{q} = \lambda\mathbf{q}, \quad \lambda^2 - \lambda\,\mathrm{Sp}\,\hat{P} + \mathrm{Det}\,\hat{P} = 0. \tag{1}$$

The matrices of any arbitrarily arranged elements can be represented in the form of a product of three very simple matrices, two of which—the matrices for the partial polarizer

$$\hat{K}(k) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \tag{2}$$

and the linear phase plate

$$\hat{\Phi}(\varphi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \tag{3}$$

are diagonal and the third—the rotation matrix

$$\hat{S}(\alpha) = \begin{pmatrix} c_\alpha & -s_\alpha \\ s_\alpha & c_\alpha \end{pmatrix} \tag{4}$$

is nondiagonal. Here and below we use the notation: $s_\alpha = \sin\alpha$ and $c_\alpha = \cos\alpha$.

2. Let us find the relations between the resonator matrices $\hat{P}_r$ and $\hat{P}_l'$ for the opposing waves in the co-

ordinate systems connected with the directions of propagation of the waves. In constructing them, we must remember that the order in which the resonator elements act on the wave vector of one wave is the reverse of the order in which they act on the counterwave vector:

$$\hat P_r=\prod_{i=N}^{1}\hat A_i,\quad \hat P_l'=\prod_{i=1}^{N}\hat A_i.$$

If one of the resonator elements is a Faraday rotator that rotates the wave vector through an angle h, i.e., if $\hat A_k=\hat S(h)$, then in constructing the matrix $\hat P_l'$ we must replace h by −h, i.e., we must set $(\hat A_k)_l=\hat S(-h)$, which corresponds to the presence of a preferred magnetic-field defined direction. Below we shall consider the case when h = 0. The results can easily be generalized to the h ≠ 0 case with the aid of the above-indicated rule for allowing for magnetic rotation.

Using the properties oof the product of transposed matrices $\hat B^+\hat C^+=(\hat C\hat B)^+$, we obtain

$$\hat P_l'=\Big(\prod_{i=N}^{1}\hat A_i^+\Big)^+. \qquad (5)$$

For the polarization-element matrices (2)–(4), we have the following equality, obtained with the aid of the reflection matrix:

$$\hat A^+=\hat T\hat A\hat T^{-1},\quad \hat T=\hat T^{-1}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

From the equality (5) we obtain

$$\hat P_l'=\Big(\hat T\prod_{i=1}^{N}\hat A_i\hat T^{-1}\Big)^+=\hat T^{-1}\hat P_r^+\hat T.$$

It is convenient to use the resonator matrices for the opposing waves when they are written in one coordinate system, connected, for example, with the r direction:

$$\hat P_l=\hat T\hat P_l'\hat T^{-1}=\hat P_r^+. \qquad (6)$$

3. It is evident from the equality (6) that

$$\mathrm{Sp}\,\hat P_r=\mathrm{Sp}\,\hat P_l',\quad \mathrm{Det}\,\hat P_r=\mathrm{Det}\,\hat P_l' \qquad (7)$$

and, in accord with the Eq. (1) for the eigenvalues, we obtain

$$\lambda_r^{(k)}=\lambda_l^{(k)}=\lambda^{(k)}\quad (k=1,2). \qquad (8)$$

If there is a nonreciprocal Faraday rotator in the resonator, then

$$\lambda_r^{(k)}(h)=\lambda_l^{(k)}(-h)\quad (k=1.2).$$

The determinants of the polarization-element matrices (2)–(4) are real and, therefore, since $\mathrm{Det}\,\hat P=\prod_{i=1}^{N}\mathrm{Det}\,\hat A_i$, the determinant $\mathrm{Det}\,\hat P$ of the resonator matrix is also real. Since the $\mathrm{Det}\,\hat P=\lambda^{(1)}\lambda^{(2)}$, the eigenvalues can be represented in the form

$$\lambda^{(k)}=|\lambda^{(k)}|e^{\pm i\psi/2}\quad (k=1,2). \qquad (9)$$

4. Let us find the relation between the eigenvectors of the opposing waves in the general coordinate system.

The eigenvectors of the resonator matrices $\hat P_r$ and $\hat P_l$ can be found from Eq. (1) and, with allowance for the equality $\lambda_1+\lambda_2=\mathrm{Tr}\,\hat P$ and the relations (6) and (8), they are obtainable in the form

$$\mathbf q_r^{(1)}=\Delta_1\begin{pmatrix}1\\a_1\end{pmatrix},\quad \mathbf q_r^{(2)}=\Delta_2\begin{pmatrix}-a_2\\1\end{pmatrix},$$
$$\mathbf q_l^{(1)}=\Delta_2\begin{pmatrix}1\\a_2\end{pmatrix},\quad \mathbf q_l^{(2)}=\Delta_1\begin{pmatrix}-a_1\\1\end{pmatrix}, \qquad (10)$$

where $a_i=p_{ki}/(\lambda_1-p_{22})$ and $\Delta_i=(1+|a_i|^2)^{-1/2}$ (i, k = 1, 2; i ≠ k).

It can be seen that the vectors $\mathbf q_r^{(1)}$ and $\mathbf q_r^{(2)}$ ($\mathbf q_l^{(1)}$ and $\mathbf q_l^{(2)}$) are nonorthogonal and that the eigenvectors of the opposing waves do not coincide with each other.

5. Thus far, the properties of the matrices and the eigenvectors have been considered in Cartesian coordinates. It is, however, more convenient to solve the nonlinear equations for the ring laser in angular coordinates. Upon going ovtr to the angular basis, the matrices $\hat P_{r,l}$ go over into the matrices $\hat U_{r,l}$:

$$\hat U_r=\begin{pmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{pmatrix},\quad \hat U_l=\begin{pmatrix}u_{22}&u_{12}\\u_{21}&u_{11}\end{pmatrix}. \qquad (11)$$

From the form of the matrices $\hat U_{r,l}$ follow the relations for the eigenvectors of the opposing waves:

$$\mathbf q_n^{(k)}=\begin{pmatrix}q_{1n}^{(k)}\\q_{2n}^{(k)}\end{pmatrix},\quad q_{mn}^{(1)}=(-1)^m q_{mn'}^{(2)},$$
$$(n,n'=r,l;\ n\neq n';\ m,k=1,2), \qquad (12)$$

where

$$q_{1r}^{(k)}=(-1)^{k+1}e^{-i\varphi_k/2}c_{\pi/4-\varepsilon_k},\quad q_{2r}^{(k)}=e^{i\varphi_k/2}s_{\pi/4-\varepsilon_k}.$$

Here $|\tan\epsilon_{1,2}|$ stands for the ratios of the semiaxes of the corresponding polarization ellipses and $2\varphi_{1,2}-\pi/4$ are the azimuths of the major semiaxes of these ellipses. The sign of $\tan\epsilon$ indicates the direction of rotation in the preferred coordinate system. The orthogonality conditions for the vectors $\mathbf q_r^{(1)}$ and $\mathbf q_r^{(2)}$ can then be written in the form

$$\varepsilon_1+\varepsilon_2=2\Delta\varepsilon=0,\quad \varphi_1-\varphi_2=2\Delta\varphi=0. \qquad (13)$$

Their relation with the elements of the matrix $\hat U$ is of the form

$$\mathrm{tg}\Big(\frac{\pi}{4}-\varepsilon_k\Big)e^{i\varphi_k}=\frac{u_{21}}{\lambda_i-u_{11}}\quad (i,k=1,2;\ i\neq k).$$

The above-considered properties of the resonator eigenvalues and eigenvectors indicate that in the absence of nonreciprocal Faraday rotators in the resonator it is impossible to produce a frequency difference for, and bring about losses of, the opposing waves (see formula (8)), but that the opposing-wave polarization states are nonetheless nonreciprocal.

2. COMPUTATION OF THE FREQUENCIES AND INTENSITIES OF GENERATION OF THE DIFFERENTLY POLARIZED OPPOSING WAVES

1. Let us consider the single-mode regime of generation in a gas ring laser. The field vector in angular coordinates has the form

$$\mathbf E=E_{0r}\begin{pmatrix}e_{1r}\\e_{2r}\end{pmatrix}\exp\{-i(\omega_rt-kz)\}+E_{0l}\begin{pmatrix}e_{1l}\\e_{2l}\end{pmatrix}\exp\{-i(\omega_lt+kz)\}+\text{c.c.} \qquad (14)$$

where E_{0n}^2 is the total wave intensity in the direction n = (r, l); e_{1n} and e_{2n} are the complex components of the normalized polarization vector:

$$|e_{1n}|^2+|e_{2n}|^2=1.$$

To determine the frequencies and intensities of the running waves, we must solve the system of stationary equations of generation for the field vectors of the opposing waves in an angular basis:

$$\hat Q_r\begin{pmatrix}e_{1r}\\e_{2r}\end{pmatrix}=\mu_r\begin{pmatrix}e_{1r}\\e_{2r}\end{pmatrix},\quad \hat Q_l\begin{pmatrix}e_{1l}\\e_{2l}\end{pmatrix}=\mu_l\begin{pmatrix}e_{1l}\\e_{2l}\end{pmatrix}, \qquad (15)$$

where

$$\mu_n=-\alpha-i(\omega_n-\nu-\sigma), \quad n=(r, l) \quad (i, k=1, 2;\ i\neq k),$$
$$Q_{ii}{}^r=M_{ii}{}^r-I_r(R|e_{ir}|^2+F|e_{kr}|^2)-I_l(L|e_{il}|^2+L_1|e_{kl}|^2),$$
$$Q_{ik}{}^r=M_{ik}{}^r-WI_le_{il}e_{kl}^*, \tag{16}$$
$$I_n=\frac{3G}{\hbar^2\gamma_{ab}}\left(\frac{1}{\gamma_a}+\frac{1}{\gamma_b}\right)|(j_a\|d\|j_b)|^2E_{0n}{}^2.$$

The elements of the matrix $\hat{Q}_l$ for the counterwave are obtained by means of the substitution $r \rightleftarrows l$.

For the field (14), the complex interaction coefficients take, in the absence of a magnetic field, the form

$$\alpha-i\sigma=\alpha_0\left[1-\eta^2f^2+i\eta\frac{f}{\sqrt{\pi}}\right], \quad \eta=\frac{\gamma_{ab}}{ku}\ll 1,$$
$$R=\alpha_0, \quad F=2\alpha_0(K'+K''), \quad L=\alpha_0/(1-if),$$
$$L_1=\frac{2\alpha_0}{1-if}\frac{K'\gamma_b+K''\gamma_a}{\gamma_a+\gamma_b}, \quad \alpha_0=\frac{2\pi^{3/2}\omega|(j_a\|d\|j_b)|^2N_0}{3\hbar ku\varepsilon_0},$$
$$W=\frac{2\alpha_0}{1-if}\frac{K'\gamma_a+K''\gamma_b}{\gamma_a+\gamma_b}, \quad f=\frac{\omega_r+\omega_l-2\omega_0}{2\gamma_{ab}}.$$

Here ω_n denotes the frequency of generation of the wave in the direction n; $\nu = (\nu_1 + \nu_2)/2$, $\nu_{1,2}$ being the resonator frequencies for the polarizations 1, 2; K′, G, and K″ are functions, given in [5], of the angular momenta j_a and j_b of the generation levels. The remaining symbols have the usual meaning (see [5]).

The real parts of the interaction coefficients are even, while the imaginary parts are odd, functions of the detuning of the generation frequency ω relative to the center ω_0 of the amplification contour.

2. The complex Q-factor matrix $\hat{M}_r$ of the resonator determines the losses, the frequencies, and the linear relation between the angular components of the resonator field. For waves propagating in different directions, the matrices $\hat{M}_r$ and M_l are different. To construct the matrix $\hat{M}_r$, let us write the Q-factor matrix $\hat{M}_0$ acting on the field-vector components in the eigenvector representation. In this representation, it is diagonal, and is expressible in terms of the resonator eigenvalues $\lambda_{1,2}$, (9):

$$M_0=\begin{pmatrix} M_{01} & 0 \\ 0 & M_{02}\end{pmatrix}=M\hat{I}+\Delta M\hat{T}, \tag{17}$$

where $\hat{I}$ is the unit matrix:

$$M={}^1/_2(M_{01}+M_{02}), \quad \Delta M={}^1/_2(M_{01}-M_{02}).$$

The elements $\hat{M}_{0k}$ can, with allowance for (9), be represented in the form

$$M_{0k}=\frac{c}{L_0}\left[\ln|\lambda_k|-i(-1)^k\frac{\psi}{2}\right] \quad (k=1.2) \tag{18}$$

and do not, in accord with (8), depend on direction (L_0 is the optical length of the resonator).

The Q-factor matrix $\hat{M}_r$ can be obtained from $\hat{M}_0$ by means of the following transformation: $\hat{M}_r = \hat{A}_r\hat{M}_0\hat{A}_r^{-1}$, where $\hat{A}_r$ is the matrix that transforms the field-vector components from the natural to the angular basis and that is expressible in terms of the eigenvector components in the angular basis. The components of the matrix $\hat{A}_n$ are of the form

$$\{\hat{A}_n\}_{mk}=q_{mn}^{(k)} \quad (n=r,l;\ m,k=1.2). \tag{19}$$

Since the eigenvectors for the opposite directions are different (see (12)), the transformation matrices $\hat{A}_r$ and $\hat{A}_l$ are also different (this is the cause of the difference between the matrices $\hat{M}_r$ and $\hat{M}_l$). The Q-factor matrix M_r is, in accord with (17) and (19), equal to

$$\hat{M}_r=M\hat{I}+\frac{\Delta M}{D}\hat{T}_r. \tag{20}$$

Here $D = c_{2\Delta\epsilon}c_{\Delta\varphi} - is_{2\epsilon}s_{\Delta\varphi}$ is the determinant (19) of the matrix $\hat{A}_r$. The components of the matrix $\hat{T}_r$ are of the form

$$\{\hat{T}_r\}_{11}=-\{\hat{T}_r\}_{22}=s_{2\varepsilon}c_{\Delta\varphi}-ic_{2\Delta\varepsilon}s_{\Delta\varphi},$$
$$\{T_r\}_{12}=2e^{-i\varphi}c_{\pi/4-\varepsilon_1}s_{\pi/4+\varepsilon_2}, \quad \{\hat{T}_r\}_{21}=2e^{i\varphi}s_{\pi/4-\varepsilon_1}c_{\pi/4+\varepsilon_2},$$

where $\epsilon = {}^1/_2(\epsilon_1 - \epsilon_2)$, $\Delta\epsilon = {}^1/_2(\epsilon_1 + \epsilon_2)$, $\varphi = {}^1/_2(\varphi_1 + \varphi_2)$, and $\Delta\varphi = {}^1/_2(\varphi_1 - \varphi_2)$.

The matrix $\hat{M}_l$ is obtained from the matrix $\hat{M}_r$ by means of the substitutions $\epsilon_1 \rightleftarrows \epsilon_2$ and $\varphi_1 \rightleftarrows \varphi_2$. Under these substitutions $\Delta\epsilon$ and φ do not change, while ϵ and $\Delta\varphi$ go over into $-\epsilon$ and $-\Delta\varphi$. It is clear from the method of constructing the matrix $\hat{M}_r$ that M_{01} and M_{02} [Eq. (18)] are its eigenvalues and $q_r^{(1)}$ and $q_r^{(2)}$ [Eq. (12)] are its eigenvectors. For the Q-factor matrix $\hat{M}_l$ of the opposite direction, the eigenvalues remain the same, but the eigenvectors are $q_l^{(1)}$ and $q_l^{(2)}$, (12). As will be shown below, the difference between the eigenvectors, which define the polarization state of the opposing-wave fields in the resonator, determines the differences in frequency and intensity of generation of the opposing waves.

3. We shall seek the solution to Eqs. (15) under the condition that the gain exceed the losses only for the type of oscillation with the highest Q factor, i.e., under the condition that

$$-\mathrm{Re}\, M_{02}>\alpha>-\mathrm{Re}\, M_{01}. \tag{21}$$

Equations (15) have been written in the weak-field approximation (i.e., under the assumption that $I_n \ll 1$), and are valid up to the third order in the field. Accordingly, in order not to exceed the accuracy of the matrix element, $\hat{Q}_n$ should not be of order higher than the second in the field. This means that in fulfilling the condition for smallness of the nonlinear deformation of the polarization state

$$|M_{01}-M_{02}|\gg\alpha+\mathrm{Re}\, M_{01} \tag{22}$$

we can replace the components e_n of the normalized field vector in (16) by the components $q_n^{(1)}$ [Eq. (12)] (n = r, l), of the eigenvector of the linear problem. The vectors $q_r^{(1)}$ and $q_l^{(1)}$ correspond to the eigenvalue with the highest Q factor M_{01}. We obtain

$$Q_{11}{}^r=M_{11}{}^r-I_r(\bar{R}-\Delta Fs_{2\varepsilon_1})-I_l(\bar{L}-\Delta Ls_{2\varepsilon_2}),$$
$$Q_{12}{}^r=M_{12}{}^r-\frac{W}{2}I_lc_{2\varepsilon_2}e^{-i\varphi_2},$$
$$Q_{21}{}^r=M_{21}{}^r-\frac{W}{2}I_lc_{2\varepsilon_2}e^{i\varphi_2}, \tag{23}$$
$$Q_{22}{}^r=M_{22}{}^r-I_r(\bar{R}+\Delta Fs_{2\varepsilon_1})-I_l(\bar{L}+\Delta Ls_{2\varepsilon_2}),$$

where

$$\bar{R}={}^1/_2(R+F), \quad \Delta F={}^1/_2(F-R),$$
$$\bar{L}={}^1/_2(L+L_1), \quad \Delta L={}^1/_2(L_1-L).$$

The solution of (15) with the matrix elements (23) is valid up to terms linear in the intensities I_r and I_l, i.e., up to and including terms linear in the $(\alpha + \mathrm{Re}M_{01})$ pump excess over the threshold.

The replacement of the matrix elements given in (16) by the elements given in (23), which do not depend on the components of the eigenvector of Eq. (15), allows us to determine the eigenvalues and eigenvectors of the system (15) by the standard methods of solving linear systems of equations.

The frequencies and intensities are determined from the secular equation

$$\mu_n^2-\mu_n \operatorname{Sp}\hat{Q}_n+\operatorname{Det}\hat{Q}_n=0, \quad n=(r, l), \tag{24}$$

with the aid of the relations

$$\omega_n=\nu+\sigma-\operatorname{Im}\mu_n, \quad \alpha+\operatorname{Re}\mu_n=0, \tag{25}$$

which are the stationarity conditions for the amplitudes of the opposing waves (14). In accord with the condition (21) for single-mode generation, $\mu_n = \mu_{0n} + \mu_{1n}$, where μ_{1n} is the nonlinear change, proportional to the intensities I_r and I_l, in the eigenvalue. Neglecting terms of order μ_{1n}^2, we obtain from Eq. (24) that

$$\mu_{1n}=\frac{M_{01}}{M_{01}-M_{02}}[\operatorname{Sp}\hat{Q}_n-\operatorname{Sp}\hat{M}_n]-\frac{\operatorname{Det}\hat{Q}_n-\operatorname{Det}\hat{M}_n}{M_{01}-M_{02}}. \tag{26}$$

Using the formulas (17), (20), and (26), we find the expressions for $\Delta\mu = \mu_r - \mu_l$ and $\mu = \mu_r + \mu_l$:

$$\Delta\mu=I(Va+\Gamma_1\delta), \quad \mu=I(\Gamma_2+G_0a\delta)+2M_{01}, \tag{27}$$

where

$$I=\tfrac{1}{2}(I_r+I_l), \quad \delta=(I_r-I_l)/(I_r+I_l),$$
$$\Gamma_1=-2(\bar{R}-\bar{L})+(\Delta F+\Delta L)2b+Wd+if_0V=\Gamma_0+if_0V,$$
$$\Gamma_2=-2(\bar{R}+\bar{L})+(\Delta F-\Delta L)2b-Wd+if_0G_0,$$
$$V=\Delta F+\Delta L-\tfrac{1}{2}W, \quad G_0=\Delta F-\Delta L+\tfrac{1}{2}W,$$
$$a=|D|^{-2}[\tfrac{1}{2}s_{4\varepsilon}s_{4\Delta\varepsilon}+is_{2\Delta\varphi}s_{2\Delta\varepsilon}c_{2\varepsilon}(s_{2\varepsilon}^2-c_{2\Delta\varepsilon}^2)],$$
$$b=|D|^{-2}c_{2\Delta\varepsilon}^2s_{2\varepsilon}^2, \quad d=|D|^{-2}(c_{\Delta\varphi}^2c_{2\varepsilon}^2c_{2\Delta\varepsilon}^2+s_{\Delta\varphi}^2s_{2\varepsilon}^2s_{2\Delta\varepsilon}^2),$$
$$f_0=|D|^{-2}s_{2\Delta\varphi}s_{2\varepsilon}c_{2\Delta\varepsilon}(s_{2\varepsilon}^2-c_{2\Delta\varepsilon}^2),$$

D being the determinant of the matrix $\hat{A}_r$ defined in (19).

We shall hereafter denote the real parts of quantities by one prime and the imaginary parts by two primes, e.g., a = a′ + ia″. From the relations (25) and (27), we obtain expressions for the mean intensity and the frequency of generation:

$$I=\frac{2(\alpha+\operatorname{Re}M_{01})}{-\Gamma_2'+(G_0'a'-G_0''a'')(V'a'-V''a'')/\Gamma_1'},$$
$$\omega=\frac{1}{2}(\omega_r+\omega_l)=\nu-\frac{\psi}{2}\frac{c}{L_0}+\sigma- \tag{28}$$
$$-\frac{I}{2}\left[\Gamma_2''+\frac{1}{\Gamma_1'}(G_0'a''+G_0''a')(V''a''-V'a')\right].$$

Of greatest interest are the expressions for the differences between the intensities and the frequencies of the opposing waves:

$$\delta=(V''a''-V'a')/\Gamma_1',$$
$$\Delta\omega=\omega_l-\omega_r=\operatorname{Im}\Delta\mu=I\left[a'\left(V''-V'\frac{\Gamma_1''}{\Gamma_1'}\right)+a''\left(V'+V''\frac{\Gamma_1''}{\Gamma_1'}\right)\right]. \tag{29}$$

The frequency difference $\Delta\omega$ is proportional to the intensity, and depends on the opposing-wave polarization states, which are determined by the parameters of the ring resonator, and on the nonlinear-interaction coefficients. The expression (29) contains terms that are even, as well as terms that are odd, functions of the generation-frequency detuning relative to the center of the amplification contour.

The frequency difference $\Delta\omega$ vanishes in the case of linear polarizations of the opposing waves (ϵ_1, $\epsilon_2 = 0$, $\pm\pi/2$) with an arbitrary difference between the azimuths, in the case of circular polarizations ($\epsilon_1 = -\epsilon_2 = \pm\pi/4$), and in the case when the sum of the ellipticities ($\epsilon_1 + \epsilon_2$) vanishes; in all these cases a′ = a″ = 0.

Let us consider the case when the polarizations of the opposing waves differ little from each other (i.e., when $\Delta\epsilon$, $\Delta\varphi \ll 1$). We obtain, correct to second order in smallness in the polarization parameters $\Delta\epsilon$ and $\Delta\varphi$, the expression

$$\Delta\omega=2I\Delta\varepsilon\left[s_{4\varepsilon}\left(V''-V'\frac{\Gamma_0''}{\Gamma_0'}\right)-2\Delta\varphi c_{2\varepsilon}^3\left(1+s_{2\varepsilon}^2\frac{V''}{\Gamma_0'}\right)\left(V'+V''\frac{\Gamma_0''}{\Gamma_0'}\right)\right]. \tag{30}$$

The coefficient attached to $s_{4\epsilon}$ is an odd, while the coefficient attached to $\Delta\varphi$ is an even, function of the detuning, the odd component of $\Delta\omega$ being a quantity of first order in smallness, the even component a quantity of second order. Notice that $\Delta\omega$ is, as a rule, different from zero at the line center.

In the expression (30), the ellipticity parameter ϵ has an arbitrary value. Let us compare the nonlinear frequency shifts $\Delta\omega_l$ and $\Delta\omega_c$ for nearly linear ($\epsilon \ll 1$) and nearly circular ($\pi/4 - \epsilon = \beta \ll 1$) polarizations:

$$\Delta\omega_l=4\alpha_0\varkappa\Delta\varepsilon\left(\frac{4\varepsilon\Phi_1-\Delta\varphi\Phi_2}{\Phi_3}\right), \quad \Delta\omega_c=4\alpha_0\varkappa\Delta\varepsilon\frac{\Phi}{\Phi_4}\beta, \tag{31}$$

where

$$\Phi_1=\operatorname{Im}\{(F-L-W)(R^*-L_1^*)\}, \quad \Phi_2=|F-L-W|^2-|R-L_1|^2,$$
$$\Phi_3=[\operatorname{Re}(R+F)]^2-[\operatorname{Re}(L+L_1+W)]^2, \quad \Phi_4=(\operatorname{Re}R)^2-(\operatorname{Re}L_1)^2, \tag{32}$$

$\alpha_0 = \Delta\omega_p N_0/2N_{thr}$ is the amplification factor at the line center, and

$$\varkappa=\frac{\alpha+\operatorname{Re}M_{01}}{\alpha_0}=1-\left(\frac{\omega-\omega_0}{ku}\right)^2-\frac{N_{thr}}{N_0}$$

is the relative excess of the amplification factor at the generation frequency over its threshold value. In the pure isotope.

$$\Delta\omega_l=\frac{4\alpha_0\varkappa\Delta\varepsilon}{1+2K_0}\left(\frac{1+f^2}{2+f^2}\right)\left[\frac{4\varepsilon(K_1-1)}{f}+\Delta\varphi(1-2K_0)\right],$$
$$\Delta\omega_c=4\alpha_0\varkappa\Delta\varepsilon\beta f(1+f^2)\frac{(2K_0+1)(K_1-1)}{(1+f^2)^2-K_1^2}, \tag{33}$$

where

$$K_0=K'+K'', \quad K_1=\frac{2(K'\gamma_b+K''\gamma_a)}{\gamma_a+\gamma_b}.$$

In a 50% isotopic mixture,

$$\Delta\omega_l=\frac{4\alpha_0\varkappa\Delta\varepsilon}{1+2K_0}\left[16\varepsilon\left(\frac{\gamma_{ab}}{\Delta\omega_{is}}\right)^2(1+2y^2)(1-K_1)f+\Delta\varphi(1-2K_0)\right],$$
$$\Delta\omega_\kappa=16\alpha_0\varkappa\Delta\varepsilon\beta\left(\frac{\gamma_{ab}}{\Delta\omega_{is}}\right)^2\left(\frac{1-K_1}{1+2K_0}\right)f, \tag{34}$$

where

$$f=\frac{\omega_r+\omega_l-(\omega_{01}+\omega_{02})}{2\gamma_{ab}}$$

It can be seen from the expressions for $\Delta\omega_l$ and $\Delta\omega_c$ that they are both of second order in smallness in the polarization parameters, but that in the case of nearly circular polarization the frequency difference $\Delta\omega_c$ contains only terms that are odd functions of the detuning, and usually vanishes at the line center.

The nonlinear deformation of the polarization states can be obtained by comparing the eigenvectors of the matrices $\hat{Q}$ and $\hat{M}$, and turns out, in accord with the condition (22), to be small (proportional to I).

APPENDIX (December 20, 1973)

The complex coefficients of the interaction between circularly polarized running waves in an active mono-isotopic gaseous medium are given by[2)]

$$\begin{gathered}
\alpha-i\sigma=\alpha_0\{1-(1-2y^2)\eta^2f^2-i2[-\chi(1+2\eta\chi)\eta f+(1-2y^2)\eta^2f^2]\},\\
R=\alpha_0\{1+(1-2y^2)\eta^2(1+f^2)+i2(1-2y^2)\eta^2 f\}, \quad F=2K_0R,\\
L=4\alpha_0\left(\frac{\gamma_{ab}}{\Delta\omega_{is}}\right)^2\left\{1+\frac{2\gamma_a\gamma_b y^2}{(\gamma_a+\gamma_b)\gamma_{ab}}-i(1+2y^2)f\right\},\\
L_1=4\alpha_0\left(\frac{\gamma_{ab}}{\Delta\omega_{is}}\right)^2\left\{2K_0\frac{\gamma_a\gamma_b y^2}{(\gamma_a+\gamma_b)\gamma_{ab}}+K_1-iK_1(1+2y^2)f\right\},\\
W=4\alpha_0\left(\frac{\gamma_{ab}}{\Delta\omega_{is}}\right)^2\left\{2K_0\left(1+\frac{\gamma_a\gamma_b y^2}{(\gamma_a+\gamma_b)\gamma_{ab}}\right)-K_1-i(2K_0-K_1)(1+2y^2)f\right\},
\end{gathered} \tag{A.1}$$

where

$$2y=\frac{\Delta\omega_{is}}{ku}\approx 1, \quad \chi=\frac{1-2yF(y)}{\sqrt{\pi}\,e^{-y^2}}, \quad F(y)=e^{-y^2}\int_0^y e^{t^2}dt,$$

$\Delta\omega_{is}$ is the isotope line shift,

$$K_0=K'+K'', \quad K_1=\frac{2(K'\gamma_b+K''\gamma_a)}{\gamma_a+\gamma_b}, \quad \eta=\frac{\gamma_{ab}}{ku}\ll 1.$$

For $Ne^{20}:Ne^{22}$, $\lambda = 0.6328\ \mu$,

$$y=0.5, \quad \chi\approx 0.42, \quad K_0=0.24, \quad K_1\approx 0.32.$$

Note added in proof (February 20, 1974). Our attention has been drawn to G. S. Kruglik and É. G. Pestov's paper (Zh. Prikl. Spektrosk. 16, 985 (1972)), in which the influence of the polarization of the opposing waves on their competition in the ring laser is discussed. The expression (11) obtained in their paper for the polarization-induced frequency splitting differs from our formula (29) in that it depends on the detuning, the total angular momenta j_a and j_b, and the difference between the azimuths of the ellipses of polarization of the opposing waves.

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2) The authors thank V. A. Sokolov for computing the coefficients (A.1).

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