Saturated fluctuations in the laser radiation intensity in a turbulent medium

K. S. Gochelashvili and V. I. Shishov

P. N. Lebedev Physical Institute, USSR Academy of Sciences
(Submitted September 18, 1973)
Zh. Eksp. Teor. Fiz. 66, 1237–1247 (April 1974)

We develop an asymptotic theory of saturated fluctuations in the intensity of quasi-monochromatic radiation propagating over large distances in a turbulent medium with a power-law spectrum of refractive index fluctuations. We obtain an analytic expression for the correlation function of the intensity fluctuations for the case where the radiation starts out as a plane wave. We study the nature of the saturation of magnitude of the relative intensity fluctuations.

1. INTRODUCTION

The study of the statistics of the light intensity under conditions of multiple scattering of the radiation in a randomly inhomogeneous medium is of interest in many fields of physics and this problem therefore is the subject of intensive studies. Experimental data on the propagation of laser radiation[1–3] indicate strong distortion of the parameters of light wave produced by a turbulent medium. This effect manifests itself, in particular, in the fact that at large propagation distances the magnitude of the relative fluctuations in the radiation intensity (scintillation index) reaches values of the order unity and saturates at that level.

So far nobody has obtained a satisfactory description of the correlation properties of saturated fluctuations in the radiation intensity in a turbulent medium although recently much attention has been paid[4–8] to the problem of a theoretical study of the intensity statistics when radiation propagates along long trajectories. In[4,5], the propagation of a wave in a randomly inhomogeneous medium with a single characteristic size of the inhomogeneities was studied. Klyatskin[6] found that at large propagation distances the n-th moment of the radiation intensity $\langle I^n\rangle$ is proportional to n, where $n = 1, 2, \ldots$. However, if we evaluate the dispersion of the quantity I^n we find, in accordance with[6] when $n \geq 2$ that the quantity $\sigma_I^2 n = \langle I^{2n}\rangle - \langle I^n\rangle^2 \leq 0$, which is impossible. When $n = 1$ we have $\sigma_I^2/\langle I\rangle^2 = 1$, but this result can also not be taken to be proven in[6] as the same method was used there for all $n = 1, 2, \ldots$.

The present authors[7] studied the asymptotic behavior of the spectrum of the intensity fluctuations of a plane wave at large distances in a medium with a power-law spectrum of turbulence. The form of the intensity fluctuations was there established only in the low and high spatial frequency regions; the important region of the intermediate frequencies remained unstudied in[7]. Brown[8] gave a numerical solution of the equation for the fourth moment of the field in a turbulent medium with a two-dimensional geometry, but the results of that paper describe the transition of the scintillation index to the saturation regime incorrectly.

The present paper is devoted to a detailed study of the spectra and correlation properties of saturated fluctuation in the radiation intensity, when it propagates through a turbulent medium with a power-law spectrum of refractive-index fluctuations. We present a method to evaluate the asymptotic expansion of the solution of the equation for the fourth moment of the field which uses in an essential way the symmetry properties of that equation. As a result we obtain an exact estimate of how the correlation function of the intensity fluctuations tends to its asymptotic form and we establish the exact nature of the saturation of the magnitude of the relative intensity fluctuations for the case when the original type of the radiation is a plane wave.

2. SPECTRAL FUNCTION OF THE INTENSITY FLUCTUATIONS

When high-frequency quasi-monochromatic radiation propagates through a medium with large-scale fluctuations in the refractive index it is convenient to characterize the statistical properties of the medium by the function

$$\frac{\partial}{\partial z'}D(u')=4\pi k^2\int d\mathbf{q}[1-\exp(i\mathbf{q}\mathbf{u}')]\Phi(0,q),$$

which is formally the same as the longitudinal gradient of the phase structure function evaluated in the geometric optics approximation. Here $\Phi(q)$ is the three-dimensional spectrum of the fluctuations in the refractive index of the medium, $k = 2\pi/\lambda$ is the wave number, z' the coordinate in the direction of the propagation of the radiation, u' a two-dimensional vector in a plane at right angles to the direction of propagation. For a power-law spectrum of the refractive index fluctuations of the form[9]

$$\Phi(q)=(2\pi)^{-2}\Gamma(\alpha+1)\sin[(\alpha-1)\pi/2]C_n^2q^{-(\alpha+2)}\quad(1<\alpha<2),$$

where C_n^2 is a structure constant characterizing the strength of the turbulence in a layer, we have

$$\frac{\partial}{\partial z'}D(u')=Ck^2u'^\alpha,$$

$$C=-(\pi/2^\alpha)\Gamma(\alpha+1)\Gamma^{-2}(\alpha/2+1)\operatorname{ctg}(\alpha\pi/2)C_n^2.$$

We shall use in what follows dimensionless variables

$$z=z'C^{2/(2+\alpha)}k^{(4-\alpha)/(2+\alpha)},\quad u=u'C^{1/(2+\alpha)}k^{3/(2+\alpha)},$$

in terms of which the dimensionless gradient of the phase structure function has the form

$$\gamma(u)=u^\alpha.$$

The variable z is connected with the parameter $z^{(\alpha+2)/2}$ which has an explicit physical meaning: this quantity is the phase structure function evaluated at the radius of the first Fresnel zone. The parameter $z^{(\alpha+2)/2}$ characterizes the degree of distortion of the coherent properties of the radiation produced by the medium.

The spectral and correlation properties of the fluctuations in the radiation intensity in a randomly inhomogeneous medium are determined by the fourth moment of the complex field amplitude $E(\mathbf{u}, z)$:

$$W(\mathbf{u},\mathbf{v},z)=\langle E(\mathbf{u},z)E(\mathbf{v},z)E^*(0,z)E^*(\mathbf{u}+\mathbf{v},z)\rangle.$$

The Fourier transform of the function $W(\mathbf{u}, \mathbf{v}, z)$ with respect to $\mathbf{u}$

$$M(\mathbf{q}, \mathbf{v}, z) = (2\pi)^{-2}\int d\mathbf{u} \exp[-i\mathbf{q}\times\mathbf{u}]W(\mathbf{u}, \mathbf{v}, z)$$

satisfies in a weakly inhomogeneous medium the equation[7]

$$\left[\frac{\partial}{\partial z} - \mathbf{q}\nabla_v + \gamma(v)\right]M(\mathbf{q}, \mathbf{v}, z) = \gamma(v)\int d\mathbf{q}_1 p(\mathbf{q}-\mathbf{q}_1, \mathbf{v})M(\mathbf{q}_1, \mathbf{v}, z), \qquad (1)$$

where

$$p(\mathbf{q}, \mathbf{v}) = \gamma^{-1}(v)[1-\cos(\mathbf{qv})]Aq^{-(\alpha+2)}, \quad A = 2^\alpha\pi^{-2}\sin(\alpha\pi/2)\Gamma^2(\alpha/2+1). \qquad (2)$$

Equation (1) has the same form as the equation of radiative transfer where z is the time, q the velocity, v the coordinate, γ the absorption coefficient, and p the scattering indicatrix.[10]

We consider as the initial type of radiation a plane wave with unit amplitude:

$$M(\mathbf{q}, \mathbf{v}, z)|_{z=0} = \delta(\mathbf{q}). \qquad (3)$$

We can write the solution of the problem (1) to (3) in the form of an iteration series. We write it down for the function

$$M(\mathbf{q}, z) = M(\mathbf{q}, \mathbf{v}, z)|_{\mathbf{v}=0} = (2\pi)^{-2}\int d\mathbf{u}\cdot\exp[-i\mathbf{q}\times\mathbf{u}]\langle I(0, z)I(\mathbf{u}, z)\rangle, \qquad (4)$$

which has the meaning of the spectral function of the intensity fluctuations,

$$M(\mathbf{q}, z) = \sum_{n=0}^{\infty} M_n(\mathbf{q}, z), \qquad (5)$$

where

$$M_0(\mathbf{q}, z) = \delta(\mathbf{q}),$$

$$M_n(\mathbf{q}, z) = \int_0^z dz_1 \int_0^{z_1} dz_2 \ldots \int_0^{z_{n-1}} dz_n \int d\mathbf{q}_1 \ldots \int d\mathbf{q}_{n-1}$$

$$\times\exp[-\mathscr{L}_n]\prod_{i=1}^{n}\gamma(v_i)p(\mathbf{q}_{i-1}-\mathbf{q}_i, \mathbf{v}_i), \quad n = 1, 2, \ldots, \qquad (6)$$

$$\mathbf{v}_i = \sum_{k=1}^{i}\mathbf{q}_{k-1}(z_{k-1}-z_k), \quad i = 1, 2, \ldots, n,$$

$$\mathscr{L}_n = \sum_{i=0}^{n}\int_{z_{i+1}}^{z_i} dz'\gamma(\mathbf{v}_i + \mathbf{q}_i[z_i - z']).$$

We assumed in (6) that

$$\mathbf{q}_0 = \mathbf{q}, \quad \mathbf{q}_n = 0, \quad z_0 = z, \quad z_{n+1} = 0.$$

The expansion (5) is according to the terminology used in transfer theory[10] an expansion in the multiplicity of the scattering. In the region of low spatial frequencies (we shall below establish the boundary of that region) the spectrum of the intensity fluctuations can be described by the first terms of the series (5):

$$M^{LF}(\mathbf{q}, z) = \delta(\mathbf{q}) + M_1(\mathbf{q}, z) + M_2(\mathbf{q}, z) + \ldots \qquad (7)$$

In the single scattering approximation we have here

$$M_{(1)}^{LF}(\mathbf{q}, z) = \delta(\mathbf{q}) + M_1(q, z) = \delta(\mathbf{q}) + Aq^{-(\alpha+2)}z\int_0^1 d\zeta\{1-\cos[q^2z\zeta]\}$$
$$\times\exp\left[-q^\alpha z^{\alpha+1}\zeta^\alpha\left(1-\frac{\alpha}{\alpha+1}\zeta\right)\right]. \qquad (8)$$

In[11], a solution of Eq. (1) was found which was close to the single scattering approximation (8). However, the quantity $\mathscr{L}_1$ which determines the "absorption" was evaluated incorrectly in[11]: only the absorption after the scattering process was taken into account, but the absorption before the scattering took place was neglected. The region of applicability of the expressions obtained in[11] does not go beyond the region of applicability of the smooth perturbations method.

If $q < q^*$, where

$$q^* = z^{-(\alpha+1)/\alpha},$$

we can in the integrand in (8) put the exponent equal to unity and hence we get[1)]

$$M_1(q, z) \approx Aq^{-(\alpha+2)}z\{1-(q^2z)^{-1}\sin(q^2z)\}. \qquad (9)$$

If in that case also $q < q_{Fr}$, where $q_{Fr} = z^{-1/2}$ is the spatial frequency which is the inverse of the radius of the first Fresnel zone, we have from (9)

$$M_1(q, z) \approx {}^1/_6 Az^3q^{(2-\alpha)}, \qquad (10)$$

while if $q_{Fr} < q < q^*$ (which clearly occurs only if $z < 1$)

$$M_1(q, z) \approx Aq^{-(\alpha+2)}z.$$

Expression (9) has a maximum when $q \approx q_{Fr}$. When $q < (u^*)^{-1}$ where

$$u^* = z^{-1/\alpha},$$

we can expand the cosine in the integrand in (8) in a power series and restrict ourselves to the first two terms. Accordingly we get

$$M_1(q, z) \approx \frac{1}{2}Aq^{2-\alpha}z^3\int_0^1 d\zeta\zeta^2\exp\left[-q^\alpha z^{\alpha+1}\zeta^\alpha\left(1-\frac{\alpha}{\alpha+1}\zeta\right)\right]. \qquad (11)$$

Hence we find for $q \lesssim q^*$ that $M_1(q, z)$ is given by Eq. (10) while for $q \gg q^*$

$$M_1(q, z) \approx \frac{1}{2\alpha}A\Gamma\left(\frac{3}{\alpha}\right)z^{-3/\alpha}q^{-(\alpha+1)}. \qquad (12)$$

Expression (11) has a maximum for $q \sim q^*$.

In the region of small values of the phase structure function evaluated at the radius of the first Fresnel zone, $z < 1$, use of the expansion (7) allows us to describe the spatial spectrum of the intensity fluctuations with a given accuracy. However, when z increases random interference leads to the appearance of all smaller scales in the field distribution; the range of frequencies occupied by the spectrum widens and when $z \gg 1$ the main part of the spectrum is described by the high-frequency asymptotic form. In the high-frequency region the spectrum is basically described by the terms in the series with the largest n lying in the range $n_0 - \sqrt{n_0} < n < n_n + \sqrt{n_0}$, where $n_0 = z^{(\alpha+2)/2}$. An analysis of the contribution from these terms to the sum (5) shows that in the frequency range $q > q^*$ the spatial spectrum of the intensity fluctuations can be written in the form of an asymptotic expansion:

$$M^{HF}(q, z) = M_{(0)}^{HF}(q, z) + M_{(1)}^{HF}(q, z) + \ldots. \qquad (13)$$

Here

$$M_{(0)}^{HF}(q, z) = (2\pi)^{-1}\int_0^\infty du\, u\exp[-\gamma(u)z]J_0(qu).$$

The quantity $M_{(0)}^{HF}$ has in the region $q \lesssim (u^*)^{-1}$ the form

$$M_{(0)}^{HF}(q, z) \approx (2\pi\alpha)^{-1}\Gamma(2/\alpha)z^{-2/\alpha}, \qquad (14)$$

and in the frequency range $q \gg (u^*)^{-1}$

$$M_{(0)}^{HF}(q, z) \approx (2\pi^3)^{-1/2}\Gamma\left(\alpha+\frac{3}{2}\right)\sin(\alpha\pi/2)zq^{-(\alpha+2)}. \qquad (15)$$

The first correction to the asymptotic form of the spectrum in the high-frequency region is given by the expression (see the Appendix)

$$
\begin{aligned}
M^{HF}_{(1)}(\mathbf{q}, z) = (2\pi)^{-2} Az \int_0^1 d\zeta \int d\mathbf{q}' \int d\mathbf{u} \exp(-i\mathbf{q}\mathbf{u})(q')^{-(\alpha+2)}\{1 \\
-\cos[\mathbf{q}'(\mathbf{u}+\mathbf{q}'z\zeta)]\}\exp\left[-|\mathbf{u}+\mathbf{q}'z\zeta|^{\alpha} z(1-\zeta) - z\zeta \int_0^1 ds|\mathbf{u}+\mathbf{q}'z\zeta s|^{\alpha}\right].
\end{aligned}
\tag{16}
$$

In the frequency range $q \lesssim (u^*)^{-1}$ we have

$$
M^{HF}_{(1)}(q, z) \propto \begin{cases} z^{-(6-\alpha^2)/\alpha}, & \alpha > \sqrt{2}, \\ z^{-4/\alpha} q^{-(2-\alpha^2)}, & \alpha < \sqrt{2}, \end{cases}
$$

and when $q \gg (u^*)^{-1}$ we have

$$
\begin{aligned}
M^{HF}_{(1)}(q, z) \approx -(\sqrt{2/\pi}) 2^{-\alpha} A \Gamma\left(1-\frac{\alpha}{2}\right) \Gamma^{-1}\left(\frac{\alpha}{2}\right) \Gamma\left(\alpha+\frac{7}{2}\right) \sin\frac{\alpha\pi}{2} \\
\times (\alpha-1)^{-1} {}_2F_1\left(-1, \alpha-1; \alpha; \frac{\alpha}{\alpha+1}\right) z^{\alpha} q^{-6}.
\end{aligned}
$$

For $z \ll 1$ we have the relation

$$
(u^*)^{-1} \ll q_{Fr} \ll q^*, \tag{17}
$$

and the essential range of frequencies is described by Eq. (9); the main power of the spectrum occurs at frequencies $q \lesssim q_{Fr}$.

When $z \gg 1$ the relation which is the opposite of (17) holds:

$$
q^* \ll q_{Fr} \ll (u^*)^{-1},
$$

and $M^{LF}_{(1)}(q^*, z) \gg M^{HF}_{(0)}(q^*, z)$. This means that for $q \lesssim q^*$ the main contribution to $M(q, z)$ is given not by the terms in series (5) with n in the vicinity of $n_0 \approx z^{(\alpha+2)/2}$ but by the first term which is given explicitly by Eq. (11). An estimate shows that the second term of this series $M_2(q, z)$ is of the order

$$
z^{-(4-\alpha^2)/\alpha} M_1(q, z) \ll M_1(q, z).
$$

The contribution from the terms with n in the range $2 \leq n \ll n_0 - \sqrt{n_0}$ will be less than the sum $M^{LF}_{(1)} + M^{HF}_{(0)}$ in the whole of the range $q > q^*$ and does not exceed, as to order of magnitude, $M_2(q, z)$ or the first correction to $M^{HF}_{(0)}$ so that we can neglect them.

For $q > q^*$ the quantity $M^{LF}_{(1)}(q, z) \propto q^{-(\alpha+1)}$ according to (12) and decreases with increasing q. Hence, $M^{LF}_{(1)}$ determines (for $z \gg 1$) the behavior of $M(q, z)$ up to frequencies q^{**}, where

$$
M^{LF}_{(1)}(q^{**}, z) = M^{HF}_{(0)}(q^{**}, z).
$$

Using (12) and (14) we get

$$
q^{**} \approx \{A\pi\Gamma^{-1}(2/\alpha)\Gamma(3/\alpha)\}^{1/(\alpha+1)} z^{-1/\alpha(\alpha+1)}.
$$

In the frequency range $q \lesssim (u^*)^{-1}$ the ratio $M^{HF}_{(1)}(q, z)/M^{HF}_{(0)}(q, z)$ is of order $z^{-(4-\alpha^2)/\alpha}$. $M(q, z)$ approaches $M^{HF}_{(0)}$ when $q > q^{**}$. According to (14) the spectrum becomes constant when the frequency increases up to $q \sim z^{1/\alpha}$; when $q > z^{1/\alpha}$ the spectrum decreases as $M(q) \propto \Phi(q)$ according to (15). The main power of the spectrum occurs at frequencies $q^* \lesssim q \lesssim z^{1/\alpha}$. We show the qualitative form of the spectrum for $z \gg 1$ in Fig. 1.

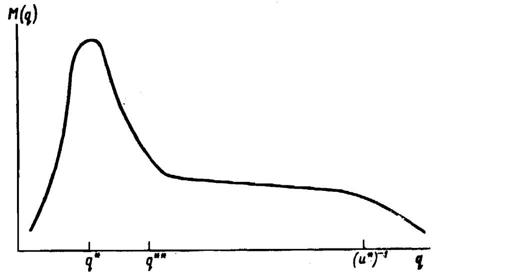

FIG. 1. Qualitative form of the spatial spectrum of the intensity fluctuations M(q,z) for large distances z.

3. SCINTILLATION INDEX AND CORRELATION FUNCTION OF THE INTENSITY FLUCTUATIONS

The scintillation index (the magnitude of the relative intensity fluctuations) is determined by the equation

$$
m^2 = (\langle I^2 \rangle - \langle I \rangle^2)/\langle I \rangle^2.
$$

In the region of small values of the phase structure function evaluated at the radius of the first Fresnel zone, $z < 1$, it is sufficient when evaluating the inverse of the Fourier transform (4) to restrict ourselves to integrate over the low-frequency section of the spectrum (8) and to use the approximate expression (9) for $M_1(q, z)$. As a result we get

$$
\begin{aligned}
m^2 = \frac{2^{(\alpha+1)}}{2+\alpha} \Gamma\left(1+\frac{\alpha}{2}\right) \\
\times \cos\left(\frac{\alpha\pi}{4}\right) z^{(\alpha+2)/2}.
\end{aligned}
\tag{18}
$$

It is clear that in the region of weak fluctuations ($z < 1$) the relative intensity fluctuations are proportional to the magnitude of the phase structure function evaluated at the radius of the first Fresnel zone. When $\alpha = 5/3$ the expression obtained is the same as the one evaluated using the method of smooth perturbations.[9]

In the case of two-dimensional inhomogeneities ($\gamma(u)$ depends only on u_x) we have

$$
m^2 = \frac{2}{2+\alpha} \Gamma(1+\alpha) \Gamma^{-1}\left(1+\frac{\alpha}{2}\right) \cos\left(\frac{\alpha\pi}{4}\right) z^{(\alpha+2)/2}, \quad z < 1. \tag{18$'$}
$$

In the region of large values of the phase structure function evaluated at the radius of the first Fresnel zone, $z \gg 1$, the scintillation spectrum is, in accordance with what was said earlier, determined for $q < q^*$ by M^{LF} only, and when $q > q^*$ by the sum $M^{LF} + M^{HF}$. We write m^2 as the sum of two components:

$$
m^2 = m_{LF}^2 + m_{HF}^2,
$$

where

$$
m_{LF}^2 = \int d\mathbf{q} \{M^{LF}(\mathbf{q}, z) - \delta(\mathbf{q})\}
$$

and

$$
m_{HF}^2 = \int_{q \geq q^*} d\mathbf{q}\, M^{HF}(\mathbf{q}, z).
$$

When evaluating m^2_{LF} we restrict ourselves in the expansion (7) to the single scattering approximation (8) and use the approximate expression (11). After integrating we find

$$
m_{LF}^2 \approx B_{LF}(\alpha) z^{-(4-\alpha^2)/\alpha},
$$

where

$$
\begin{aligned}
B_{LF}(\alpha) = \frac{2^{\alpha}}{\pi\alpha(\alpha-1)} \Gamma^2\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{4}{\alpha}-1\right) \sin\frac{\alpha\pi}{2} \\
\times {}_2F_1\left[\frac{4-\alpha}{\alpha}; \alpha-1; \alpha; \frac{\alpha}{\alpha+1}\right].
\end{aligned}
\tag{19}
$$

When $\alpha = 5/3$ the quantity $B_{LF} = 0.6$ with a relative error of not more than 5%. The accuracy of Eq. (19) is determined by the contribution from the next term $M_2(q, z)$ in the expansion (7); that contribution is of order $z^{-2(4-\alpha^2)/\alpha}$.

When evaluating m^2_{HF} we get by integrating the quantity $M^{HF}_{(0)}(q, z)$ from the expansion (13) over $\mathbf{q}$

$$m^2_{HF(0)}=1+O\{z^{-2(\alpha+2)/\alpha}\}.$$

The term, next in order of smallness, in the expansion in m^2_{HF} is determined by the contribution from $M^{HF}_{(1)}(q, z)$ given by (16). It is equal to

$$m^2_{HF(1)}=B_{HF}(\alpha)z^{-(4-\alpha^2)/\alpha}, \tag{20}$$

where $B_{HF}(\alpha) = B_{LF}(\alpha)$. The accuracy of Eq. (20) is of order $z^{-2(4-\alpha^2)/\alpha}$. Details of the evaluation of $m^2_{HF(1)}$ are given in the Appendix.

The scintillation index is thus for $z \gg 1$ equal to

$$m^2=1+B(\alpha)z^{-(4-\alpha^2)/\alpha}+O\{z^{-2(4-\alpha^2)/\alpha}\}, \tag{21}$$

where $B(\alpha) = 2B_{LF}(\alpha)$. Similar calculations for the case of two-dimensional inhomogeneities in the refractive index lead to the relation

$$m^2=1+\tilde{B}(\alpha)z^{-(4-\alpha^2)/\alpha}, \quad z>1, \tag{22}$$

where the quantity $\tilde{B} = 0.88$ when $\alpha = 5/3$.

We show in Fig. 2 the theoretical values of the scintillation index obtained for two-dimensional inhomogeneities using Eqs. (18′) and (22) (dotted curves) and the results of numerical calculations by Brown.[8] For large z Brown's curve lies below the curve obtained by us. This disagreement is caused by the following fact. When solving the equation for the fourth moment of the field numerically boundary conditions were given in[8] at $u_{x,0} = v_{x,0} = 5.36$ which in fact meant the introduction of an external scale of turbulence of $\sim u_{x,0}, v_{x,0}$.

The asymptotic behavior $\tilde{B}z^{-(4-\alpha^2)/\alpha}$ in (22) calculated by us for a purely power-law spectrum of the refractive index fluctuations is determined by frequencies $q \lesssim q^*$. When $z > 3$ the quantity $(q^*)^{-1} > u_{x0}, v_{x0}$ and in Brown's calculations the asymptotic behavior of $m^2(z)$ will be determined by the presence of boundary conditions.

We give in Fig. 3 the theoretical values of the scintillation index for the case of three-dimensional inhomogeneities calculated, using Eqs. (18) and (21) for $\alpha = 5/3$. These results agree well with the experimental data of[2].

Let us consider the correlation properties of the intensity fluctuations. Performing calculations similar to the ones given above we get easily for $z \ll 1$ for the correlation function of the intensity fluctuations results which are the same (for $\alpha = 5/3$) with those evaluated using the smooth perturbations method.[7]

For $z \gg 1$ we have

$$\begin{gathered}\langle I(0,z)I(u,z)\rangle-1\\ =\exp[-\gamma(u)z]+B_{LF}(\alpha)z^{-(4-\alpha^2)/\alpha}\{b_1(\alpha,uq^*)+b_2(\alpha,u/u^*)\}.\end{gathered} \tag{23}$$

Here

$$b_1(\alpha,\rho)=\alpha(\alpha-1)\Gamma^{-1}\left(\frac{4}{\alpha}-1\right){}_2F_1^{-1}\left[\frac{4-\alpha}{\alpha},\alpha-1;\alpha;\frac{\alpha}{\alpha+1}\right]$$
$$\times\int_0^1 d\zeta\,\zeta^2\int_0^\infty dx\,x^{3-\alpha}J_0(x\rho)\exp\left[-x^\alpha\zeta^\alpha\left(1-\frac{\alpha}{\alpha+1}\zeta\right)\right],$$

$b_1(\alpha, 0) = 1$. When $\rho \gg 1$

$$b_1(\alpha,\rho)\approx\left(-\frac{\alpha}{3}\right)(\alpha-1)\Gamma^{-1}\left(\frac{4}{\alpha}-1\right){}_2F_1^{-1}\left[\frac{4-\alpha}{\alpha},\alpha-1;\alpha;\frac{\alpha}{\alpha+1}\right]$$
$$\times\sqrt{\frac{2}{\pi}}\Gamma\left(\frac{7}{2}-\alpha\right)\sin\left(\frac{\alpha\pi}{2}\right)\rho^{-(4-\alpha)}.$$

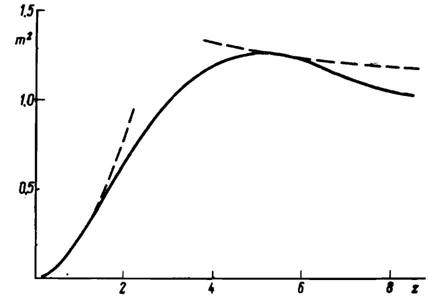

FIG. 2. The scintillation index m^2 as function of the distance z in a medium with two-dimensional inhomogeneities. The full-drawn curve shows Brown's numerical calculations, the dotted curves were obtained using approximate asymptotic expressions.

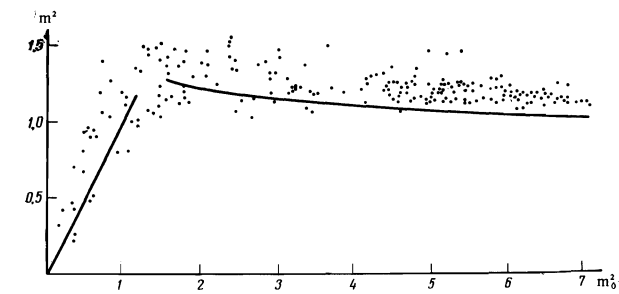

FIG. 3. The scintillation index of a collimated laser beam in a surface layer of the atmosphere as function of the scintillation index m_0^2 of a plane wave evaluated in the first approximation using the smooth perturbations method. [2] The full-drawn curves were constructed using the symptotic formulae (18) and (21) with $\alpha = 5/3$.

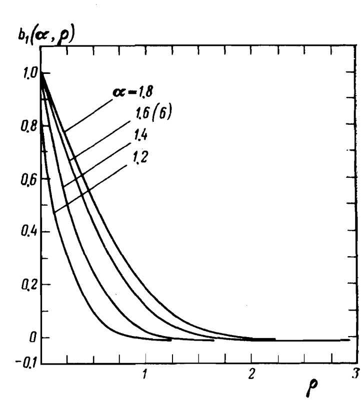

FIG. 4. The component $b_1(\alpha,\rho)$ of the correlation function of the intensity fluctuations, $z > 1$.

We show in Fig. 4 the graphical form of the functions $b_1(\alpha, \rho)$. The function

$$b_2\left(\alpha,\frac{u}{u^*}\right)=\frac{\alpha}{\pi}(\alpha-1)\Gamma^{-1}\left(\frac{4}{\alpha}-1\right){}_2F_1^{-1}\left[\frac{4}{\alpha}-1,\alpha-1;\alpha;\frac{\alpha}{\alpha+1}\right]$$
$$\times z^{2(2+\alpha)/\alpha}\int_0^1 d\zeta\int d\mathbf{n}\,n^{-(\alpha+2)}\left\{1-\cos\left[\mathbf{n}\left(\frac{\mathbf{u}}{u^*}+\mathbf{n}\zeta\right)z^{-(2+\alpha)/\alpha}\right]\right\}$$
$$\times\exp\left[-\left|\frac{\mathbf{u}}{u^*}+\mathbf{n}\zeta\right|^\alpha(1-\zeta)-\zeta\int_0^1 ds\left|\frac{\mathbf{u}}{u^*}+\mathbf{n}\zeta s\right|^\alpha\right],$$
$$b_2(\alpha,0)=1,$$

describes the corrections to the asymptotic form of the correlation function of the intensity fluctuations in the range of transverse scales of $\sim u^*$.

The first term in Eq. (23) is determined by $M^{HF}_{(0)}$, the second term by $M^{LF}_{(1)}$ and $M^{HF}_{(1)}$. It is clear that in the range $z \gg 1$ the correlation function contains two scale-lengths. The main energy occurs at fluctuations with a scale of $\sim u^*$ which decreases when the distance and strength of the turbulence increase. The fluctuations with a scale length $\sim (q^*)^{-1}$ which increases with increasing z are weaker in amplitude.

The decrease in the correlation radius of the intensity fluctuations (at the level e^{-1}) with increasing propagation distances (in the region of large values of the phase structure function evaluated at the radius of the first Fresnel zone) is clearly observed experimentally.[3]

The results of the present analysis allow us to assume that for sufficiently large values of z the field is distributed according to the normal law in a turbulent medium. However, the deviations of m^2 from unity and the behavior of the correlation function of the intensity fluctuations in the region of transverse scale lengths $\sim (q^*)^{-1}$ are just connected with the deviation of the field distribution from the normal law.

Let us indicate a number of effects which may lead to a disagreement between the actually observed behavior of $m^2(z)$ from that evaluated in the present paper. We neglected in our analysis the presence of external L_0 and internal l_0 scale lengths of the turbulence. The effect of the external scale length leads to a faster saturation of the intensity fluctuations and the influence of the internal scale length, on the other hand, will lead to a slower saturation of the fluctuations. The effect of the boundedness of the beam can also lead to a slowing down of the saturation of the intensity fluctuations.

The multiple mode nature of actual wave beams leads to a cut-off of the high spatial frequencies in the scintillation spectrum. If the direction of wave propagation lies within a cone of opening angle φ_0 the size of φ_0 begins to manifest itself for $z > [\varphi_0 c^{1/(\alpha+2)} k^{(1-\alpha)/(\alpha+2)}]^{\alpha/(\alpha+1)}$. This inequality follows from the condition $\varphi_0 z' > b_0$ (z' is the dimensionless distance and b_0 the dimensionless scale of the diffraction picture) which was obtained in[12] for a medium with a single characteristic inhomogeneity scale.

In conclusion we express our deep gratitude to A. M. Prokhorov for his interest in the paper. We are grateful to E. A. Zubov for doing the calculations on the ÉVM.

APPENDIX

It follows from the form of the equation for the fourth moment of the complex field amplitude W(u, v, z) in the coordinate representation[7] and the boundary conditions that W is symmetric under an interchange of the variables **u** and **v**. All terms in the asymptotic series for W for large z must have the same symmetry.

We shall write down the main term in the asymptotic expansion of the solution of the equation for W for $v \neq 0$:[7]

$$W_{(0)}(u, v, z) = \exp[-\gamma(u)z] + \exp[-\gamma(v)z]. \qquad (A.1)$$

The first correction to the asymptotic solution of (A.1) is given by the relation

$$W_{(1)}(\mathbf{u},\mathbf{v},z) = W_{(1)}^{(1)}(\mathbf{u},\mathbf{v},z) + W_{(1)}^{(2)}(\mathbf{u},\mathbf{v},z),$$

where

$$W_{(1)}^{(1)}(\mathbf{u},\mathbf{v},z) = \int d\mathbf{q}\exp(i\mathbf{qu})M_1(\mathbf{q},\mathbf{v},z),$$

$$W_{(1)}^{(2)}(\mathbf{u},\mathbf{v},z) = \int d\mathbf{q}\exp(i\mathbf{qu})M_{(1)}^{HF}(\mathbf{q},\mathbf{v},z).$$

$W_{(1)}^{(1)}$ and $W_{(1)}^{(2)}$ do here not possess separately the symmetry under an exchange of the variables **u** and **v** and do not contain components possessing this symmetry. Hence we must have the relation

$$W_{(1)}^{(2)}(\mathbf{u},\mathbf{v},z) = W_{(1)}^{(1)}(\mathbf{v},\mathbf{u},z);$$

whence we find

$$M_{(1)}^{HF}(\mathbf{q},\mathbf{v},z) = (2\pi)^{-2}\int d\mathbf{u}\int d\mathbf{q}'\exp(-i\mathbf{qu}+i\mathbf{q}'\mathbf{v})M_1(\mathbf{q}',\mathbf{u},z), \qquad (A.2)$$

where

$$M_1(\mathbf{q},\mathbf{v},z) = Aq^{-(\alpha+2)}z\int_0^1 d\zeta\{1-\cos[\mathbf{q}(\mathbf{v}+\mathbf{q}z\zeta)]\}\cdot$$
$$\times\exp\left[-|\mathbf{v}+\mathbf{q}z\zeta|^\alpha z(1-\zeta) - z\int_0^\zeta d\eta|\mathbf{v}+\mathbf{q}z\eta|^\alpha\right].$$

Using (A.2) we can evaluate

$$\int d\mathbf{q}\exp(i\mathbf{qu})M_{(1)}^{HF}(\mathbf{q},0,z)$$
$$= Az^{\alpha+2}\int_0^1 d\zeta\int d\mathbf{n}\, n^{-(\alpha+2)}\left\{1-\cos\left[\mathbf{n}\left(\frac{\mathbf{u}}{u^*}+\mathbf{n}\zeta\right)z^{-(2+\alpha)/\alpha}\right]\right\}$$
$$\times\exp\left[-\left|\frac{\mathbf{u}}{u^*}+\mathbf{n}\zeta\right|^\alpha(1-\zeta)-\zeta\int_0^1 ds\left|\frac{\mathbf{u}}{u^*}+\mathbf{n}\zeta s\right|^\alpha\right].$$

For u = 0 we have

$$\int d\mathbf{q}\, M_{(1)}^{HF}(\mathbf{q},0,z) = \pi Az^{-(4-\alpha^2)/\alpha}\alpha^{-1}(\alpha-1)^{-1}$$
$$\times\Gamma\left(\frac{4}{\alpha}-1\right){}_2F_1\left[\frac{4}{\alpha}-1,\alpha-1;\alpha;\frac{\alpha}{\alpha+1}\right],$$

whence it follows immediately that $B_{HF}(\alpha) = B_{LF}(\alpha)$.

---

1)We note that the region of applicability of the expression for the low-frequency approximation of the spectrum in [7] was limited by the condition $q < q^*$.

1V. Ya. S'edin, S. S. Khmelevtsov, and M. F. Nebol'sin, Izv. vuzov, Radiofizika **13**, 44 (1970) [Radiophys. Qu. Electron. **13**, 32 (1972)].

2M. E. Gracheva, A. S. Gurvich, S. S. Kashkarov, and V. V. Pokasov, Preprint OOFAG, Moscow, 1973.

3G. R. Ochs, R. R. Bergman, and J. R. Snyder, J. Opt. Soc. Am. **59**, 231 (1969).

4I. M. Dagkesamanskaya and V. I. Shishov, Izv. vuzov, Radiofizika **13**, 16 (1970) [Radiophys. Qu. Electron. **13**, 9 (1972)].

5V. I. Shishov, Zh. Eksp. Teor. Fiz. **61**, 1399 (1971) [Sov. Phys.-JETP **34**, 744 (1972)].

6V. I. Klyatskin, Zh. Eksp. Teor. Fiz. **60**, 1300 (1971) [Sov. Phys.-JETP **33**, 703 (1971)].

7K. S. Gochelashvili and V. I. Shishov, Optica Acta **18**, 767 (1971).

8W. P. Brown, J. Opt. Soc. Am. **62**, 966 (1972).

9V. I. Tatarskii, Rasprostranenie voln v turbulentnoĭ atmosfere (Wave Propagation in a Turbulent Atmosphere) Nauka, 1967.

10V. V. Sobolev, Perenos luchistoĭ energii v atmosferakh zvezd i planet (Radiant Energy Transfer in the Atmospheres of Stars and Planets) Gostekhizdat, 1956.

11V. I. Tatarski, The effects of the Turbulent Atmosphere on Wave Propagation, Jerusalem, 1971; V. I. Tatarskii, Rasprostranenie korotkikh voln v srede so sluchaĭnymi neodnorodnostyami v priblizhenii markovskogo sluchaĭnogo protsessa (Propagation of Short Waves in a Medium with Random Inhomogeneities in the Random Markov Process Approximation) Preprint OOFAG, Moscow, 1970.

12V. I. Shishov, Izv. vuzov, Radiofizika **15**, 1278 (1972).

Translated by D. ter Haar
127