Evolutionality of magnetohydrodynamic discontinuities with allowance for dissipative waves

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It is shown that dissipative effects must be taken into account in investigations of the evolutionary properties of singular inclined waves and rotational discontinuities in magnetohydrodynamics. Dissipative waves must be taken into account in the system of small-amplitude waves that travel away from the discontinuity. It turns out as a result that a unique solution of the small-perturbation problem exists for the switching-off wave, i.e., such a wave is evolutionary; the switching-on shock wave in the linear approximation and the rotational discontinuity are not evolutionary.

The question of the evolutionality of switching-on and switching-off waves has a long history (see, e.g.[1-3]), but no satisfactory solution has been obtained so far. We recall that the switching-on and switching-off waves, or singular oblique shock waves, are respectively limiting cases of evolutional rapid and slow magnetohydrodynamic shock waves; in a coordinate system in which $\mathbf{V} \parallel \mathbf{H}$, the magnetic field is perpendicular to the wave front ahead of the fast magnetohydrodynamic shock wave and equal to the Alfven velocity behind it, whereas for the slow magnetohydrodynamic shock wave the situation is reversed. The problem of interaction of fast and slow shock waves with Alfven waves of infinitesimally small amplitude has a unique solution in the form of a system consisting of a reflected and transmitted wave, and consequently these waves are evolutional (see[4]). In the limiting case of a switching-on wave, the reflected wave has zero velocity relative to the shock-wave front, and an infinitesimally small wavelength; for the switching-on wave, a similar situation obtains for one of the transmitted waves. Such reflected and transmitted waves were regarded in[1] as inadmissible, so that the switching-on and switching-off shock waves were assumed to be non-evolutional.

Anderson[2] arrived at the same conclusion on the basis of a calculation of the amplitudes of the outgoing waves, one of which is infinite for the switching-on and switching-off waves.

These conclusions are unsatisfactory because the method employed can be used only when the wavelength of the perturbations is much larger than the width of the discontinuity. This condition, however, is violated on going to singular shock waves: as indicated, the length of one of the outgoing waves tends to zero. In addition, the possible appearance of very short waves makes it necessary to take into account the dissipation and the corresponding wave damping.

In this article, the evolutionality of the switch-on and switch-off waves is considered with dissipation taken into account. This gives rise to dissipative waves that propagate in the system and attenuate rapidly with increasing distance from the discontinuity. With dissipation taken into account, it becomes possible to determine rigorously the number of waves that emerge from the singular discontinuity. It turns out as a result that the switching-off wave is evolutional while the switching-on wave is non-evolutional in the linear approximation.

Alfven perturbations in a homogeneous medium, when all the dissipative processes are taken into account, are described by the following system of equations:

$$\frac{\partial h_z}{\partial t} + V_x\frac{\partial h_z}{\partial x} - H_x\frac{\partial v_z}{\partial x} - \beta\frac{\partial^2 h_z}{\partial x^2} = 0,$$
$$\frac{\partial v_z}{\partial t} + V_x\frac{\partial v_z}{\partial x} - \frac{H_x}{4\pi\rho}\frac{\partial h_z}{\partial x} - \nu\frac{\partial^2 v_z}{\partial x^2} = 0. \qquad (1)$$

The unperturbed vectors of the plasma velocity $\mathbf{V}$ and of the magnetic field intensity $\mathbf{H}$ lie in this case in the (x, y) plane; the wave propagates along the x axis; v_x and h_z are perturbations of the z-components of the velocity and of the magnetic field in the wave, while ρ, ν, and β are respectively the unperturbed plasma density and kinematic and magnetic viscosities. Substituting in the system (1) the relations

$$v_z = v\exp[i(\omega t - kx)], \quad h_z = \sqrt{4\pi\rho}\, b\exp[i(\omega t - kx)],$$

we obtain from the condition that there exist a nontrivial solution the following dispersion law for the Alfven perturbations, as well as the connection between b and v in the wave:

$$k^2V_A^2 - (\omega - kV - i\beta k^2)(\omega - kV - i\nu k^2) = 0, \qquad (2)$$

$$v = -\frac{\omega - kV - i\beta k^2}{kV_A}b, \qquad (3)$$

where $V_A^2 = H_x^2/4\pi\rho$, and we have left out the x index of V, since the other components of V will not be used. Being interested in long waves, we assume ω to be small (a criterion of smallness will be presented below); then, expanding k in powers of ω, we obtain from (2):

at $V = V_A$

$$k_{1,2} = \pm\sqrt{\frac{\omega}{\nu+\beta}}(1-i), \quad k_3 = \frac{\omega}{2V} - i\frac{(\nu+\beta)\omega^2}{16V^3}, \qquad (4)$$
$$k_4 = -\frac{\omega(\nu^2+\beta^2)}{V(\nu+\beta)^2} + i\frac{V(\beta+\nu)}{\beta\nu};$$

at $V \neq V_A$

$$k_{1,2} = \frac{\omega}{V\pm V_A} - i\frac{(\nu+\beta)\omega^2}{2(V\pm V_A)^3}, \qquad (5)$$

$$k_{3,4} = -\frac{\omega[(\beta-\nu)^2V \pm (\beta+\nu)B]}{4V_A^2\beta\nu + V^2(\beta-\nu)^2 \pm V(\beta+\nu)B} + i\frac{V(\beta+\nu)\pm B}{2\beta\nu},$$

where B denotes for brevity the arithmetic value of $(V^2(\beta-\nu)^2 + 4\beta V_A^2\nu)^{1/2}$.

Let us consider the properties of the waves corresponding to the obtained values of k. We note beforehand that the switching-off wave, as the particular case of slow shock waves, has a finite intensity, while the switching-on wave exists only in the interval

$$V_{A1}^2 < V_1^2 < V_{A1}^2 + \frac{2}{\gamma-1}(V_{A1}^2 - C_1^2),$$

where C is the speed of sound in the medium, γ is the adiabatic exponent, and the subscript 1 labels quantities ahead of the shock wave. Therefore the width of the front of the singular shock waves is bounded from below by a quantity on the order of $(\nu + \beta)/|V - V_A|$. (Here and below, the difference $V - V_A$ is taken naturally on that side of the discontinuity where it does not vanish. We assume V and V_A to be positive; in the rest system of the shock wave, the plasma moves in the positive x direction.) It follows from the foregoing that the waves corresponding to k_4 from (4) and to $k_{3,4}$ from (5) attenuate significantly over a length on the order of the width of the discontinuity, and do not exist outside the shock-wave front. The amplitudes of these waves will therefore not enter in the conservation laws that connect the plasma perturbations outside the shock-wave front.

Let us determine the propagation direction of the remaining waves, i.e., the energy-transport direction. In the presence of wave damping, the velocity, and possibly also the direction of the energy transport do not coincide with the group velocity. In a stable medium, however, such as the homogeneous plasma on both sides of the discontinuity, all the waves attenuate in the propagation direction, and consequently waves with positive imaginary part of k transport energy in the negative x direction, and vice versa. We note also that the waves corresponding to $k_{1,2}$ from (4) are dissipative, namely, they vanish in the absence of dissipation and attenuate over the proper wavelength. Further, the condition that the width of the discontinuity be small in comparison with the wavelength of the perturbations yields for waves with k_3 from (4) and $k_{1,2}$ from (5), just as for the waves with $k_{1,2}$ from (4),

$$\omega \ll (V-V_A)^2/(\nu+\beta). \qquad (6)$$

This condition coincides with the condition used in the derivation of expressions (4) and (5).

Knowing $k(\omega)$, we can find the proportionality coefficient in (3) between b and v in the perturbation wave. For undamped Alfven waves, it is equal to ± 1; the upper and lower sign corresponds to waves traveling upstream and downstream, respectively. The deviation of the proportionality coefficient from ± 1 for damped waves is of importance to us only for waves with $k_{1,2}$ from (4). For these we have, at the assumed accuracy,

$$v=\left(1\mp\frac{\nu-\beta}{V}\sqrt{\frac{i\omega}{2(\nu+\beta)}}\right)b. \qquad (7)$$

We now proceed directly to the investigation of the evolutionality of the switching-on and switching-off waves. As is well known, the evolutionality of a discontinuity means that a solution of the small-perturbation problem exists and is unique[5], i.e., if we know the amplitude of the wave incident on the discontinuity we can simultaneously determine the amplitude of the outgoing waves: the amplitudes of the incident waves, naturally, are assumed given. To find the unknown amplitudes in the case of normal incidence of a plane Alfven wave, we have two equations obtained by linearizing the relations on the shock wave[1]

$$\{\rho V v-\rho V_A b\}=0, \quad \{\rho^{1/2} V_A v-\rho^{1/2} V b\}=0, \qquad (8)$$

where the curly brackets denote the difference between the quantities on both sides of the discontinuity. It follows from the foregoing that the switching-on wave and the switching-off wave each has two outgoing waves, one of which is in the general case dissipative. The waves outgoing from the switching-on wave for which $V_1 > V_{A1}$ and $V_2 = V_{A2}$ are those with wave vectors k_1 and k_3 from (4), and both travel downstream. The waves outgoing from the switching-off wave, for which $V_1 = V_{A1}$ and $V_2 < V_{A2}$ have the wave vectors k_2 from (4) upstream and k_1 from (5) downstream.

Simple calculations yield the amplitudes of the outgoing waves in the case when an Alfven wave of unit amplitude is incident from $+\infty$ and $-\infty$ on the switching-off wave: the dissipative-wave amplitude is

$$A_r=-\frac{2V_1}{\beta-\nu}\sqrt{\frac{2(\beta+\nu)}{i\omega}}$$

and the amplitude A_l of the traveling wave is equal to -1 in the former case and to 0 in the latter. To obtain the amplitudes we used the condition (6) and the relations $V_1V_2 = V_{A2}^2$ and $\rho_2/\rho_1 = V_{A2}^2/V_2^2$ on the switching-off wave. We shall use analogous relations on the switching-on wave, $V_1V_2 = V_{A1}^2$ and $\rho_2/\rho_1 = V_1^2/V_{A1}^2$. Thus, the problem of small perturbations for the switching-off wave, with dissipation taken into account, has a unique solution although one of the outgoing Alfven waves does have anomalous properties, being dissipative. Consequently, the switching-off wave is evolutional.

The situation is different with the switching-on wave. Although two outgoing waves exist for it, too, both propagate in the medium behind the shock wave. The system (8) with allowance for the relations on the switching-on wave can be written in the form

$$V_1(v_2-b_2)=V_1v_1-V_{A1}b_1, \quad V_{A1}(v_2-b_2)=V_{A1}v_1-V_1b_1. \qquad (9)$$

The system (9) is contradictory at nonzero incident-wave amplitude, i.e., at nonzero v_1 and b_1, and has a nontrivial solution for the amplitudes of the outgoing waves at zero amplitude. Consequently, in the linear approximation the switching-on wave is subject to instability of two types: decay into several discontinuities and spontaneous emission of Alfven waves.

We note in conclusion that when the viscosity is taken into account, the additional dissipative wave appears only under the condition $V = V_A$. This means that for the fast and slow shock waves, on both sides of which $V \neq V_A$, allowance for the viscosity does not change the known results[1,4]. At the same time, a new situation arises with respect to the rotational discontinuities, in which all the quantities are continuous, with the exception of the direction of the tangential components of the magnetic field and of the velocity. On both sides of the rotational discontinuity we have $V = V_A$, and consequently dissipative waves should exist. The boundary conditions (8) for the rotational discontinuity are linearly dependent and reduce to a single condition for the admissible system of one Alfven wave and two dissipative waves of small amplitude. Therefore the rotational discontinuity will spontaneously radiate waves, and in this sense it should be regarded as non-evolutional. (The arguments presented above pertain directly to a 180° rotational discontinuity, but it can be easily seen that this conclusion is valid also for rotational discontinuities with arbitrary angle of magnetic-field rotation.)

It is also interesting to note that the particular case $\nu = \beta$ is exceptional in that the connection between the amplitudes v and b turns out to be the usual one, as in the absence of dissipation (see (7) at $\nu = \beta$; the same holds true in the exact solution without the use of the approximation (6)). The system of boundary conditions

(8) has no solution in this case (the amplitude of the dissipative wave becomes infinite). In this special case it is necessary to take into account the next higher derivatives with respect to the coordinates in the equations of the system (1), and this leads to a finite result for the amplitudes of the Alfven waves that go out of the switching-off wave.

We note also that allowance for the dissipation in the dispersion equation for magnetoacoustic and entropy waves leads to the appearance of dissipative magnetosonic waves when the velocity of the plasma relative to the discontinuity coincides with the velocity of the corresponding waves, and as a consequence, leads to nonevolutionality of the tangential, contact, and weak discontinuities.

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137