Theory of the resistivity and magnetoresistance of metals at low temperatures

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The effect of the crystal-structure-dependent anisotropy of the electron distribution function on the resistivity ρ, magnetoresistance $\Delta\rho_H$ and Hall coefficient R at low temperatures in metals is analyzed. It is shown that the principal features in the temperature and concentration dependences of $\Delta\rho_H$, R and the impurity part $\Delta\rho_T$ of the resistivity have a common physical nature, associated with the special role of the anisotropy of the electron distribution function in metals. It is shown, in particular, that the temperature dependence of the resistivity of a metal in the limit of high magnetic fields is very close to the temperature dependence of the resistivity (at $\mathbf{H} = 0$) in the so-called "dirty limit" (this also applies to the absolute values of these quantities). The results obtained make it possible to explain in a unified way the previously discovered anomalous dependences of $\Delta\rho_H$, $\Delta\rho_T$ and the total resistivity on the temperature and impurity concentration, in particular, in the case of aluminum. The nature of the universal T^3 dependence found experimentally for $\Delta\rho_T$ and ρ is analyzed.

1. INTRODUCTION

A number of experimental papers have recently appeared which demonstrate the anomalous character of the dependence of the resistivity and magnetoresistance of non-transition metals with diamagnetic impurities on the concentration c of the latter and on the temperature T. We are concerned here with the low-temperature region and with a broad range of low values of the concentrations. The greatest interest here attaches to the results obtained for aluminum-based alloys, which are in fact the system most fully investigated up to the present time, although all the features are qualitatively the same for other alloys too (cf., e.g., the review [1]).

The first puzzling result was obtained in the work of Borovik et al. [2], who discovered an increase, instead of a decrease, in the magnetoresistance $\Delta\rho_H$ of aluminum on going from liquid-helium to liquid-hydrogen temperatures. In the later work of Tsien (Chiang) et al. [3] and Fickett [4], not only was this result confirmed but it was also established that, with increasing T in high fields, the magnetoresistance goes through a maximum, the magnitude of which depends on the impurity concentration. In general, the whole pattern of the temperature behavior of $\Delta\rho_H$ is found to be very sensitive to the impurity concentration, displaying, in other words, a nonlinear dependence of the actual shape of the curve $\Delta\rho_H(T)$ on the concentration for small c.

A maximum in the curve of the temperature dependence of the Hall coefficient has also been observed recently (in intermediate magnetic fields) in the alkali metals [5] and copper [6]. Up to now, none of these results has received a theoretical explanation.

Of the work in which the ordinary resistivity of metals is studied, the papers [7] of Caplin and Rizzutto have stimulated the most interest. In these papers it was established that the temperature-dependent part of the total resistivity of aluminum in the "dirty limit" (cf. below) in the temperature range 10–50°K has a universal dependence of the form T^3 for any type of impurity, whereas the resistivity of pure aluminum in this range essentially obeys the classical T^5 law. The concentration dependence is found to be very weak and, at fixed T, becomes noticeable, on the other hand, only on going to low concentrations. It is interesting that the impurity part $\Delta\rho_T$ of the resistivity in the temperature and concentration ranges under consideration also has the same universal T^3 dependence.

The anomalous dependence of $\Delta\rho_T$ on the impurity concentration in aluminum was also clearly exhibited in the work of Fickett [4]. On the other hand, it was found experimentally even earlier that the impurity part of the resistivity as a function of T in alloys based on aluminum has, at somewhat higher temperatures, a sharp maximum whose magnitude depends nonlinearly on the concentration and is of the same scale as the residual resistivity (cf., e.g., [8,9]). Similar behavior has also been found in many other alloys (cf. the review [1]). Actually, we shall be concerned with the general pattern of the anomalous temperature and concentration dependences of the resistivity of metals with impurities.

The data of Caplin and Rizzutto led to a series of publications [10-12] containing attempts to give a theoretical explanation of the dependences observed in the experiment of [7] by turning away from the usual picture of the low-temperature resistivity of metals. Unfortunately, these publications were found to be in error. Criticism of the papers [10,11] has already been published (cf., e.g., [13,14]). The paper by Dworin [12], in which the observed results are associated with the drag effect, simply contains a mathematical error. A correct analysis of formula (8) of the text of his article leads to a T^5 law, corresponding to the result obtained earlier by Gurzhi [15] in the same problem, instead of T^3. Moreover, the entire anomalous pattern is observed in regions of temperature and concentration that are too high for the drag effect to play any role.

In fact, all the above-mentioned effects in the low-temperature behavior of the resistivity and magnetoresistance have a common physical nature and can be explained in a unified way. The starting point here turns out to be the fact that the electron distribution function $f(\mathbf{p}, n)$ in metals without impurities has, as a rule, sharply expressed anisotropy, which reflects the symmetry of the lattice. The reasons for the appearance of

this anisotropy are the Umklapp processes in the electron-phonon interaction, and the anisotropy of the phonon spectrum and Fermi surface. It was shown in a paper by one of the authors and Zhernov [16] that, even if the Fermi surface is assumed spherical, the scale of the anisotropy of the electron distribution function in polyvalent metals is so great that taking it into account can change the resistivity of an ideal metal in a given temperature range by an order of magnitude. In this case, any scattering mechanism that decreases the anisotropy will strongly influence the magnitude of the resistivity. Elastic scattering by impurities or by any other lattice imperfections turns out to be just such a mechanism. As was shown in the same paper [16], allowance for this distinctive interference of inelastic scattering by phonons and elastic scattering by impurities leads to a sharp deviation from Matthiessen's rule, of the same order as the residual resistivity, and to a nonlinear concentration dependence of the resistivity at low temperatures. The latter is found at any low concentration—only the temperature range in which the non-linear dependence appears is shifted.

All these results enable us to explain the pattern of the behavior of the impurity resistivity in the temperature range in which the characteristic peak in the dependence $\Delta\rho_T(T)$ is observed, and (in combination with the results of an earlier paper [17]) the behavior of $\Delta\rho_T$ in the entire range of higher temperatures. As will be shown below, these results can also be used to explain fully, without invoking any additional ideas, the anomalous behavior, found in the experiment of [7], of the resistivity in alloys.

Recognizing that the magnetoresistance $\Delta\rho_H$ in the isotropic approximation is in general equal to zero (cf., e.g., [18]), it is easily understood that a finite $\Delta\rho_H$ exists only to the extent that the electron distribution function possesses anisotropy, depending on the crystal structure. In accordance with this, the temperature dependence of the magnetoresistance, especially at saturation, should reflect the temperature dependence of the generation, by scattering, of anisotropy in the distribution of electrons, and the concentration dependence should reflect the "isotropization" of this distribution. As will be seen from the analysis carried out below, the dependences of $\Delta\rho_H$ and $\Delta\rho_T$ on T and c are both determined by the same quantities. This establishes the internal correlation between these, at first sight, different quantities, and makes it possible to understand the anomalous character of the behavior of $\Delta\rho_H$ and $\Delta\rho_T$.

Thus, allowance for the anisotropy of the electron distribution enables us to explain consistently the observed pattern of the electrical conductivity and magnetoresistance in metals with impurities. Moreover, a clear experimental check of the ideas developed appears possible.

We begin with the determination of the exact asymptotic value of the resistivity of a metal as $H \to \infty$ in an arbitrary case, and then compare these results with the resistivity in zero magnetic field. Allowance for the anisotropy of the distribution function requires, naturally, that we go outside the framework of the usual single-moment approximation in solving the kinetic equation. From a purely formal point of view, this presents no fundamental difficulties (cf., e.g., the work of Bross[19] and García-Moliner[20]).

2. RESISTIVITY OF A METAL IN A STRONG MAGNETIC FIELD.

The semi-classical Boltzmann kinetic equation for electrons in a metal in a magnetic field can be represented conveniently in the operator form

$$N^\alpha(k)=\hat{P}(\varphi^\alpha(k))+\hat{M}(\varphi^\alpha(k)). \tag{2.1}$$

Here,

$$N^\alpha(k)=-ev^\alpha(k)\partial f^{(0)}(k)/\partial\varepsilon(k),$$

$$\hat{M}=\frac{e}{c}\frac{\partial f^{(0)}(k)}{\partial\varepsilon(k)}e^{\alpha\beta\gamma}H^\gamma v^\alpha(k)\frac{\partial}{\partial p^\beta}; \tag{2.2}$$

$k\equiv \mathbf{p}, n$, where n is the band index and $\mathbf{p}$ is the quasi-momentum; $\mathbf{v}(k)$ is the group velocity of the electron, $e^{\alpha\beta\gamma}$ is the unit antisymmetric tensor of the rank three, and $\hat{P}$ is the standard collision operator, which takes into account the scattering by impurities and phonons. The correction to the equilibrium electron distribution function in these expressions is assumed to have the form

$$f^{(1)}(k)=-\frac{\partial f^{(0)}(k)}{\partial\varepsilon(k)}(\varphi(k)\mathbf{u}),\quad \mathbf{u}=\frac{\mathbf{E}}{|\mathbf{E}|}; \tag{2.3}$$

In this paper we exclude from consideration the region of extremely low temperatures, in which drag effects can become important, and the phonon distribution appearing in $\hat{P}$ will be assumed to be the equilibrium distribution.

We shall expand the function $\varphi^\alpha(k)$ in a certain complete set of functions $\{\psi_i^{(n)}(\mathbf{p})\}$, chosen separately for each n:

$$\varphi^\alpha(k)=\sum_i a_i^{(n)}(\alpha)\psi_i^{(n)}(\mathbf{p}),\quad \alpha=1,2,3. \tag{2.4}$$

The scalar operator $\hat{M}$ with an arbitrary direction of the magnetic field connects functions of arbitrary symmetry; because of this, when considering the general problem we should, in principle, understand by $\{\psi_i^{(n)}(\mathbf{p})\}$ a complete set that is not restricted by the crystal symmetry.

It is clear from general considerations, and also from the explicit form of (2.3), that we shall be interested only in values of the function $\varphi^\alpha(k)$ at the Fermi surface. We shall assume that the latter does not intersect itself, so that, in principle, we can introduce in place of $\varphi^\alpha(k)$ a single function $\varphi^\alpha(\mathbf{p})$ for the entire Brillouin zone, ignoring the details of its behavior outside the Fermi surface. For these functions we shall have an expansion of the type (2.4) with functions $\psi_i(\mathbf{p})$ defined in the entire Brillouin zone and can omit the index n; we shall construct the $\psi_i(\mathbf{p})$ with the aid of the polynomials $\psi_i^{(n)}(\mathbf{p})$.

Such a representation for the distribution function is especially convenient when it is clear from physical considerations that it is fundamentally meaningful to make a single assignment of trial function for all the bands. Thus, in the case when the drift of particles plays the determining role, as, e.g., in a strong magnetic field or in the presence of the drag effect, three piecewise-linear functions $\psi_\gamma(\mathbf{p})$ of the quasi-momentum should be chosen as the principal trial functions.

In the framework of the repeated-zone scheme, the function $\varphi^\alpha(\mathbf{p})$ should be periodic in $\mathbf{p}$ in the reciprocal-lattice space, and at the same time continuous. With a view to using the method of moments (in other words,

using a limited number of functions in the representation (2.4)), we should, generally speaking, require that the functions $\psi_i(\mathbf{p})$ also satisfy periodicity and continuity conditions. For such a choice of functions, the operator $\hat{M}$ turns out to be antisymmetric in the indices i and j.

We shall consider the case of a closed Fermi surface. For the first three functions $\psi_\gamma(\mathbf{p})$ ($\gamma = 1, 2, 3$), we choose piecewise-linear functions of the γ-th quasimomentum component p^γ, measured, on each segment of the Fermi surface, from zero or from the appropriate value of $\mathbf{G}/2$, where $\mathbf{G}$ is a reciprocal-lattice vector. (If the Fermi surface goes outside the Brillouin-zone boundary, a piecewise-linear function makes it possible to ensure the periodicity of $\psi_\gamma(\mathbf{p})$—for more detail, cf. [21].) Then,

$$\partial\psi_\nu(\mathbf{p})/\partial p^\alpha = \delta^{\alpha\nu}$$

and

$$M_{\gamma\gamma'} = \frac{e}{c} e^{\alpha\gamma'\delta} \int \frac{dS_F}{4\pi^3} \frac{v^\alpha(\mathbf{p})}{v_F(\mathbf{p})} \psi_\gamma(\mathbf{p}) = \sum_n (\pm)\frac{e}{c} e^{\alpha\gamma'\delta} H^\delta \int_{(n)} \frac{d\mathbf{p}}{4\pi^3} \delta^{\alpha\gamma} = \sum_n \frac{e}{c} e^{\gamma\gamma'\delta} H^\delta (\pm N_n) = \frac{e}{c} e^{\gamma\gamma'\delta} H^\delta (N_e - N_h). \tag{2.5}$$

Here the second integral is taken over the volume delimited by the close portions of the Fermi surface, and N_n is the number of electrons (+) or holes (−) in the band with index n; $N_{e,h}$ is the total number of electrons or holes, respectively.

We note that the choice of one set of functions or another in the expansion of $\varphi^\alpha(k)$ is dictated primarily, especially in using the method of moments, by considerations of convenience. If, e.g., the drift is insignificant, the representation (2.4) with an independent expansion for each energy band may turn out to be more convenient. This is especially clear in the case of compensated metals, in which $N_e = N_h$ (although a single representation for all the bands can be used effectively in this case too).

In order not to complicate the problem, we shall confine ourselves below to treating only noncompensated metals and shall use an expansion with the common functions $\psi_i(\mathbf{p})$. For the first three functions we choose the $\psi_\gamma(\mathbf{p})$ introduced above, and for $i \geq 4$ we choose a set of polynomials that are mutually orthogonal and orthogonal to $v^\alpha(\mathbf{p})$ with weight $\partial f^{(0)}(\mathbf{p})/\partial\epsilon(\mathbf{p})$ (which is equivalent to integrating only over the Fermi surface). Such a choice is always possible, inasmuch as the scalar product of $\psi_\gamma(\mathbf{p})$ and $v^\alpha(\mathbf{p})$ with this weight is equal to $N_e - N_h \neq 0$. It is important that, with this choice,

$$\langle\psi_\nu|\hat{M}|\psi_i\rangle = 0 \qquad (\gamma = 1, 2, 3; \quad i \geq 4). \tag{2.6}$$

We shall substitute the expansion of the distribution function $\varphi^\alpha(\mathbf{p})$ into Eq. (2.1). Taking the scalar product of (2.1) with $\psi_i(\mathbf{p})$, we obtain a system of algebraic equations for the coefficients $a_i(\alpha)$:

$$\langle N^\alpha|\psi_i\rangle = \sum_{j=1} a_j(\alpha)\,(P_{ij} + M_{ij}). \tag{2.7}$$

The left-hand side of (2.7) is nonzero only in the case $i = \alpha$, and the corresponding scalar product with the above choice of functions $\psi_\gamma(\mathbf{p})$ ($\gamma = 1, 2, 3$) does not depend on the label of the axis in a crystal of arbitrary symmetry:

$$\langle N^\alpha|\psi_\alpha\rangle \equiv j \quad (\alpha = 1, 2, 3). \tag{2.8}$$

Then for the coefficients $a_j(\alpha)$ we find

$$a_j(\alpha) = (\hat{P} + \hat{M})^{-1}_{j\alpha}\, j. \tag{2.9}$$

On the other hand, for the conductivity tensor we have

$$\sigma^{\alpha\beta} = e\int \frac{2d\mathbf{p}}{(2\pi)^3} v^\alpha(\mathbf{p})\varphi^\beta(\mathbf{p}) \left(-\frac{\partial f^{(0)}(\mathbf{p})}{\partial\epsilon(\mathbf{p})}\right) = j a_\alpha(\beta).$$

As a result,

$$\sigma^{\alpha\beta} = j^2(\hat{P} + \hat{M})^{-1}_{\alpha\beta} \qquad (\alpha, \beta = 1, 2, 3). \tag{2.10}$$

When (2.6) is taken into account, the matrix $\hat{K} = \hat{P} + \hat{M}$ has the following structure:

$$\hat{K} = \begin{pmatrix} \hat{K}_1 & \hat{P}' \\ \tilde{\hat{P}}' & \hat{K}_2 \end{pmatrix}. \tag{2.11}$$

Here $\hat{K}_1$ is a square third-order matrix, $\hat{P}'$ is a rectangular three-row matrix, being the corresponding part of the matrix $\hat{P}$, and $\tilde{\hat{P}}'$ is the transpose of the matrix $\hat{P}'$. If we denote the inverse of the matrix $\hat{K}$ by $\hat{T}$, we have for the square third-order matrix $\hat{T}_1$ in the upper left corner of $\hat{T}$

$$\hat{T}_1 = (\hat{K}_1 - \hat{P}'\hat{K}_2^{-1}\tilde{\hat{P}}')^{-1}.$$

It is clear that $\hat{T}_1$ coincides with the matrix appearing in the right-hand side of (2.10). Therefore, we obtain for the resistivity tensor in its most general form

$$\rho^{\alpha\beta} = j^{-2}(\hat{K}_1 - \hat{P}'\hat{K}_2^{-1}\tilde{\hat{P}}')_{\alpha\beta} \quad (\alpha, \beta = 1, 2, 3). \tag{2.12}$$

We remark that the expression (2.12) is very convenient, inasmuch as it only requires inversion of the matrix $\hat{K}_2$.

We shall analyze the question of the asymptotic behavior of the resistivity tensor in strong magnetic fields. We shall confine ourselves to treating crystals of high symmetry and shall assume for simplicity that the metal possesses three mutually perpendicular symmetry planes, defined by the principal axes of the crystal, with the coordinate origin fixed at both the center of the Brillouin zone and the center of the individual segments of the Fermi surface (in the periodically-repeated zone representation). We introduce a coordinate frame rigidly attached to the principal axes of the crystal. Then the functions of the complete system should either be odd in p^α (for one of the α) or contain combinations of the type $p^1p^2p^3$. It is interesting that in the case when the magnetic field is directed along one of the principal axes of the crystal, combinations of the type $p^1p^2p^3$ are not generated in the transverse components of the distribution function $\varphi^\alpha(\mathbf{p})$.

To make the subsequent analysis easier, we shall not take these functions into account. In this case we obtain an exact result for the asymptotic form of the transverse components of the resistivity tensor for $\mathbf{H}$ parallel to one of the principal axes, and an approximate result, equivalent to the method of moments, for an arbitrary direction of $\mathbf{H}$. In this case, the set of functions $\{\psi_i\}$ can be divided into three subsystems $\{\psi_{\alpha\lambda}\}$ ($\alpha = 1, 2, 3$), each of which contains polynomials that are odd only in a certain component p^α. The only nonzero matrix elements of the operator $\hat{M}$ are those between functions belonging to different subsystems, i.e., with $\alpha \neq \alpha'$. It can be shown that, by an appropriate choice of the three orthogonal subsystems of functions, it is possible to ensure that the conditions

$$\langle\psi_{\alpha\lambda}|\hat{M}|\psi_{\alpha'\lambda_1}\rangle = 0, \quad \lambda \neq \lambda_1. \tag{2.13}$$

are fulfilled. The condition (2.13) is found to be fulfilled automatically for λ or $\lambda_1 = 1$, when the corresponding set of three functions is chosen in a form that is piece-

wise-linear in p (as defined above) (cf. (2.6)). It should be specially emphasized that, when a system of independent functions with $\lambda > 1$, orthogonal to $\mathbf{v}(\mathbf{p})$, is being chosen, the condition (2.13) ensures that the choice of the first set of three functions in precisely the form taken above is unique.

In this case, the matrix $\hat{K}_2$ has a comparatively simple structure. In fact, with the condition (2.13), matrix elements of the operator $\hat{M}$ appear only in the square third-order matrices lying along the diagonal of the matrix $\hat{K}_2$ in (2.11). Simple analysis then leads to the result that all the elements of the inverse of the matrix $\hat{K}_2$ tend, in the general case, to constant values as $H \to \infty$. This leads to the result that the second term in formula (2.12) has a finite value, determining the angular dependence of the resistivity on the direction of the magnetic field. This result, naturally, is in full agreement with the general result obtained for the asymptotic behavior of the resistivity tensor and found in the paper [22] by Lifshitz et al.

Here, however, the case when H is directed along one of the principal axes, e.g., along the axis 3, is found to be quite distinct from the others. In this case, the three-row matrix lying on the diagonal of the matrix $\hat{K}_2$ has the form

$$\begin{pmatrix} P_{1\lambda|1\lambda} & M_{1\lambda|2\lambda} & 0 \\ M_{2\lambda|1\lambda} & P_{2\lambda|2\lambda} & 0 \\ 0 & 0 & P_{3\lambda|3\lambda} \end{pmatrix},$$

which enables us to conclude that, for the resistivity-tensor components transverse to the direction of the magnetic field, the second term in (2.12) disappears completely as $H \to \infty$. Hence follows the important result that the transverse magnetoresistance is uniquely determined by the diagonal matrix element of the scattering operator, calculated for a piecewise-linear function of a component of the quasi-momentum. In this case,

$$\rho^{\alpha\beta} = j^{-2} P_{\alpha_1|\alpha_1} \delta^{\alpha\beta} + R_0 H e^{\alpha\beta 3} \quad (\alpha, \beta = 1, 2), \qquad R_0 = 1/(N_e - N_h)ec. \tag{2.14}$$

An expression for the resistivity in the absence of a magnetic field is obtained from (2.12) if we put the operator $\hat{M}$ equal to zero. Comparing this result with (2.14), we find an expression for the transverse magnetoresistance, in the direction of the principal axes:

$$\Delta\rho_H = \frac{1}{j^2} \sum_{\lambda,\, \lambda_1 \geq 2} P_{\alpha_1|\alpha\lambda} (\hat{P}^{-1})_{\alpha\lambda|\alpha\lambda_1} P_{\alpha\lambda_1|\alpha_1}.$$

Thus, when H is oriented along one of the principal axes, the magnetoresistance is strictly equal to the correction to the resistivity for $H = 0$ that arises when one goes beyond the standard single-moment approximation. As already noted in the Introduction (cf. also [16]), because of the anisotropy arising in the electron distribution function as a consequence of anisotropy of the scattering and anisotropy of the electron spectrum, this correction turns out to be large, changing the resistivity by an order of magnitude in many cases. Inasmuch as the sharp deviation of the resistivity from Matthiessen's rule and, in particular, the presence of the maximum in the curve of the dependence of the impurity part of the resistivity on T are due to precisely this quantity, it is clear that these features of the behavior should also be clearly manifested in the asymptotic form of the magnetoresistance for large values of H.

We have here specially singled out the magnetoresistance in the case when the magnetic field is directed along a principal axis of the crystal, since for an arbitrary field direction there is no longer any such simple connection between $\Delta\rho_H$ and the resistivity. However, in all cases, the magnetoresistance arises only when one goes beyond the framework of the usual three-moment approximation ($\lambda = 1$). (For $H = 0$, this corresponds to the single-moment approximation for each of the components of the resistivity tensor.)

As was shown earlier [16], all the qualitative features of the behavior of the resistivity that are associated with anisotropy of the distribution function are already picked up in the two-moment approximation. In the six-moment approximation, which is here equivalent to the latter, the resistivity tensor (2.12) in a cubic crystal has the following form:

$$\rho^{\alpha\beta} = \rho\delta^{\alpha\beta} + \rho_1(\delta^{\alpha\beta} - h^\alpha h^\beta) + RHe^{\alpha\beta\nu}h^\nu;$$

$$\rho = \frac{P_{11}}{j^2}\left(1 - \frac{P_{12}^2}{P_{11}P_{22}}\right), \quad \rho_1 = \frac{P_{12}^2}{j^2 P_{22}} \frac{(\gamma H)^2}{1+(\gamma H)^2}, \tag{2.15}$$

$$R = R_0\left[1 + \frac{\beta (P_{12}/P_{22})^2}{1+(\gamma H)^2}\right], \quad \gamma = \frac{\beta R_0 j^2}{P_{22}}, \quad P_{\lambda\lambda_1} \equiv P_{\alpha\lambda|\alpha\lambda_1}.$$

The quantity $\beta = M\alpha_2|\alpha_2'/M\alpha_1|\alpha_1'$ depends on the form of the functions with $\lambda = 2$ and on the shape of the Fermi surface. If these functions are chosen in the same form as in [16]:

$$\psi_{\alpha 2} = (p^\alpha)^3 - {}^3/_5 p^\alpha p_\nu^2,$$

then, in the case of a spherical fermi surface, $\beta = 18/175$.

It can be seen from the expression (2.15) that, in the case of the magnetoresistance, the six-moment approximation turns out to be fairly crude, leading, in contrast to the general case, to the absence of longitudinal magnetoresistance and to a transverse magnetoresistance that is independent of the orientation of the magnetic field with respect to the crystallographic axes. However, it picks up the character of the dependence of $\Delta\rho_H$ on the temperature and impurity concentration, and this is of very great interest for the subsequent analysis.

3. TEMPERATURE AND CONCENTRATION DEPENDENCE OF THE RESISTIVITY AND MAGNETORESISTANCE.

We now consider a real situation, in which scattering of electrons by phonons and impurities occurs simultaneously. Assuming the impurity concentration c to be small, for the matrix elements of the collision operator we have (for a cubic crystal)

$$P_{\lambda\lambda_1} = P_{\lambda\lambda_1}^{(0)} + cR_{\lambda\lambda_1}. \tag{3.1}$$

With the object of simplicity, and in order to distinguish the effect greatest in magnitude in the intermediate region of low temperatures, we shall ignore the change, associated with the introduction of the impurities, in the phonon spectrum, and also the vibrations of the impurity atoms themselves (cf. [17]). Then $R_{\lambda\lambda_1}$ in (3.1) is a matrix element of the static-scattering operator.

It follows from the form of the expression (2.14) that, by measuring the asymptotic behavior of the transverse resistivity in a magnetic field directed along one of the principal axes as a function of the temperature, we can determine the quantities

$$P_{11}^{(0)}(T), \quad R_{11}. \tag{3.2}$$

On the other hand, in the two-moment (six-moment for $H \neq 0$) approximation, we have for the change in resistivity in a magnetic field:

$$\Delta\rho_H^{\perp} = (P_{12}^{(0)} + cR_{12})^2 / j^2 (P_{22}^{(0)} + cR_{22}). \tag{3.3}$$

We shall consider the so-called "clean limit," i.e., those temperatures and impurity concentrations for which

$$P_{\lambda\lambda_1}^{(0)}(T) \gg cR_{\lambda\lambda_1}. \tag{3.4}$$

Bearing in mind that, in the limit (3.4), the resistivity in the absence of a magnetic field is simply

$$\rho^{(0)}(T) = \frac{1}{j^2}\left(P_{11}^{(0)} - \frac{P_{12}^{(0)2}}{P_{22}^{(0)}}\right) \equiv \frac{P_{11}^{(0)}}{j^2}(1-\eta^{(0)}), \tag{3.5}$$

we obtain for the magnitude of the relative magnetoresistance in this case

$$\Delta\rho_H^{\perp}(T)/\rho^{(0)}(T) = \eta^{(0)}/(1-\eta^{(0)}). \tag{3.6}$$

As was shown earlier [16], the function $\eta^{(0)}(T)$ arising by virtue of the anisotropy of the electron distribution function increases with T at low temperatures, passes through a maximum, and then decreases. It follows from (3.6) that the magnetoresistance will also behave in this way in the "clean limit." This result is general in character; specific cases and different models will correspond only to different positions and magnitudes of the maximum. The presence of defects, which, in particular, lead to a "residual" magnetoresistance with the same form (3.6) but with $\eta^{(0)}$ replaced by

$$\eta_i = R_{12}^2 / R_{11} R_{22}, \tag{3.7}$$

leaves the pattern qualitatively unchanged, although, as a result of the "isotropization" of the distribution function, the pattern becomes less dramatic.

The character of the behavior with temperature of $\Delta\rho_H^{\perp}(T)/\rho(T)$, where the total resistivity for $H = 0$ is equal to

$$\rho(T) = \rho_0 + \rho^{(0)}(T) + \Delta\rho_T,$$
$$\Delta\rho_T = P_{22}^{(0)}\left(\frac{P_{12}^{(0)}}{P_{22}^{(0)}} - \frac{R_{12}}{R_{22}}\right)^2 \Big/ j^2\left(1 + \frac{P_{22}^{(0)}}{cR_{22}}\right), \tag{3.8}$$

can be followed from the example of the curves shown in Fig. 1. The solid curves were constructed using the same model for the electron-phonon system of the metal as in [16], for the case $p_F/q_0 = 1.45$, which is very close to the corresponding value for aluminum. The curves differ in the values of the residual resistivity,

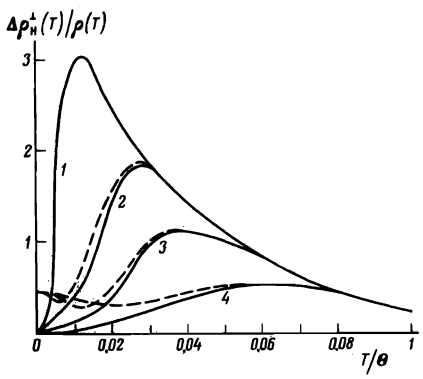

FIG. 1. The dependences $\Delta\rho_H^{\perp}(T)/\rho(T)$ for different values of the residual resistivity: 1) the "clean limit", 2) $\rho_0/\rho^{(0)}(\Theta_D) = 10^{-5}$, 3) $\rho_0/\rho^{(0)}(\Theta_D) = 10^{-4}$, 4) $\rho_0/\rho^{(0)}(\Theta_D) = 10^{-3}$.

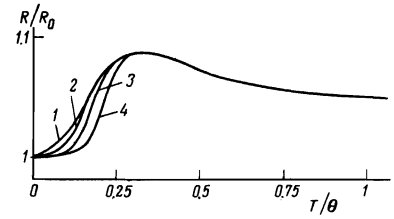

FIG. 2. The dependence $R(T)/R_0$ for different values of the residual resistivity: 1) the "clean limit", 2) $\rho_0/\rho^{(0)}(\Theta_D) = 10^{-3}$, 3) $\rho_0/\rho^{(0)}(\Theta_D) = 10^{-2}$, 4) $\rho_0/\rho^{(0)}(\Theta_D) = 10^{-1}$.

which is characterized by the ratio $\rho_0/\rho^{(0)}$ $(T = \Theta_D)$. In the same figure and purely illustratively, dashed curves corresponding to the parameter $R_{12}/R_{22} = 0.3$ are given (in the framework of the model considered in [16], $R_{12} = 0$). It can be seen that allowance for the anisotropy of the scattering (as a result of the nonspherical shape of the Fermi surface) in the residual resistivity leaves the pattern qualitatively unchanged.

The character of the dependence obtained for the relative magnetoresistance fully explains, at least qualitatively, the results obtained in the experiments of Borovik et al. [2], Tsien (Chiang) et al. [3] and Fickett [4]. As is clear from the preceding discussion, the data from these experiments essentially determine directly the scale of the anisotropy of the electron distribution function. The better samples in the experiments [3,4] gave a value of the residual resistivity that corresponds approximately to the value taken for the curve 3 in Fig. 1. It is interesting that the positions of the maximum in the two cases are extremely close, although the absolute value of the effect is several times greater in the experiment. Thus, in reality, the anisotropy of the electron distribution function in the case of aluminum evidently turns out to be greater than with the assumed parameters of the model, and the resistivity of the pure metal is reduced by almost an order of magnitude as a result of this anisotropy. We remark that, to judge from the curves of Fig. 1, the experimentally found maximum of the magnetoresistance is still appreciably lower than that attainable in the "clean limit."

We must draw attention to the fact that measurement of the "residual" magnetoresistance in the limit of high magnetic fields makes it possible to determine the quantity η_i (cf. (3.3) and (3.2)). In practice, this opens up the possibility of a direct experimental estimate of the effect on the residual resistivity of anisotropy of the scattering. It is interesting that, from the results of the papers [4,3], it follows that $\eta_i \sim 0.6$; thus, the role of anisotropy is great even in the case of the residual resistivity (in aluminum we certainly have $1-\eta_{max}^{(0)} \ll 1-\eta_i$).

The anisotropy of the scattering also has an effect on the magnitude of the Hall coefficient. It leads to the appearance of a correction to the value $R_0 = 1/(N_e - N_h)ec$, and the temperature dependence of this correction in intermediate magnetic fields (cf. (2.15)) also has the form of a function with a maximum of the same origin. Such a dependence of the Hall coefficient has been observed in alkali metals in [5]. In Fig. 2 the quantities R/R_0, found within the framework of the same model [16] but with the parameter value $p_F/q_0 = 0.6$, which corresponds exactly to the alkali metals, are given as functions of the temperature. It has been assumed here that $\gamma H \ll 1$. It is interesting that the dependence found gives the correct position of the maximum, the characteristic very

slow decrease of the Hall constant with temperature to the right of the maximum, and even an accurate absolute value of the effect at the maximum. (It should be noted that the model used is most adequate for precisely the alkali metals, the Fermi surfaces of which are almost spherical.) The small magnitude of the effect is, of course, a specific feature of the alkali metals, in which the anisotropy of the distribution function plays a small role. The effect can be much more striking in the polyvalent metals. We remark that, as can be seen from the figure, the impurities here have a weaker effect, inasmuch as the maximum is displaced into the region of higher temperatures.

We now consider the range of temperatures and concentrations that corresponds to the inequality opposite to (3.4)–the so-called "dirty limit." In this case, the resistivity in the absence of a magnetic field has the following value:

$$\rho-\rho_0=\frac{P_{11}^{(0)}}{j^2}\left(1-\frac{R_{12}}{R_{22}}\frac{2R_{22}P_{12}^{(0)}-P_{22}^{(0)}R_{12}}{P_{11}^{(0)}R_{22}}\right). \qquad (3.9)$$

It can be seen from this expression that in the "dirty limit" this quantity does not depend on the impurity concentration and its dependence on T is close to that of $P_{11}^{(0)}(T)$.

Returning now to the result (3.2), we can state that the temperature dependence of the transverse resistivity of a metal in the limit of high magnetic fields should be close to the temperature dependence of the resistivity in the "dirty limit" (for H = 0). It should be noted that this statement is also approximately true for polycrystalline samples, since the corrections to the result (2.14) for an arbitrary direction of the field H turn out, apparently, to be relatively small. We remark also that the assumptions used about the character of the symmetry in the derivation of the asymptotic form of the magnetoresistance are satisfied in practice in most metals, at least with regard to an overwhelming proportion of the carriers (in aluminum, in particular).

With the object of checking this statement, we have made use of the data on the measurement of $\rho(T)$ in aluminum in the limit of high fields, contained in a dissertation by Tsien (Chiang) [23]. These data are shown in Fig. 3. In this same figure we give the results obtained by Caplin and Rizzutto [7] from resistivity measurements for precisely the "dirty-limit" case. It can be seen that both series of results lie on straight lines (on a logarithmic scale), with similar slopes. The actual absolute values of the resistivity in the two cases also differ relatively little. Thus, the experimental results confirm the statement made above. On the other hand, the above analysis enables us to understand that the universal T^3 dependence found by Caplin and Rizzutto[7] for the resistivity in the "dirty limit" is connected, in the temperature range studied, with the character of the temperature dependence of the matrix element $P_{11}^{(0)}(T)$.

In connection with this result, it was of interest to elucidate independently the character of the temperature dependence of the other diagonal matrix element $P_{22}^{(0)}(T)$. For this, as follows from (3.8), we can use the dependence of $\rho(T)-\rho_0$ on the inverse impurity concentration at a fixed value of T. The hyperbolic character of this dependence can be traced very clearly if we make use of, e.g., the corresponding curve first given in the paper [4] by Fickett. An analysis of the concentration dependence, carried out on the basis of the experimental

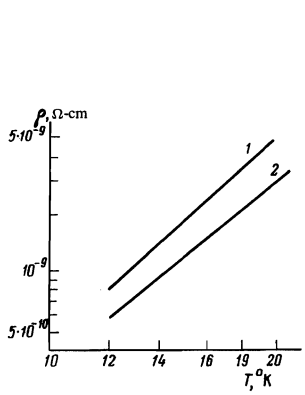

FIG. 3

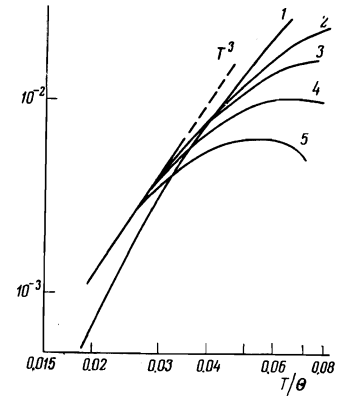

FIG. 4

FIG. 3. Dependence of $\rho(T,\mathbf{H})-\rho_0(\mathbf{H})$ on temperature in a strong magnetic field, from the data of Tsien (Chiang) [23] (the straight line 1), and the dependence $\rho(T)-\rho_0$ in the "dirty limit" from the data of Caplin and Rizzutto [7] (the straight line 2).

FIG. 4. The dependence $\rho^{(0)}(T)/\rho^{(0)}(\Theta_D)$ (curve 1) and curves of $\Delta\rho_T(T)/\rho^{(0)}(\Theta_D)$ for different values of the residual resistivity: 2) the "dirty limit", 3) $\rho_0/\rho^{(0)}(\Theta_D)=10^{-2}$, 4) $\rho_0/\rho^{(0)}(\Theta_D)=4\times10^{-3}$; 5) $\rho_0/\rho^{(0)}(\Theta_D)=10^{-3}$; for clarity the dependence $\sim T^3$ is shown by a dashed line.

data of [4,7-9], showed that the dependence $P_{22}^{(0)}(T)$ in the temperature range 10–40°K is also comparatively close to T^3.

Such a dependence of $P_{11}^{(0)}(T)$ and $P_{22}^{(0)}(T)$ on temperature is evidently connected with Umklapp processes in the electron-phonon scattering, which in aluminum are not frozen out until comparatively low temperatures are reached. Attention was specially drawn to this fact in the paper by Barabanov and Maksimov [24], in which, in a treatment of the electrical resistivity of the pure metal, an attempt was made to take into account the real character of the electron and phonon spectra of aluminum. It is interesting that if, on the basis of the results obtained by these authors, we analyze separately the behavior of the diagonal matrix elements, it turns out that $P_{11}^{(0)}(T)$ and $P_{22}^{(0)}(T)$ behave like T^3 in precisely the interval 10–30°K (private communication). At the same time, the total resistivity (3.8) of the pure metal in their calculations in this temperature interval was proportional to T^n with $n>4$.

Thus, the results given above apparently fully explain the mysterious universal temperature dependence found by Caplin and Rizzutto for the resistivity of aluminum in the "dirty limit."

The results obtained above evidently also give a good description of the character of the temperature and concentration dependences of the resistivity in the whole range of variation of the parameters, in particular, in passing from the "dirty" to the "clean limit." This is also clearly visible from the curves for the impurity part $\Delta\rho_T$ (3.8) of the resistivity, which were obtained in the framework of the same model calculations [16] ($p_F/q_0=1.45$) and are given in Fig. 4; the curves fully reproduce, qualitatively, the pattern observed by Caplin and Rizzutto [7]. It is interesting that, even for a spherical Fermi surface, the model calculations give a dependence close to T^3 for the temperature behavior of $\Delta\rho_T$ in the "dirty limit" (and separately for $P_{11}^{(0)}(T)$) and a dependence close to T^5 for the total resistivity in the "clean limit."

In conclusion, we remark that, as is made clear by the preceding discussion, a combination of measurements of the resistivity and magnetoresistance of a metal with impurities makes it possible to solve the problem of extracting individual matrix elements of the scattering operators or combinations of them.

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