External fields in the gravitational collapse of rotating masses

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(Submitted August 20, 1973)
Zh. Eksp. Teor. Fiz. 66, 1505-1514 (May 1974)

The behavior of the electromagnetic and gravitational fields is investigated for the metric of a rotating black hole. We show that the rotation leads to a $(2l+1)$-fold splitting of the frequency of the lth spherical harmonic of the field. In addition, in the case of the electromagnetic field the "off-diagonal" terms of the metric lead to the appearance of both "electric" and "magnetic" components of the electromagnetic field tensor, independent of the character of the source. Here the invariant $\mathbf{E}\cdot\mathbf{H}\neq 0$. We show that the decay laws of the external fields produced by a rotating collapsing mass, as well as the "tails" of wave packets propagating in the metric of a rotating black hole, have the same form as in the case of absence of rotation.

1. The behavior of the external fields created by a collapsing object are of fundamental interest in the theory of gravitational collapse. It is known[1] that for an external observer the final stage of evolution of a collapsing object is a stationary state. The laws of "expiration" of all nonstationary external fields created by a collapsing object show how this transition is realized and how the formation of a stationary capturing surface S_{Sch} (Schwarzshild horizon) is formed in the process of collapse.

In the case when the black hole has a vanishing total angular momentum this problem was solved in[2,3]. Here we consider the behavior of the fields for which the sources are rotating collapsing objects. We shall consider the deviations from spherical symmetry as small (i.e., the contribution to the total energy of the black hole from nonstationary perturbations should be much smaller than the self-energy $E_0 \sim Mc^2$). Then the problem of behavior of the external fields of a collapsing body reduces to finding the answer to two questions[2,3]: 1) one must find the solutions to the field equations in a given external metric in vacuum, 2) the solution must satisfy matching conditions at the surface of the collapsing object and the condition of absence of incoming waves at spacelike infinity.

The "external metric" for the case of a rotating black hole is the Kerr metric[4]

$$ds^2=\frac{\Sigma-r_g r}{\Sigma}dt^2-\frac{\Sigma}{\Delta}dr^2-\Sigma\,d\theta^2-\left(r^2+a^2+\frac{r_g r a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\,d\varphi^2-\frac{2ar_g r\sin^2\theta}{\Sigma}dt\,d\varphi. \quad (1.1)$$

Here we have used the notations

$$\Sigma=r^2+a^2\cos^2\theta,\quad \Delta=r^2-r_g r+a^2 \quad (1.2)$$

and a system of units with $c = G = 1$. The total angular momentum of the black hole is $K = aMc = aM$, where M is its mass; $r_g = 2M$. The Schwarzschild surface S_{Sch} of the metric (1.1) is defined by the equation $\Delta = (r - r_+)(r - r_-) = 0$; the solutions of this equation are

$$r_+=M+(M^2-a^2)^{1/2},\quad r_-=M-(M^2-a^2)^{1/2}. \quad (1.3)$$

Unfortunately, the degree of generality of the exact solution (1.1) is unknown. At any rate, the metric obtained from (1.1) in the linear approximation in a/r_g

$$ds^2=\frac{r-r_g}{r}dt^2-\frac{r}{r-r_g}dr^2-r^2(d\theta^2+\sin^2\theta\,d\varphi^2)-\frac{2ar_g\sin^2\theta}{r}dt\,d\varphi, \quad (1.4)$$

is the general expression for the external metric of a slowly rotating black hole[1].

2. We consider some properties of the solutions of Maxwell's equations in a space with the metric (1.4). Since the total angular momentum of the majority of stars is small and their electromagnetic fields are more accessible to observation, this problem is of independent interest. Moreover, the results given in this section remain valid also in the general case. The system of equations for the electromagnetic field in the metric of a slowly rotating ball in the vacuum is written out in the Appendix. Equations (A.3) differ from the equations in the Schwarzschild metric by a change of the "potential" and the appearance of inhomogeneous right-hand sides. For small a (as will be seen from the result in the general case too) one can take into account the corrections related to these effects independently.

We first consider static axially-symmetric fields (it will be shown below that a solution which is static in the whole space is necessarily axially symmetric). For $k = n = 0$ the system (A.3) takes the form (in this section we set $r_g = 1$)

$$E_l''+\frac{E_l'}{r(r-1)}-\frac{l(l+1)}{r(r-1)}E_l=\frac{3ial(l+1)}{r^3(r-1)}\left[\frac{H_{l+1}}{2l+3}-\frac{H_{l-1}}{2l-1}\right],$$
$$H_l''+\frac{H_l'}{r(r-1)}-\frac{l(l+1)}{r(r-1)}H_l=\frac{3ial(l+1)}{r^3(r-1)}\left[\frac{E_{l+1}}{2l+3}-\frac{E_{l-1}}{2l-1}\right]. \quad (2.1)$$

The general solution of homogeneous equations can be expressed in terms of hypergeometric functions and equals

$$C_1r^2F(-l+1,\ l+2;\ 3;\ r)+C_2r^{-l}F(l,\ l+2;\ 2l+2;\ r^{-1}), \quad (2.2)$$

where C_1 and C_2 are arbitrary constants.

The inhomogeneous terms of the equations lead, in general, to the appearance in the solutions of all spherical multipole harmonics. To first order in a, however, one does not have to take into account the "feedback" and the chain of equations decouples. As an example we list the solution which for $a = 0$ goes over into the field of a magnetic dipole d parallel to the Z axis:

$$F_{23}=3d\psi(r)\sin\theta\cos\theta,\quad F_{13}={}^3/_2 d\psi'(r)\sin^2\theta,$$
$$F^{01}=\frac{9ad\chi(r)}{r^2}(3\cos^2\theta-1),\quad F^{02}=-\frac{9ad\chi'(r)}{r^2}\sin\theta\cos\theta; \quad (2.3)$$

$$\psi(r)=2r+1+2r^2\ln\frac{r-1}{r},$$
$$\chi(r)=\frac{1}{3r}+\frac{5}{9}+\frac{10}{3}r-\frac{40}{3}r^2+\left(\frac{1}{3}+10r^2-\frac{40}{3}r^3\right)\ln\frac{r-1}{r}. \quad (2.4)$$

For $r \gg r_g$ we have

$$F_{23}\approx-2d\frac{\sin\theta\cos\theta}{r},\quad F_{13}\approx d\frac{\sin^2\theta}{r^2},$$

$$F^{01}\approx\frac{ad}{r^5}(3\cos^2\theta-1),\quad F^{02}\approx\frac{3ad}{r^6}\sin\theta\cos\theta.$$

Going over to locally Euclidean coordinates for $r = r_0 \gg r_g$ we obtain for the "physical" electromagnetic field strengths the expressions

$$H^{(r)}=2d\frac{\cos\theta}{r^3},\quad H^{(\theta)}=d\frac{\sin\theta}{r^3},$$
$$E^{(r)}=-ar_gd\frac{3\cos^2\theta-1}{r^5},\quad E^{(\theta)}=-ar_gd\frac{3\sin\theta\cos\theta}{r^5}. \tag{2.5}$$

Similarly any $2l$-pole source leads to the appearance of both "electric" and "magnetic" components of the electromagnetic field tensor. In the linear approximation in a the solution contains terms of tensorial dimension from $l - 1$ to $l + 1$. Finally, we note that the invariant $\mathbf{E}\cdot\mathbf{H} \neq 0$ (except at the equator). Therefore in distinction from the well-known unipolar effect in the case under discussion there exists no reference frame in which the electric field should be absent.

Obviously, the same situation occurs also for the wave solutions. The corrections to E and H which appear in this case decrease like $1/r^4$ for $r \to \infty$.

Let us now consider the effects related to the change of the "potential." The equation in which we are interested is of the form

$$E''+\frac{1}{r(r-1)}E'+\left[\frac{r^2k^2}{(r-1)^2}-\frac{2ank}{r(r-1)^2}-\frac{l(l+1)}{r(r-1)}\right]E=0. \tag{2.6}$$

Or going over to the new coordinate $x = r + \ln(r - 1)$, we have

$$\frac{d^2E}{dx^2}+\left[k^2-\frac{2ank}{r^3}-\frac{(r-1)l(l+1)}{r^3}\right]E=0. \tag{2.7}$$

For $n \neq 0$ we consider first the solutions of Eq. (2.7) in the two asymptotic regions:

a) for $r\gg r_g,\ x\gg r_g$
$$E\approx C_1e^{ikx}+C_2e^{-ikx}, \tag{2.8}$$

b) for $r\sim r_g,\quad -x\gg r_g$
$$E\approx C_3e^{i(k-an)x}+C_4e^{-i(k-an)x}. \tag{2.9}$$

Let a source with frequency k be situated at infinity. In this case for $k \gg 1$ we may set approximately $C_1 \approx C_2 = 0$, $C_4 \approx C_2 = C$. Substituting (2.8), (2.9) into (A.2) and setting for simplicity $C(l_0) = 1$, $C(l \neq l_0) = 0$, we obtain

$$E\approx\sum_{n=-l_0}^{l_0}P_{0n}{}^{l_0}e^{-in\varphi}e^{-ik(t+x)},\quad r\gg r_g, \tag{2.10}$$

$$E\approx\sum_{n=-l_0}^{l_0}P_{0n}{}^{l_0}e^{-in(\varphi-ax)}e^{-ik(t+x)},\quad r\sim r_g.$$

Going over to a locally Lorentzian reference frame at the point r involves the following transformation[1)]

$$\varphi\to\tilde\varphi=\varphi-ar_gt/r^3, \tag{2.11}$$

for the metric (1.1), (1.4). Therefore, for a local inertial observer (more precisely, for a local Schwarzschild observer) the solution (2.10) has the form

$$E\approx\sum_n P_{0n}^l e^{-in\tilde\varphi}e^{-i(k-an)(t+x)},\quad r\sim r_g. \tag{2.12}$$

Thus, the frequency k of a monochromatic wave at infinity will be split 2k + 1-fold near r_g, and the solution has the form of a superposition of waves with the frequencies k − an. Making in (2.7) and (A.2) the substitution $k = \Omega + an$, it is easy to find that conversely, a frequency Ω given near r_g will be split into $2l + 1$ components at infinity. If the source of radiation is at the point $r = r_0$, the corresponding substitution is obviously $k = \Omega + anr_g/r_0^3$ and the magnitude of the splitting at infinity is $K = ar_gc/r_0^3$ (we transformed to dimensional units).

Taking into account the term $l(l + 1)/r(r - 1)$ in the equations (2.6)–(2.7) does not change the result, since the "potential barrier" leads only to the appearance of a reflected wave and does not affect the frequency. The first paper in which the splitting of frequencies in the Kerr metric was pointed out is[5]. There it was also pointed out that the splitting is related to the different redshifts for quanta with different projections of the angular momentum, M_z.

One must consider separately the case of frequencies $k \lesssim a$. As can be seen from (2.6), the approximation linear in a becomes insufficient for such frequencies and one must take into account further terms in the expansion. However, the splitting scheme is more general and valid for low frequencies also. In particular, the zero frequency also splits, therefore a static solution in one asymptotic region of space may turn out to be a wave solution in another. It is obvious that the only solution which is static for all r is the axially-symmetric one, since for n = 0 there is no splitting.

3. We now go over to an investigation of the behavior of the fields created by a collapsing rotating object. In this exposition we shall make ample use of the results of[3], quoted as I in the sequel. As in I, we use the standard notation $\Psi(t, x)$ for the field function in terms of which are expressed the "physical" field components and for which one can derive a closed expression. The behavior of the function

$$\Psi(t,x)=\int_{-\infty}^{\infty}f_k\Psi_k(x)e^{-ikt}dk \tag{3.1}$$

for large times is determined by the analytic properties of its Fourier transform in the complex k plane. The spectral function f_k in (3.1) is determined from the initial data and Ψ_k, is the solution of the corresponding field equations, having unit normalization of an incident wave and satisfying the boundary condition

$$\Psi_k(x\to\infty)\sim e^{ikx}. \tag{3.2}$$

In the case of vanishing total angular momentum $\Psi_k(x)$ satisfies the equation (cf. I):

$$\Psi_k''+[k_2-U_l(x)]\Psi_k=0. \tag{3.3}$$

All the potentials $U_l(x)$ which appear in external field problems for nonrotating objects have the following asymptotic behaviors:

$$U_l(x\gg r_g)\approx\frac{l(l+1)}{x^2}\left(1+\frac{2r_g}{x}\ln x\right), \tag{3.4}$$

$$U_l(-x\gg r_g)\approx\text{const}\cdot\exp(x/r_g). \tag{3.5}$$

As was shown in I, the slowly decreasing "tail"

$$V(x)=\frac{2r_gl(l+1)}{x^3}\ln x$$

leads to branch points of the type $k^n \ln k$ for $k \to 0$ in the function $\Psi_k(x)$. In going over to the t-representation according to (3.1), the contribution of terms containing ln k turns out to be essential for large times and leads to power-laws for the decay, of the type $\Psi(t, x) \propto (t - x)^{-n}$.

In the first order with respect to the angular momentum of the electromagnetic field one can obtain equations which are close in type to (3.3) (cf. the Appendix). However, as we have seen in the previous section, in the region of small k the linear approximation in a/r_g is insufficient and one must take into account the next terms in the expansion. It is convenient to take directly the equations in the Kerr metric (1.1). Such equations have been derived by Teukolskiĭ [6]. We make use of the following form of the equation for the function Ψ_k:

$$\frac{d^2\Psi_k}{dx^2}+\left\{k^2+\frac{1}{(r^2+a^2)^2}\left[-2r_g rank+a^2n^2+ians(2r-r_g)-ikr_g s(r^2-a^2)\right.\right.$$
$$\left.\left.+2ikrs\Delta-a^2k^2\Delta-l(l+1)\Delta-\frac{2anks^2}{l(l+1)}+a^2k^2\Delta\delta_{l,n}\right]-U_s(x)\right\}\Psi_k=0, \quad (3.6)$$

where the following notations have been introduced

$$U_{-1}=\frac{r_g^2(r^2-a^2)}{4(r^2+a^2)^4}+\frac{r_g r\Delta}{(r^2+a^2)^3}\left[\frac{2(r^2-a^2)}{r^2+a^2}-1\right],$$
$$U_{-2}=\frac{r_g(r^2-a^2)}{(r^2+a^2)^3}\left(r-\frac{r_g a^2}{r^2+a^2}\right)-\frac{a^2\Delta}{(r^2+a^2)^3}\left(3+\frac{5r_g r}{r^2+a^2}\right),$$
$$\delta_{l,n}^{(s)}=\frac{(l^2-n^2)(l^2-s^2)}{l^2(4l^2-1)}+\frac{n^2s^2}{l^2(l+1)^2}+\frac{[(l+1)^2-n^2][(l+1)^2-s^2]}{(l+1)^2(2l+1)(2l+3)}$$
$$-2s^2\left[\frac{((l+1)^2-s^2)((l+1)^2-n^2)}{(l+1)^3(2l+1)(2l+3)}-\frac{(l^2-n^2)(l^2-s^2)}{l^3(4l^2-1)}\right]$$

The coordinate x is related to the radial coordinate r in (1.1) by means of the relation

$$x=r+M\left(\frac{M}{(M^2-a^2)^{1/2}}+1\right)\ln(r-r_+)-M\left(\frac{M}{(M^2-a^2)^{1/2}}-1\right)\ln(r-r_-) \quad (3.7)$$

The parameter s can take on the values s = 0 (scalar field), s = −1 (the electromagnetic field), s = −2 (gravitational field). The function $\Psi_k(s, r)$ in (3.6) is related to the radial function R(r) introduced by Teukolskiĭ [6] in the following manner:

$$\Psi_k(s=0, r)=R(r), \quad \Psi_k(s=-1, r)=(r^2+a^2)^{1/2}\Delta^{-1/2}R(r),$$
$$\Psi_k(s=-2, r)=(r^2+a^2)^{1/2}\Delta^{-1}R(r).$$

Equation (3.6) is written for different field components than (3.3) and goes over into (3.3) for a = 0 only in the case of the scalar field (s = 0), which has only one component.

In order to construct the solutions of (3.6) we use an approximation method analogous to the one used in I (cf. Appendix 2). As zeroth approximation, taking into account the frequency splitting, we use the functions

$$u_1=\begin{cases} e^{i\Omega x}+b_1e^{-i\Omega x}, & x<x_0 \\ a_1h_l^{(1)}(kx), & x>x_0 \end{cases}, \quad (3.8)$$
$$u_2=\begin{cases} a_2e^{-i\Omega x}, & x<x_0 \\ h_l^{(2)}(kx)+b_2h_l^{(1)}(kx), & x>x_0 \end{cases}, \quad (3.9)$$

where $\Omega = k - an/r_g r_+$, and $h_l^{(1,2)}(z)$ are the Riccati-Hankel (spherical Hankel) functions. The coefficients $a_{1,2}(k)$, $b_{1,2}(k)$ in the low frequency region ($|kx_0| \ll 1$) are

$$\frac{\Omega}{k}a_2=a_1\approx\frac{2i\Omega x_0(2ikx_0)^l\exp(-ianx_0/r_+r_g)l!}{(2l)![il+(k+\Omega)x_0]},$$
$$b_1\approx-\exp(2i\Omega x_0)\left[1+\frac{2i\Omega x_0}{l-i(k+\Omega)x_0}\right],$$
$$b_2\approx1+\frac{2i(l+1+i\Omega x_0)(kx_0)^{2l+1}}{(2l+1)!!(2l-1)!!(l-i\Omega x_0)}. \quad (3.10)$$

The functions $u_1(x)$ and $u_2(x)$ satisfy the equation

$$u''+(k^2-V_0)u=0,$$
$$V_0=\begin{cases} 2ank/r_+r_g-a^2n^2/(r_gr_+)^2, & x<x_0 \\ l(l+1)/x^2, & x>x_0 \end{cases}. \quad (3.11)$$

We search for a solution of (3.6) in the form

$$\Psi_k(s, x)=A_k(s, x)[u_1+B_k(s, x)u_2],$$
$$\Psi_k'(s, x)=A_k(s, x)[u_1'+B_k(s, x)u_2']. \quad (3.12)$$

The requirement that there be no incoming waves at $x \to \infty$ imposes on B_k the condition

$$B_k(s, x\to\infty)\to0. \quad (3.13a)$$

In order to determine the boundary condition for $A_k(s, x)$ we consider a solution of (3.6) in the asymptotic region $x \to -\infty (r \to r_+)$. In the leading order with respect to $y = r - r_+ \ll r_g$, Eq. (3.6) can be rewritten in the form

$$\frac{d^2\Psi_k}{dx^2}+\left(\Omega-\frac{is}{2r_+}\right)^2\Psi_k+O\left(\frac{a}{M}\right)\Psi_k=0, \quad a<M. \quad (3.14)$$

For a = M (3.14) goes over continuously into $\Psi'' + \Omega^2\Psi = 0$. However, this case is physically uninteresting, and for $(M - a)/M \ll 1$ the ratio $|(M - a)x|/M^2$ is important; this ratio can be made arbitrarily large for $x \to -\infty$, corresponding to the case (3.14). A solution of (3.14) is (we normalize the incident wave to one)

$$\Psi_k(s,x)=\exp\left[i\left(\Omega-\frac{is}{2r_+}\right)x\right]+C\exp\left[-i\left(\Omega-\frac{is}{2r_+}\right)x\right]. \quad (3.15)$$

The divergences in (3.15) are fictitious; they disappear when going over from Ψ to physical components of the fields. Let us expand the equation (3.12) for $x \to -\infty$:

$$\Psi\approx Ae^{i\Omega x}+A(b_1+Ba_2)e^{-i\Omega x}.$$

Comparing with the expression (3.15) we find

$$A_k(s, x\to-\infty)=\exp(sx/2r_+). \quad (3.13b)$$

The equations for the functions A_k and B_k are obtained by substituting (3.12) into (3.6); they coincide in form with analogous equations in I, namely

$$B_k'(s,x)=-\frac{V}{2ika_1}[u_1+B_ku_2]^2, \quad (3.16)$$
$$A_k'(s,x)=A_k\frac{V}{2ika_1}u_2[u_1+B_ku_2]. \quad (3.17)$$

We have denoted by V the difference between the "potentials" of the equations (3.6) and (3.11). For $x > x_0$ the "potential" V(x) equals

$$V(x)=-\frac{2iks}{x}-\frac{2iksr_g}{x^2}\ln x-\frac{2iksr_g^2}{x^3}\ln x-\frac{2iksr_g^2}{x^3}\ln^2 x$$
$$+\left(iksr_g+\frac{2anks^2}{l(l+1)}-a^2k^2\delta_{l,n}^{(s)}\right)\left(\frac{1}{x^2}+\frac{2r_g}{x^3}\ln x\right)$$
$$+\frac{2r_gl(l+1)}{x^3}\ln x+O\left(\frac{1}{x^3}\right). \quad (3.18)$$

The branch points of the form $k^m \ln k$ in the coefficients A_k and B_k appear in integrating the equations (3.16) and (3.17) with boundary conditions (3.13), the region of integration $x > x_0$ being the determinant one. For small k the contribution in the leading order in k to the coefficient of ln k comes from the following terms of the expansion (3.18):

$$V=-\frac{2iks}{x}-\frac{2iksr_g}{x^2}\ln x+\frac{ikr_gs}{x^2}+\frac{2anks^2}{x^2l(l+1)}+\frac{2r_gl(l+1)}{x^3}\ln x. \quad (3.19)$$

In the case a = 0 the solution of (3.16) and (3.17) should lead to the same result as the solution of the corresponding equations in I. It is obvious that the term $2anks^2/x^2 l(l+1)$, the unique term in (3.19) which is proportional to the total angular momentum of the collapsing mass, contributes only to the coefficient of $k^m \ln k$ and the exponent m coincides with its value for $a = 0$. A change in the exponent produced by the "frequency splitting" is taken into account in the zeroth approximation functions u_1 and u_2.

Now the solution $\Psi_k(x)$ can be written in the form (cf. I)

$$\Psi_k(s, x>x_0)=C_1(\Omega r_g)(kr_g)^l[1+C_2(kr_g)\ln(kr_g)]e^{ikx}+\ldots, \quad (3.20)$$

for $|kx| \gg 1$,

$$\Psi_k(s,x>x_0)=C_3\left(\frac{r_g}{x}\right)^l(kr_g)+\ldots+C_4\left(\frac{x}{r_g}\right)^{l+1}(\Omega r_g)(kr_g)^{2l+2}\ln(kr_g)+\ldots \quad (3.21)$$

for $|kx| \ll 1$, and

$$\Psi_k(s, x\to-\infty)=A_k(s,-\infty)[e^{i\Omega x}+b_1e^{-i\Omega x}+\ldots +C_5(\Omega r_g)(kr_g)^{2l+2}\ln(kr_g)\exp(-i\Omega x-sx/r_+)]. \quad (3.22)$$

There remains to calculate the spectral function f_k. For this we note that in the coordinates $r, t, \theta, \tilde{\varphi}$, where $\tilde{\varphi} = \varphi - at/r_g r_+$, the law of fall of the particle onto the capturing surface $r = r_+$ of the Kerr metric coincides with the laws of motion near r_g in the Schwarzschild metric[4]. The measurable frequency in this reference frame is Ω. Repeating the calculations carried out in Appendix 3 of I, it is easy to see that one obtains for the spectral function f_Ω exactly the same expressions as for f_k in I. There are small differences for perturbations with $n \neq 0$: for such perturbations the case of the initially static field is not realized. This is physically obvious, since a rotating body which does not have axial symmetry relative to the rotation axis must radiate.

The decay laws for the external fields produced by a rotating collapsing mass are obtained by substituting the values for Ψ_k and for the spectral function into (3.1). The calculations do not differ from the analogous ones of I and lead to the same results.

4. Thus, we have shown that the qualitative behavior of the signals emitted by a rotating black hole is the same as in the absence of rotation. An observer situated at a distance $r \gg r_g$ will first record a sharp change of the field amplitude with a characteristic time $\tau_0 \sim r_g/c$. This is followed by a decay

$$\Psi\approx\text{const}/\tau^{l+2}, \quad r_g\ll\tau=t-r\ll r, \quad (4.1)$$

which pertains to the wave field. For even larger times $t \gg r$ the decay law takes on the form

$$\Psi\approx\text{const}/t^{2l+3}. \quad (4.2)$$

Near the Schwarzschild horizon $r = r_+$ right after the jump of the amplitude the law (4.2) is valid. The coefficients in (4.1)–(4.2) depend on the order l of the spherical harmonic and on the total angular momentum of the black hole.

In[7] (cf. also[8]) it was shown that the problem of the decay laws of external fields in gravitational collapse is part of the more general problem on the form of wave signals propagating in the gravitational field of an isolated mass. The interaction of the wave with the curvature of spacetime leads to an extension of the posterior wave front and to the appearance of "tails" of the form (4.1), (4.2) following the main field pulse. The investigation carried out above shows that "tails" of the same type also appear in wave packets which propagate in the field of a rotating mass.

In the case of the electromagnetic field it is easy to obtain more detailed information on the changes introduced by the rotation of the central body. From the results of the previous section it can be seen that in the leading order the angular momentum enters linearly into the answers. Therefore, for an analysis of electromagnetic disturbances of a rotating black hole one can also use the system (A.3).

Let us consider, for instance, the first of these equations

$$\frac{d^2E_l}{dx^2}+\left[k^2-\frac{2ankr_g}{r^3}-\frac{(r-r_g)l(l+1)}{r^3}\right]E_l = \frac{3iar_g(r-r_g)}{r^5}\left[\frac{l[(l+1)^2-n^2]^{1/2}}{2l+3}H_{l+1}-\frac{(l+1)[l^2-n^2]^{1/2}}{2l-1}H_{l-1}\right].$$

As was already remarked in Sec. 2, the contribution of inhomogeneous terms to the fundamental wave train is small in the parameter $1/r$. A different situation arises in the computation of the "tails." It is easy to see that in the leading approximation the "tails" of E_l can be found from the solution of the equation

$$E_l''+\left[k^2-\frac{l(l+1)}{x^2}\right]E_l=\frac{2r_gl(l+1)\ln x}{x^3}E_l-\frac{3iar_g(l+1)(l^2-n^2)^{1/2}}{x^4(2l-1)}H_{l-1},$$

$x > x_0$, i.e., the rotation leads to the fact that the electromagnetic field of a wave of "magnetic type" of order $l-1$ also participates in the formation of the tail of the wave emitted by an electric multipole of order l.

The author is indebted to A. Z. Patashinskiĭ and V. K. Pinus for numerous stimulating discussions.

APPENDIX

The equations for the electromagnetic field in the metric (1.3) have been derived in[8]. Here we list the main formulas. We introduce the following combinations of the components of the electromagnetic field tensor F_{ik}:

$$\begin{gathered} iF_{02}\mp\frac{1}{\sin\theta}F_{03}=2E_\pm, \quad iF_{12}\mp\frac{1}{\sin\theta}F_{13}=2H_\pm, \\ r^2F^{01}=E, \quad -\frac{i}{\sin\theta}F_{23}=H, \\ iF^{02}\mp\sin\theta F^{03}=2E^\pm, \quad iF^{12}\mp\sin\theta F^{13}=2H^\pm. \end{gathered} \quad (A.1)$$

All quantities are represented in the form

$$f(r,t,\theta,\varphi)=\sum_{l,n}e^{-in\varphi}P^l_{mn}(\cos\theta)\int_{-\infty}^{\infty}f_{l,n}(k,r)e^{-ikt}dk, \quad (A.2)$$

where $m = \pm1$ for $E^\pm, H^\pm$ ($E_\pm, H_\pm$) and $m = 0$ for E and H; $P^l_{mn}(x)$ are generalized Legendre polynomials[9]. E and H satisfy the following system of equations

$$\begin{gathered} \left\{E''+\frac{r_g}{r(r-r_g)}E'+\left[\frac{r^2k^2}{(r-r_g)^2}-\frac{2ankr_g}{r(r-r_g)^2}-\frac{l(l+1)}{r(r-r_g)}\right]E\right\}_l \\ =\frac{3iar_g}{r^3(r-r_g)}\left[\frac{l[(l+1)^2-n^2]^{1/2}}{2l+3}H_{l+1}-\frac{(l+1)(l^2-n^2)^{1/2}}{2l-1}H_{l-1}\right], \\ \left\{H''+\frac{r_g}{r(r-r_g)}H'+\left[\frac{r^2k^2}{(r-r_g)^2}-\frac{2ankr_g}{r(r-r_g)^2}-\frac{l(l+1)}{r(r-r_g)}\right]H\right\}_l \\ =\frac{3iar_g}{r^3(r-r_g)}\left[\frac{l[(l+1)^2-n^2]^{1/2}}{2l+3}E_{l+1}-\frac{(l+1)(l^2-n^2)^{1/2}}{2l-1}E_{l-1}\right]. \end{gathered} \quad (A.3)$$

The remaining components of the electromagnetic field tensor are related to E and H by means of the relations

$$H_+ - H_- = -H'/\alpha, \quad H^+ + H^- = -ikE/\alpha r^2,$$
$$E_+ - E_- = ikH/\alpha, \quad E^+ + E^- = -E'/\alpha r^2, \qquad (A.4)$$

where $\alpha = [l(l+1)]^{1/2}$. The set of quantities dependent on E describe electromagnetic disturbances of "electric" type, and those dependent on H, describe disturbances of "magnetic" type, i.e., these disturbances are produced respectively by electric and magnetic multipoles.

[1)]Near the Schwarzschild surface, in the so-called ergosphere, one can realize in general only a reference frame rotating with respect to infinity.

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Translated by Meinhard E. Mayer
155