Energy distribution of electrons in a weakly ionized current-carrying plasma with a transverse inhomogeneity

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We solve the kinetic equation and find the electron energy distribution in a weakly ionized plasma which is placed in an external uniform electric field E_z and which is inhomogeneous in a direction at right angles to to the current. If the size R of the inhomogeneity exceeds the electron energy relaxation length $\lambda_T=\lambda/\delta^{1/2}$ (λ is the electron mean free path and δ the fraction of energy lost in an elastic collision) the effect of the transverse electric field on the distribution function will be small; it leads to the occurrence of a transverse energy flux $\sim nD_eT/R$, where D_e is the electron diffusion coefficient, and T the temperature (average energy), which here depends on the transverse coordinate. If, however, $R \ll \lambda_T$, the distribution function becomes non-local and its shape is determined by the complete transverse profile of the potential $\varphi(x,y)$, while the energy $\epsilon=1/2mv^2+e\varphi(x,y)$ is the argument. In that case the connection between the concentration n and φ is of the form $n\sim e^{e\varphi/T}$ so that the ambipolar diffusion equation is significantly nonlinear. We consider the effect of the nonlocal behavior on the shape of the distribution function tail.

1. INTRODUCTION

The form of the electron energy distribution function for a uniform, weakly ionized plasma in a uniform external electric field has been studied in many papers.[1-6] The collisions of electrons with heavy particles can usually be split into two kinds: quasi-elastic collisions with small loss of kinetic energy $\delta\theta \ll T$ (for atomic gases, for instance, $\delta = 2m/M$) and inelastic collisions accompanied by a large loss of energy $\epsilon_1 \gg T$. The distribution function is in the "hot" energy region $\theta \sim T$ determined by the quasi-elastic collisions which correspond to the transport frequency $\nu(\theta)$, the electron-electron collisions, occurring with a frequency $\nu_e(\theta)$, and the electric field E_z.

If $\nu_e \gg \delta\nu$ the distribution in the hot region is Maxwellian, but when $\nu_e \ll \delta\nu$ the distribution is set up as the result of the competition between two processes—diffusion in energy with a diffusion coefficient

$$D_\epsilon^0=\frac{1}{3}(eE_z\lambda)^2\frac{\nu^3}{\omega^2+\nu^2},$$

where E_z is the instantaneous value of the field when $\omega \ll \delta\nu$ and the effective field when $\omega \gg \delta\nu$,[2] and the drift velocity in energy space $V_\epsilon^0 = -\theta\delta\nu$. Zero energy flux corresponds to the function

$$f_0(\theta)=An\exp\left\{-\int_0^\theta\frac{V_\epsilon^0(\theta)}{D_\epsilon^0(\theta)}d\theta\right\}. \qquad (1)$$

The average electron energy is

$$T\sim\frac{D_\epsilon^0(T)}{V_\epsilon^0(T)}\sim\frac{eE_z\lambda(T)\nu(T)}{[\delta(\omega^2+\nu^2(T))]^{1/2}}.$$

Inelastic collisions lead to the appearance of an energy flux; when $\epsilon_1 \gg T$ their role is reduced to an important depletion of the distribution function as compared to (1) only in the tail region—at energies

$$\theta\geqslant\varepsilon_1-\Delta\varepsilon=\varepsilon_1-D_\epsilon^0(\varepsilon_1)/V_\epsilon^0(\varepsilon_1),$$

while the solution in the hot region—for $\theta \sim T$ is practically the same as (1); the role played by the inelastic collisions for the energy balance is unimportant.[6]

These results have been applied in a number of papers[7-9] to a plasma with an inhomogeneity across the current. The presence of an inhomogeneity leads to the appearance of a diffusion field

$$E_\perp(x,y)\sim\frac{T}{eR}\sim\frac{E_z\nu\lambda_T}{R(\omega^2+\nu^2)^{1/2}}$$

besides the "current" field E_z. It was assumed in these papers that the form of the distribution function has the form (1) at each point, where D_ϵ^0 is determined not by the whole field but only by its current part E_z while the profile of the concentration satisfies the ambipolar diffusion equation (the difference of the distribution from the Maxwellian manifests itself only in the values of the mobility and the diffusion coefficient; in particular, the Einstein relation is not satisfied). Such an approach may be called the local approximation. However, the characteristic time for establishing the distribution function (1) is $(\delta\nu)^{-1}$. In a non-uniform plasma a particle passes during this time through a distance $\lambda_T \sim (D_e/\delta\nu)^{1/2} \sim \lambda/\delta^{1/2}$, where $D_e \sim \frac{1}{3}v\lambda$ is the electron coordinate diffusion coefficient. The local approximation is therefore applicable only when the scale-length of the inhomogeneity $R \gg \lambda_T$[10] (in that case $E_\perp \ll E_z$).

In the present paper we analyze by using simple examples a number of kinetic effects connected with the non-locality of the distribution function. When $R \gg \lambda_T$ the deviation of the distribution function (1) from Maxwellian leads to the fact that the transverse energy flux is anomalously large: $q_\perp \sim D_enT/R$ while the transverse-particle flux is determined by the ambipolar diffusion coefficient $j_\perp \sim D_an/R \ll q_\perp/T$. The presence of an energy flux leads to the fact that the average energy of the electron gas depends on the transverse coordinates: $T = T_0 + T_1(x, y)$ where the form of the kinetic correction $T_1 \sim (\lambda_T/R)^2T_0$ is determined by the form of the function $\nu(\theta)$. If, however, $R \ll \lambda_T$ it is necessary to take the transverse electric field into account consistently.

We have obtained solutions of the kinetic equation for the case $\lambda \ll R \ll \lambda_T$. As the transverse electron diffusion proceeds faster than the energy change in the field E_z and the heat exchange with neutral particles, $\epsilon = \frac{1}{2}mv^2 + e\varphi(x, y)$ is for $\epsilon < \epsilon_1$ an approximate integral of motion and hence also an argument of the distribution function (here φ is the potential of the transverse electric field). For trapped particles (when $\epsilon > e\varphi(x, y)$)

$$f_0(\varepsilon)=A_0\exp\left\{-\int_0^\varepsilon \frac{\overline{V}_\varepsilon(\varepsilon)}{\overline{D}_\varepsilon(\varepsilon)}d\varepsilon\right\}, \tag{2}$$

where the averaging of the quantities $V_\epsilon(\theta)$ and $D_\epsilon(\theta)$ is over the cross-section accessible for particles with the given energy ϵ; for untrapped particles the argument of the exponent in (2) is

$$-\int_0^\varepsilon \frac{V_\varepsilon(\theta)}{D_\varepsilon(\theta)}d\theta.$$

In any case the distribution in kinetic energy θ in a fixed point is, when $\theta < e\varphi(x, y)$, close to a Maxwellian one with a variable temperature while the connection between n, $\varphi(x, y)$ does not have the Boltzmann form $n \sim e^{-e\varphi/T}$. The equation for the ions becomes therefore non-linear and does not reduce to the usual equation of ambipolar diffusion.

We have considered the effect of non-locality on the form of the tail of the distribution function. The proportionality between the electron concentration and the number of inelastic collisions in a given point is then violated. The steep diminution of the distribution function starts from energies $\epsilon \gtrsim \epsilon_1 - \Delta\epsilon$ so that in regions with a small concentration this decrease leads to values of the kinetic energy much smaller than the threshold ϵ_1 and the number of inelastic collisions is very small. We have obtained a solution for the case when the equation for the tail is reduced to a one-dimensional one in which the frequency of the inelastic collisions is replaced by a frequency averaged over a cross-section.

2. INITIAL EQUATIONS

For the sake of simplicity we restrict ourselves to the quasi-stationary case $\omega \ll \delta\nu$; we put the temperature of the heavy particles (ions and atoms) equal to zero, and $\nu_e \ll \delta\nu$. We also assume that the frequency of inelastic collisions $\nu^* \ll \nu$ and that second kind collisions are unimportant. The change in the profile of the concentration, and hence also the field $\mathbf{E}_\perp$, is connected with the relatively slow diffusion and recombination of ions so that we can assume the field $\mathbf{E}_\perp$ to be quasi-stationary when $\delta\nu \gg D_a/R^2 + \tau_{rec}^{-1}$.

If the electron mean free path $\lambda \ll R$ and $\delta \ll 1$ we can restrict ourselves in the expansion of the distribution function $f(r, v, t)$ in spherical harmonics to the first two terms:[1]

$$f(\mathbf{r},\mathbf{v},t)=f_0(\mathbf{r},v,t)+\frac{\mathbf{v}}{v}\mathbf{f}_1(\mathbf{r},v,t).$$

The set of equations for those has the form

$$\frac{\partial f_0}{\partial t}+\frac{v}{3}\nabla\mathbf{f}_1+\frac{e}{3mv^2}\frac{\partial}{\partial v}[v^2\mathbf{E}(\mathbf{r},t)\mathbf{f}_1]-\frac{1}{2v^2}\frac{\partial}{\partial v}(v^3\delta\nu f_0)=-\nu^*(\theta)f_0+I, \tag{3a}$$

$$\mathbf{f}_1=-\frac{e\mathbf{E}(\mathbf{r},t)}{m\nu}\frac{\partial f_0}{\partial v}-\frac{v}{\nu}\nabla f_0. \tag{3b}$$

Here I is the source of slow particles connected with excitation and ionization. Taking it into account can materially affect the form of the distribution function only in the low-velocity region $\theta \ll T$[6] when the energy diffusion coefficient may vanish and the electrons undergoing inelastic collisions are trapped for a long time in the low-energy region. We shall in what follows neglect that fact and put $I = 0$.

We change in the set (3) from the variables $\mathbf{r}$, v to the variables $\mathbf{r}$, $\epsilon = \frac{1}{2}mv^2 + e\varphi(x, y)$. Then

$$\left(\frac{\partial}{\partial v}\right)_r=mv\left(\frac{\partial}{\partial\varepsilon}\right)_r,\quad \left(\frac{\partial}{\partial\mathbf{r}}\right)_v=\left(\frac{\partial}{\partial\mathbf{r}}\right)_\varepsilon-e\mathbf{E}_\perp\left(\frac{\partial}{\partial\varepsilon}\right)_r$$

and terms with the field $\mathbf{E}_\perp$ drop out when we substitute (3b) into (3a). As a result we get an equation of diffusion in coordinate space and in energy with drift in energy and absorption:

$$-\frac{\partial f_0}{\partial t}+\frac{1}{3v}\nabla\left(\frac{v^3}{\nu}\nabla f_0\right)+\frac{m}{2v}\frac{\partial}{\partial\varepsilon}\left[v^3\left(\delta\nu f_0+\frac{2e^2E_z^2}{3m\nu}\frac{\partial f_0}{\partial\varepsilon}\right)\right]=-\frac{\partial f_0}{\partial t}$$
$$+\frac{1}{3v}\nabla(D(\theta)\nabla f_0)+\frac{m}{2v}\frac{\partial}{\partial\varepsilon}\left[V_\varepsilon(\theta)f_0+D_\varepsilon(\theta)\frac{\partial f_0}{\partial\varepsilon}\right]=\nu^*(\theta)f_0. \tag{4}$$

The coefficients in the equation are functions[1)] of $\theta(x, y, \epsilon) = \frac{1}{2}mv^2 = \epsilon - e\varphi$. We shall consider the form of the solution in the "hot" region where the right-hand side of (4) equals zero, in a few simple cases.

3. LARGE SCALE INHOMOGENEITY $R \gg \lambda_T$. THERMAL PARTICLES

For the case of interest to us of a transverse inhomogeneity in the particle and energy flux we can obtain (1) from (3b). As E_z = constant thermal diffusion and thermal conductivity (i.e., terms in the particle and energy flux proportional to $\nabla|\mathbf{E}|$) are absent in first approximation in λ_T/R. The particle flux is equal to

$$\mathbf{j}_e=\frac{4\pi}{3}\int_0^\infty \mathbf{f}_1v^3\,dv=nb_e\mathbf{E}_\perp-D_e\nabla n. \tag{5}$$

For instance, for the case of a power-law dependence of the transport cross-section on the electron velocity

$$\sigma(v)=\sigma_0(v/v_0)^\alpha \tag{6}$$

the kinetic coefficients have the form[10] (when $-2 < \alpha < 4$)

$$b_e=e\Gamma\left(\frac{2-\alpha}{4+2\alpha}\right)(2-\alpha)\Big/m\Gamma\left(\frac{3}{4+2\alpha}\right)3\nu(v_{T0}),$$
$$D_e=\Gamma\left(\frac{4-\alpha}{4+2\alpha}\right)v_{T0}^2\Big/\Gamma\left(\frac{3}{4+2\alpha}\right)3\nu(v_{T0}),\quad \nu=N_a v\sigma(v), \tag{7}$$

where the thermal velocity v_{T0} is

$$v_{T0}^{4+2\alpha}=\frac{4e^2E_z^2v_0^{2\alpha}(2+\alpha)}{3m^2\delta N_a^2\sigma_0^2}.$$

The electron energy flux is given by the expression

$$\mathbf{q}_e=\frac{2\pi m}{3}\int_0^\infty \mathbf{f}_1v^5\,dv=nb_T\mathbf{E}-D_T\nabla n, \tag{8}$$

where

$$b_T=\frac{ev^2}{6\nu(v_{T0})}(4+2\alpha)\Gamma\left(\frac{8+\alpha}{4+2\alpha}\right)\Big/\Gamma\left(\frac{3}{4+2\alpha}\right),$$
$$D_T=mv_{T0}^4\Gamma\left(\frac{6-\alpha}{4+2\alpha}\right)\Big/3\nu(v_{T0})\Gamma\left(\frac{3}{4+2\alpha}\right). \tag{9}$$

The quasi-neutrality condition $j_{e\perp} = j_{i\perp}$ connects the transverse field with the concentration profile of the plasma. Using the fact that the ion mobility is small it reduces to $j_{e\perp} = 0$ so that

$$\mathbf{E}_\perp=\frac{D_e}{nb_e}\nabla n=\frac{mv_{T0}^2}{e}\left[\Gamma\left(\frac{4-\alpha}{4+2\alpha}\right)\Big/(4+2\alpha)\Gamma\left(\frac{6+\alpha}{4+2\alpha}\right)\right]\nabla\ln n. \tag{10}$$

The transverse ion motion is determined by the equation

$$\frac{\partial n}{\partial t}+\nabla\mathbf{j}_{i\perp}=nZ-\frac{n}{\tau_{rec}}. \tag{11}$$

The terms on the right-hand side determine the creation and recombination of particles while the ion current

$$\mathbf{j}_{i\perp}=-D_i\nabla n-nb_i\mathbf{E}_\perp=-D_a\nabla n, \tag{12}$$

where the ambipolar diffusion coefficient is

$$D_a=D_i+\frac{b_iD_e}{b_e}=D_i\left[1+\frac{mv_{T0}^2}{T_i}\Gamma\left(\frac{4-\alpha}{4+2\alpha}\right)\Big/(4+2\alpha)\Gamma\left(\frac{6+\alpha}{4+2\alpha}\right)\right]. \tag{13}$$

The transverse energy flux then turns out to be non-vanishing:

$$q_{e\perp}^{(a)}=-\frac{mv_{T0}^4}{6\nu(v_{T0})}\left\{\Gamma\left(\frac{8+\alpha}{4+2\alpha}\right)\Gamma\left(\frac{4-\alpha}{4+2\alpha}\right)\Big/\Gamma\left(\frac{6+\alpha}{4+2\alpha}\right)-\Gamma\left(\frac{6-\alpha}{4+2\alpha}\right)\Big/\Gamma\left(\frac{3}{4+2\alpha}\right)\right\}\nabla n=\frac{mv_{T0}^4}{6\nu(v_{T0})}\Phi(\alpha)\nabla n\sim D_eT\nabla n\gg D_aT\nabla n. \tag{14}$$

When $-2 < \alpha < -1$ the function $\Phi(\alpha)$ is negative so that the direction of the heat current is the same as the diffusion current while for $-1 < \alpha < 4$, $\Phi(\alpha) > 0$ and the heat current is in the same direction as the field current. The reason for this is that systems close to thermodynamic equilibrium have a chemical potential so that for them the coefficients b_e and D_e, b_T and D_T are pairwise connected through the Einstein relations. However, in our case $E_\perp$ and $\nabla_\perp n$ are independent forces and therefore there is no connection between the kinetic coefficients.

The presence of an energy flux leads to the appearance of a transverse gradient in the electron temperature

$$T=\langle{}^1/_3mv^2f_0\rangle=T_0+T_1(x,y),$$

where

$$T_0=m\Gamma\left(\frac{5}{4+2\alpha}\right)v_{T0}^2\Big/3\Gamma\left(\frac{3}{4+2\alpha}\right)\sim\frac{eE_zv_{T0}}{\delta^{1/2}\nu(v_{T0})} \tag{15}$$

is the average electron energy in the homogeneous plasma. To estimate the ensuing temperature drop $T_1(x, y) \ll T_0$ we neglect the change in the form of f_0 and put

$$f_0=\frac{4+2\alpha}{4\pi v_T^3}\Gamma^{-1}\left(\frac{3}{4+2\alpha}\right)n\exp\left\{-\left(\frac{v}{v_T}\right)^{4+2\alpha}\right\},$$

$$v_T=v_{T0}(1+T_1/2T_0).$$

The energy put into the plasma per unit time is

$$b_eenE_z^2=e^2E_z^2n\Gamma\left(\frac{6+\alpha}{4+2\alpha}\right)(4+2\alpha)\Big/m\Gamma\left(\frac{3}{4+2\alpha}\right)3\nu(v_T), \tag{16}$$

while the energy given off is equal to

$$\left\langle\frac{mv^2}{2}\delta\nu(v)f_0\right\rangle=\frac{mn}{2}v_T^2\delta\nu(v_T)\Gamma\left(\frac{6+\alpha}{4+2\alpha}\right)\Big/\Gamma\left(\frac{3}{4+2\alpha}\right). \tag{17}$$

The work done by the field $E_\perp$ is of the order of $eE_\perp j_{e\perp} \sim eE_\perp D_a\nabla_\perp n$ and is small compared to $\nabla q_{e\perp}^{(a)}$. Writing the energy balance in the form

$$b_eenE_z^2-\left\langle\frac{mv^2}{2}\delta\nu f_0\right\rangle+\nabla q_{e\perp}^{(a)}=0, \tag{18}$$

we get

$$\frac{T_1(x,y)}{T_0}=\frac{\nabla q_{e\perp}^{(a)}}{(2+\alpha)enb_eE_z^2}=\frac{v_{T0}^2\nabla^2n}{3\delta\nu^2(v_{T0})n}\Phi(\alpha)\Gamma\left(\frac{3}{4+2\alpha}\right)\Big/\Gamma\left(\frac{6+\alpha}{4+2\alpha}\right)\sim\left(\frac{\lambda_T}{R}\right)^2. \tag{19}$$

4. SMALL SCALE INHOMOGENEITY $R \ll \lambda_T$. THERMAL PARTICLES

We rewrite (4) introducing dimensionless variables—the energy $\tilde\epsilon = \epsilon/\epsilon_0$, where $\epsilon_0 = 2e^2E_z^2/3m\delta\nu^2(\epsilon_0)$, and distances in units R:

$$\tilde\nabla\tilde D\tilde\nabla f_0+\rho^2\frac{\partial}{\partial\tilde\varepsilon}\left(\tilde V_\varepsilon f_0+\tilde D\frac{\partial f_0}{\partial\tilde\varepsilon}\right)=Lf_0=0, \tag{4a}$$

where

$$\rho=R/\lambda_T=R(3\delta)^{1/2}\nu(\varepsilon_0)/v(\varepsilon_0)\ll1,$$

$$\tilde D=D(0)/D(\varepsilon_0)=D_\varepsilon(0)/D_\varepsilon(\varepsilon_0),\quad \tilde V_\varepsilon=V_\varepsilon(0)/V_\varepsilon(\varepsilon_0).$$

The principal term containing the transverse gradient vanishes for any arbitrary function of ϵ, so that this quantity is an approximate constant of the motion. The form of the ϵ-dependence of f_0 depends in an essential way on whether the motion of the particles is finite or infinite.

A local rarefaction of the plasma corresponds to a hump in the electron potential so that all particles are untrapped. They stay in the inhomogeneity region for a time $\sim R^2/D_e \ll (\delta\nu)^{-1}$—a distribution function is formed in the homogeneous plasma and has the form (1) in which the kinetic energy θ is replaced by ϵ:

$$f_0^{(0)}(\varepsilon)=A\exp\left\{-\int_0^\varepsilon\frac{V_\varepsilon(\theta)}{D_\varepsilon(\theta)}d\theta\right\}. \tag{20}$$

This function corresponds to zero energy flux and does not satisfy the equation. To check that nevertheless it is a valid zeroth approximation we obtain the expression for the small correction $f_0^{(1)}(r, \epsilon) \sim \rho^2 f_0^{(0)}(\epsilon)$ which guarantees the conservation of particle number and we find the condition for its regularity.

The equation for $f_0^{(1)}$ has the form

$$Lf_0^{(1)}=\rho^2\frac{\partial}{\partial\tilde\varepsilon}\left[\frac{\tilde D(\tilde\varepsilon-\tilde\varphi)}{\tilde D(\tilde\varepsilon)}\tilde V_\varepsilon(\tilde\varepsilon)-\tilde V_\varepsilon(\tilde\varepsilon-\tilde\varphi)\right]f_0^{(0)}(\varepsilon)=\rho^2\Psi. \tag{21}$$

As the diffusion in coordinate space proceeds faster than in energy the form of $f_0^{(1)}$ is significantly different at small and large distances from the inhomogeneity.

In the region $|r| \sim R \ll \lambda_T$ we can neglect the second term in L. For an axially symmetric inhomogeneity, for instance,

$$f_0^{(1)}(\tilde r,\tilde\varepsilon)=\rho^2\int^{\tilde r}\frac{d\tilde r'}{\tilde r'\tilde D(\tilde r',\tilde\varepsilon)}\int_{\tilde r_0(\varepsilon)}^{\tilde r'}\Psi d\tilde r''+C(\tilde\varepsilon),$$

where $\tilde r_0(\tilde\epsilon)$ is the coordinate of the turning point when $\tilde\epsilon < \tilde\varphi(0)$ and zero when $\tilde\epsilon > \tilde\varphi(0)$. For a plane inhomogeneity

$$f_0^{(1)}(\tilde x,\tilde\varepsilon)=\rho^2\int^{\tilde x}\frac{d\tilde x'}{\tilde D(\tilde x',\tilde\varepsilon)}\int_{\tilde x_0(\tilde\varepsilon)}^{\tilde x'}\Psi d\tilde x''+C(\tilde\varepsilon).$$

when $\tilde\epsilon < \tilde\varphi(0)$ the lower limit $\tilde x_0(\tilde\epsilon)$ is the turning point $\tilde\epsilon = \varphi(\tilde x_0)$ while for $\tilde\epsilon > \tilde\varphi(0)$

$$\int_{\tilde x_0}^{\infty}\Psi(\tilde x,\tilde\varepsilon)\,d\tilde x=\int_{-\infty}^{\tilde x_0}\Psi(\tilde x,\tilde\varepsilon)\,d\tilde x.$$

The correction $f_0^{(1)}$ remains finite in the vicinity of the turning points if as $v \to 0$ the collision frequency $\nu(v) \sim v^{1+\alpha}$, with $-2 < \alpha < 1$. When $|r| \gg R$ we have $\Psi \to 0$ and there is an appreciable diffusion in energy so that the correction satisfies a homogeneous elliptical equation with a boundary condition at the origin: for the plane case

$$\tilde D\frac{\partial}{\partial\tilde x}f_0^{(1)}=\rho^2\int_{x_0}^{\infty}\Psi(\tilde x,\tilde\varepsilon)\,d\tilde x,$$

and for the cylindrical case

$$\tilde D\frac{\partial}{\partial\tilde r}f_0^{(1)}=\frac{\rho^2}{r}\int_{\tilde r_0}^{\infty}\Psi(\tilde r,\tilde\varepsilon)\,d\tilde r,$$

i.e., it is the solution to the Neumann problem.

If the plasma is bounded by walls in the transverse direction the wall potential exceeds the average electron energy considerably. All thermal particles then perform a finite motion. To find the form of the function $f_0^{(0)}(\epsilon)$ we integrate (4a) over the cross-section $s(\epsilon)$ which is accessible to particles with energy ϵ. The integral of the first term vanishes identically when

$\alpha < 2$ so that the solution has the form (2) where the averaging means:

$$\bar{V}_\varepsilon(\varepsilon)=\int\limits_{s(\varepsilon)} V_\varepsilon(\varepsilon-e\varphi)\,ds.$$

For this solution the diffusion and drift energy fluxes are compensated only on average over the cross-section so that the correction connected with the flux in coordinate space is (for the plane case) equal to

$$f_0^{(1)}=\rho^2\int\limits_{\tilde{x}_1}^{\tilde{x}}\frac{d\tilde{x}'}{\bar{D}(\tilde{x}',\tilde{\varepsilon})}\int\limits_{\tilde{x}_0(\tilde{\varepsilon})}^{\tilde{x}'}\Psi_1(\tilde{x}'',\tilde{\varepsilon})\,d\tilde{x}'', \tag{22}$$

where

$$\Psi_1=\frac{\partial}{\partial\tilde{\varepsilon}}\left[\frac{\bar{V}_\varepsilon(\tilde{\varepsilon})}{\bar{D}_\varepsilon(\tilde{\varepsilon})}D_\varepsilon(\tilde{\varepsilon}-\tilde{\varphi})-V_\varepsilon(\tilde{\varepsilon}-\tilde{\varphi})\right]$$

and the lower limit $\tilde{x}_1$ can be found from the normalization condition

$$\int\limits_0^\infty d\varepsilon\int\limits_{\tilde{x}_{01}(\varepsilon)}^{x_{02}(\varepsilon)}(\varepsilon-e\varphi)^{1/2}f_0^{(1)}dx=0.$$

As for untrapped particles $f_0^{(1)}(\tilde{x},\tilde{\varepsilon})$ is regular for $-2<\alpha<1$.

The distribution functions (20) and (2) for trapped and untrapped particles depend on the total energy $\epsilon=\theta+e\varphi(r)$ so that except for the case ν = constant they do not split up into a product of two factors, one of which depends only on the coordinates and the other on the kinetic energy. The kinetic energy distribution of the particles for fixed coordinate is

$$f_0(\varepsilon>e\varphi(\mathbf{r}))=A\exp\{-F(\varepsilon)\}\approx A\exp\left\{-F(e\varphi(\mathbf{r}))-\frac{\partial F}{\partial\varepsilon}\bigg|_{e\varphi(\mathbf{r})}\theta-\ldots\right\}. \tag{23}$$

Up to $\theta\lesssim e\varphi(r)$ it is nearly a Maxwellian one.

The local electron temperature depends on the coordinate. It equals

$$\left[\frac{\partial F}{\partial\varepsilon}\bigg|_{e\varphi(\mathbf{r})}\right]^{-1}=\begin{cases}D_\varepsilon(e\varphi(\mathbf{r}))/V_\varepsilon(e\varphi(\mathbf{r}))\text{ — for untrapped particles}\\ \bar{D}_\varepsilon(e\varphi(\mathbf{r}))/\bar{V}_\varepsilon(e\varphi(\mathbf{r}))\text{ — for trapped particles}\end{cases} \tag{24}$$

The particle concentration in a given point is connected with the potential through the relation

$$n(\mathbf{r})=\frac{4\pi\sqrt{2}}{m^{3/2}}\int\limits_{e\varphi(\mathbf{r})}^{\infty}(\varepsilon-e\varphi(\mathbf{r}))^{1/2}f_0(\varepsilon)\,d\varepsilon. \tag{25}$$

This expression has the Boltzmann form

$$n\sim\exp\{-e\varphi(\mathbf{r})/T\}$$

only when ν = constant. For instance, for a power-law velocity-dependence (6) of the cross-section we have in the region of strong rarefaction $e\varphi>T$:

$$n(\mathbf{r})=(2+\alpha)^{-1/2}n_0z^{-(3+3\alpha/2)}\exp(-z^{2+\alpha})\cdot\Gamma\left(\frac{3}{2}\right)\Big/\Gamma\left(\frac{3}{4+2\alpha}\right), \tag{26}$$

where $z=2e\varphi(r)/mv_{T0}^2$ and n_0 is the plasma concentration far from the inhomogeneity.

Neglecting T_i as compared to T in (11) and (12) and taking into account in (26) only the basic exponential dependence on φ we find that the equation describing the ion motion does not reduce to the usual ambipolar diffusion equation and is essentially non-linear:

$$\frac{\partial n}{\partial t}-\frac{b_imv_{T0}^2}{2e(2+\alpha)}\nabla\left(\ln\frac{n_0}{n}\right)^{-(1+\alpha)/(2+\alpha)}\nabla n=Zn-\frac{n}{\tau_{\rm rec}}. \tag{27}$$

For trapped particles the equation will be an integral equation as $f_0(\epsilon)$ depends on the potential profile.

5. ALLOWANCE FOR INELASTIC COLLISIONS. FORM OF THE DISTRIBUTION-FUNCTION TAIL

In the local approximation, when the diffusion in coordinate space is unimportant, inelastic collisions with large energy losses $\epsilon_1\gg T$ lead to a steep drop in the distribution function starting at the energy

$$\theta=\varepsilon_1-\Delta\varepsilon=\varepsilon_1-\frac{D_\varepsilon(\varepsilon_1)}{V_\varepsilon(\varepsilon_1)}=\varepsilon_1-\frac{e^2E_z^2}{m\delta\nu^2(\varepsilon_1)}, \tag{28}$$

if the frequency $\nu^*(\theta)$ of inelastic collisions is sufficiently high. The condition for this is the following:

$$\nu^*\gg\frac{m\delta^2\varepsilon_1\nu^3(\varepsilon_1)}{e^2E_z^2}\text{ when }\nu^*=\text{const},$$

$$\frac{\partial\nu^*}{\partial\theta}\bigg|_{\varepsilon_1}\frac{m^2\delta}{\varepsilon_1\nu(\varepsilon_1)}\left[\frac{eE_z}{m\delta\nu(\varepsilon_1)}\right]^4\gg1\text{ when }\nu^*=\text{const}\cdot(\theta-\varepsilon_1).$$

The number of excitations per unit volume is then proportional to the plasma concentration.[3-6]

One estimates easily that for electrons in the tail of the distribution function for which $\theta\gtrsim\epsilon_1$ the local approximation holds when

$$R\gg\lambda_T(\varepsilon_1)=(T/m\delta)^{1/2}\nu(T)/\nu^2(\varepsilon_1).$$

Here $\lambda_T(\epsilon_1)$ has the meaning of an energy relaxation length for electrons with $\theta\sim\epsilon_1$. The criterion for the applicability of the earlier results[7-9] thus has the form

$$R\gg\max\,[\lambda(T)/\delta^{1/2},\lambda_T(\varepsilon_1)].$$

Let us consider the form of the tail in the non-local case

$$R\ll\min\,[\lambda(T)/\delta^{1/2},\lambda_T(\varepsilon_1)]$$

for simple examples. In a plasma bounded by walls the solution has the form (2) when

$$\varepsilon<\varepsilon_1-\bar{D}_\varepsilon(\varepsilon_1)/\bar{V}_\varepsilon(\varepsilon_1)\approx\varepsilon_1-\Delta\varepsilon$$

while for larger values of ϵ inelastic collisions begin to manifest themselves (the kinetic energy in the point of observation may then be appreciably less than ϵ_1). If the inelastic collision frequency is not very large (for the exact criterion see below Eq. (36)) so that the argument of f_0 also in the tail region is only ϵ we can integrate (4) over the cross-section $s(\epsilon)$ accessible to particles with energy ϵ and reduce the problem to a one-dimensional one:

$$\begin{gathered}\frac{\partial}{\partial\varepsilon}\left[\bar{V}_\varepsilon(\varepsilon)f_0(\varepsilon)+\bar{D}_\varepsilon(\varepsilon)\frac{\partial f_0}{\partial\varepsilon}\right]=\frac{(2\varepsilon_1)^{1/2}}{m^{3/2}}\nu^*_{\rm eff}(\varepsilon)f_0(\varepsilon);\\ \nu^*_{\rm eff}(\varepsilon)=\int\limits_{s^*(\varepsilon)}\nu^*(\varepsilon-e\varphi(\mathbf{r}))\left(\frac{\varepsilon-e\varphi(\mathbf{r})}{\varepsilon_1}\right)^{1/2}d\mathbf{r}.\end{gathered} \tag{29}$$

Here $s^*(\epsilon)$ is that part of the x,y-plane for which ν^* is non-zero.

The main interest is in taking into account inelastic collisions for those cases when due to them the distribution function near threshold ϵ_1 drops much faster than (2) and (20). The inelastic collisions occur then basically near the minimum of the potential $\varphi(r)$ where its behavior can be approximated by the parabola $e\varphi=\beta r^2$ for the cylindrical and $e\varphi=\beta x^2$ for the plane case where the constants $\beta\sim T/R^2$ are determined by the simultaneous solution of Eqs. (11), (12), and (25). We have thus in the cylindrical case

$$\nu^*_{\rm eff}(\varepsilon)=\begin{cases}\dfrac{\pi\nu^*}{\beta}(\varepsilon-\varepsilon_1), & \nu^*=\text{const}\\[2mm] \dfrac{\partial\nu^*}{\partial\theta}\dfrac{\pi}{2\beta}(\varepsilon-\varepsilon_1)^2, & \nu^*=\text{const}(\theta-\varepsilon_1)\end{cases} \tag{30}$$

while for a plane inhomogeneity

$$\nu^*_{\text{eff}}(\varepsilon)=\begin{cases}\nu^*\left(\dfrac{\varepsilon-\varepsilon_1}{\beta}\right)^{1/2}, & \nu^*=\text{const}\\[2mm] \dfrac{\partial\nu^*}{\partial\theta}\dfrac{2\beta}{3}\left(\dfrac{\varepsilon-\varepsilon_1}{\beta}\right)^{3/2}, & \nu^*=\text{const}(\theta-\varepsilon_1)\end{cases}. \tag{31}$$

As we are interested in a relatively fast decrease in the distribution function we can neglect above threshold the first term on the left-hand side of (28) and in the second term replace $\overline{D}_\epsilon(\epsilon)$ by $\overline{D}_\epsilon(\epsilon_1)$ so that the solution above threshold in the four cases enumerated in (30) and (31) has the form

$$f_0=A_n(\varepsilon-\varepsilon_1)^{1/2}K_{1/2q_n}\left[\frac{c_n^{1/2}}{q_n}(\varepsilon-\varepsilon_1)^{q_n}\frac{(2\varepsilon_1)^{1/4}}{m^{3/4}[\overline{D}_\varepsilon(\varepsilon_1)]^{1/2}}\right], \tag{32}$$

where

$$q_1=3/2,\ c_1=\pi\nu^*/\beta;\ \ q_2=2,\ \ c_2=(\pi/2\beta)(\partial\nu^*/\partial\theta);$$
$$q_3=5/4,\ c_3=\nu^*/\beta^{1/2};\ \ q_4=7/4,\ c_4=2/3(\beta)^{-1/2}\partial\nu^*/\partial\theta;$$

K_μ is a Macdonald function.

On the other hand, below threshold the solution has the form[6]

$$f_0=A_0\exp\left\{-\int_0^\varepsilon\frac{\overline{V}_\varepsilon(\varepsilon')}{\overline{D}_\varepsilon(\varepsilon')}d\varepsilon\right\}\left[1-B_n\int_T^\varepsilon\frac{d\varepsilon'}{\overline{D}_\varepsilon(\varepsilon')}\exp\left\{\int_0^{\varepsilon'}\frac{\overline{V}_\varepsilon(\varepsilon'')}{\overline{D}_\varepsilon(\varepsilon'')}d\varepsilon''\right\}\right]. \tag{33}$$

The constant A_0 follows from the normalization while A_n and B_n follow from joining (32) and (33) at $\epsilon = \epsilon_1$. As in the problem of diffusion at an absorbing wall B_n then determines the total number of inelastic collisions:

$$B_n^{-1}\approx\int_T^{\varepsilon_1}\frac{d\varepsilon'}{\overline{D}_\varepsilon(\varepsilon')}\exp\left\{\int_0^{\varepsilon'}\frac{\overline{V}_\varepsilon(\varepsilon'')}{\overline{D}_\varepsilon(\varepsilon'')}d\varepsilon''\right\}. \tag{34}$$

Integrating (4) over the cross-section and over the energy from $\epsilon \sim T$ to infinity and using the fact that when $\epsilon < \epsilon_1 - \Delta\epsilon$ the distribution function depends on ϵ only and has the form (33) we get

$$2\pi A_0B_n=\frac{4\pi\sqrt{2}}{m^{3/2}}\int_{\varepsilon_1}^{\infty}(\varepsilon-e\varphi)^{1/2}d\varepsilon\int_{s^*(\varepsilon)}\nu^*(\varepsilon-e\varphi)f_0(\varepsilon,r)\,ds. \tag{35}$$

For the applicability of the formulae obtained it is necessary, on the one hand, that the function (32) drops faster than (22), i.e., that the value $\Delta^*(\epsilon) = \epsilon - \epsilon_1$ for which the argument of the Macdonald function becomes of order unity is much smaller than the characteristic scale $\Delta\epsilon$ of the drop in (2) when $\epsilon < \epsilon_1$. On the other hand, in order that f_0 is independent of the coordinates in the energy range $\epsilon \sim \epsilon_1 + \Delta^*\epsilon$ which is of interest to us, we need the condition

$$\nu^*(\varepsilon_1+\Delta^*\varepsilon)\ll\beta D_\varepsilon(\varepsilon_1)/\Delta^*\varepsilon.$$

The condition for the applicability of the one-dimensional approximation (32) and (29) has thus for the four cases considered in (30) and (31) the following form:

$$\begin{gathered}\frac{T\varepsilon_1}{eE_zR^3m\nu(\varepsilon_1)}\gg\nu^*\gg\frac{m^2\varepsilon_1T\delta^3\nu^5(\varepsilon_1)}{e^4E_z^4}\\ \frac{T\varepsilon_1}{e^2E_z^2R^4m\nu(\varepsilon_1)}\gg\frac{\partial\nu^*}{\partial\theta}\gg\frac{m^3\varepsilon_1T\delta^4\nu^7(\varepsilon_1)}{e^6E_z^6}\\ \frac{\varepsilon_1}{mR^2\nu(\varepsilon_1)}\left(\frac{T}{eE_zR}\right)^{4/3}\gg\nu^*\gg\frac{m^{5/2}\varepsilon_1T^{1/2}\delta^{5/2}\nu^4(\varepsilon_1)}{e^3E_z^3},\\ \frac{\varepsilon_1}{mR^3eE_z\nu(\varepsilon_1)}\left(\frac{T}{eE_zR}\right)^{5/3}\gg\frac{\partial\nu^*}{\partial\theta}\gg\frac{m^{5/2}\varepsilon_1T^{1/2}\delta^{7/2}\nu^6(\varepsilon_1)}{e^5E_z^5};\end{gathered} \tag{36}$$

$s(\epsilon_1)$ has been put $\sim R^2$ for the cylindrical case and $\sim R$ for the plane one.

If the inelastic collision frequency is less than (36) the effect of the inelastic collisions is unimportant and the distribution function is also in the region $\epsilon \gtrsim \epsilon_1$ given by Eq. (2). The number of excitations per unit time and unit volume,

$$W(\mathbf{r})=\frac{4\pi\sqrt{2}}{m^{3/2}}\int_{\varepsilon_1+e\varphi(\mathbf{r})}^{\infty}(\varepsilon-e\varphi(\mathbf{r}))^{1/2}\nu^*(\varepsilon-e\varphi(\mathbf{r}))f_0(\varepsilon)\,d\varepsilon \tag{37}$$

is even in that case not proportional to the plasma concentration (25). If f_0 falls off faster than the Maxwellian function, the ratio $W(r)/n(r)$ is maximal in the region of the largest plasma concentration. The increase in ν^* leads to a further decrease in the size of the region where the larger part of the inelastic collisions takes place.

For values of ν^* exceeding the magnitude given by Eq. (36) the distribution function depends in an essential way on r for $\epsilon \gtrsim \epsilon_1 - \Delta\epsilon$. In the region where $\nu^*(\epsilon - e\varphi(r)) = 0$ the problem reduces to the equation for diffusion in energy with drift with the boundary condition $f_0(\epsilon, r_k) = 0$, where $e\varphi(r_k) = \epsilon - \epsilon_1$ corresponding to total absorption of the particles at the boundary. The solution below the excitation threshold $\epsilon < \epsilon_1$ has, as before, the form (33) to (35) with the characteristic energy scale $\sim\Delta\epsilon$ while the scale of the drop in f_0 for $\epsilon > \epsilon_1$ is determined by the energy $\Delta_1\epsilon$ obtained by an electron from the field E_z during a time Δt which the electron spends in the region $\epsilon > \epsilon_1$ before it hits the absorbing boundary.

For the plane geometry $\Delta t \sim R^2/D_\epsilon(\epsilon_1)$ while for the cylindrical geometry

$$\Delta t\sim\frac{R^2}{D_\varepsilon(\varepsilon_1)}\ln\frac{R}{R_k(\varepsilon_1+\Delta_1\varepsilon)},$$

where $R_k(\epsilon)$ is the size of the absorbing region. Therefore

$$\Delta_1\varepsilon=[D_\varepsilon^0(\varepsilon_1)\Delta t]^{1/2}\sim\begin{cases}eE_zR & \text{— for the plane geometry}\\ eE_zR\left[\ln\dfrac{R}{R_k(\varepsilon_1+\Delta_1\varepsilon)}\right]^{1/2} & \text{— for the cylindrical geometry}\end{cases} \tag{38}$$

The linear size of the region in which the main part of the inelastic collisions takes place, $R_k(\epsilon_1 + \Delta_1\epsilon)$, is then determined by $e\varphi(R_k) \sim \beta R_k^2 \sim \Delta_1\epsilon$ so that

$$R_k(\varepsilon_1+\Delta_1\varepsilon)\sim\begin{cases}R(eE_zR/T)^{1/2} & \text{— for the plane geometry}\\ R(eE_zR/T)^{1/2}\ln^{1/4}(T/eE_zR) & \text{— for the cylindrical geometry}\end{cases} \tag{39}$$

In both cases $R_k(\epsilon_1 + \Delta_1\epsilon) \ll R$, $\Delta_1\epsilon \ll \Delta\epsilon$, independent of the magnitude or the form of the function $\nu^*(\theta)$, i.e., the effect of the inelastic collisions on f_0 leads to the fact that the main part of excitations is concentrated in a small neighborhood of the maximum of the plasma concentration.

On the other hand, inside the absorbing region the diffusion in energy is unimportant. Indeed, during the time a particle spends in the absorbing region it is shifted in energy by an amount of order $\Delta\zeta\{D_\epsilon^0(\epsilon_1)/D_\epsilon(\epsilon_1)\}^{1/2}$ which is small compared with $\Delta_1\epsilon$ and with $e\Delta\zeta\partial\varphi/\partial\zeta \sim TR_k\Delta\zeta/R^2$ a characteristic scale for changes in the boundary condition.

The solution is thus independent of the geometry and has the following form for the case $\nu^* = \text{constant}$:

$$f_0=C(\varepsilon)\exp\left(-\frac{\zeta}{\Delta\zeta}\right)=C(\varepsilon)\exp\left[-\zeta\left(\frac{\nu^*}{D_\varepsilon(\varepsilon_1)}\right)^{1/2}\right], \tag{40}$$

while when $\nu^* = \text{constant} \times (\theta - \epsilon_1)$

$$f_0=C(\varepsilon)\zeta^{1/2}K_{1/3}\left[\left(\frac{\zeta}{\Delta\zeta}\right)^{3/2}\right]=C(\varepsilon)\zeta^{1/2}K_{1/3}\left[\frac{2\sqrt{2}}{3}\zeta^{3/2}\left(\frac{\partial\nu^*}{\partial\theta}\right)^{1/2}\left(\frac{\beta(\varepsilon-\varepsilon_1)}{D_\varepsilon^2(\varepsilon_1)}\right)^{1/4}\right], \tag{41}$$

where ζ is the coordinate reckoned along the normal to the boundary into the absorbing region.

We obtain the equations determining the profile of the potential and the field E_z for the simplest model of a positive column of a gas-discharge plasma. Assuming that recombination takes place on the walls and the ionization is concentrated in a small central region while each inelastic collision leads to ionization[11] we get from (11), (12), and (35) for a plane geometry

$$b_i\frac{\partial^2\Phi}{\partial x^2}=2\pi A_0B_n\delta(x), \tag{42}$$

where

$$\Phi=\int_0^{\varphi} n(\varphi)\,d\varphi$$

with the boundary conditions

$$\Phi(x=\pm R)=\int_0^{\infty} n(\varphi)\,d\varphi. \tag{43}$$

Its solution is

$$\Phi=\pi A_0B_n|x|/b_i, \tag{44}$$

while condition (43) which determines the longitudinal field E_z reduces to the following one:

$$\frac{3m^{1/2}eR\overline{V}_\varepsilon(\varepsilon_1)}{8\pi\sqrt{2}b_i}=\int_0^{\infty}\varepsilon^{1/2}\exp\left\{\int_\varepsilon^{\varepsilon_1}\frac{\overline{V}_\varepsilon(\varepsilon')}{\overline{D}_\varepsilon(\varepsilon')}d\varepsilon'\right\}d\varepsilon. \tag{45}$$

1)The quantities D, V_ϵ, and D_ϵ differ from D_ϵ, V_ϵ^0, and D_ϵ^0 (see Introduction) through an unimportant factor v which describes the change in phase volume when the energy changes.

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170