A general method for integration of many-point ring diagrams for Fermi systems

E. G. Brovman and A. Kholas[1)]

I. V. Kurchatov Atomic Energy Institute
Institute for Nuclear Research, Swerk, Poland
(Submitted December 25, 1973)
Zh. Eksp. Teor. Fiz. **66**, 1877–1894 (May 1974)

A method is proposed which makes it possible to integrate the expression for an arbitrary ring diagram for Fermi particles, the results being expressed directly in terms of the invariants of the "many-point" diagrams. The case of the "four-point" diagram, which can be important, in particular, in the analysis of the energy of the metallic phase of hydrogen, is analyzed in detail.

1. INTRODUCTION

In recent times a theory of metals which uses the effective smallness of the electron-ion interaction has achieved widespread circulation (cf., e.g.,[1]). In the expansion of the energy of the electron-ion system in this parameter (V_K/ϵ_F), alongside the pair interionic interactions there also arise effective many-particle forces[2], associated with the higher terms of the expansion ($n \geq 3$). In a number of cases these play an important role (cf.[3,4]). An extremely interesting substance for the study of many-particle forces is metallic hydrogen, in which the electron-ion interaction is known exactly, and in the absence of experiment it is especially important to know the absolute value of the cohesive energy of the lattice and the lattice structure. In particular, since the contribution of the third-order terms has been found to be large, and in fact determined the structural type for metallic hydrogen[5], we must also estimate the contributions from higher orders.

This problem presents considerable difficulties, primarily because it is necessary to know the expressions for the irreducible many-point functions $\Lambda^{(n)}(q_1, \ldots, q_n)$[2]. In practice, we are concerned with the calculation of Fermi ring diagrams with one loop and an arbitrary number of external-field "tails." In itself, this approximation corresponds to a self-consistent field, but it is essentially also the basic "structure" in the derivation of more complicated approximations[3,4]. Until recently, an analytic expression has been known only for the two-tail diagram—the simple polarization loop[6]; this was obtained in the standard way. In[7], another representation for the ring diagrams was proposed, which made it possible, in particular, to integrate the three-tail diagram.

On the basis of the same representation, a highly nontrivial method is proposed in this paper, making it possible in principle to find the expression for an arbitrary many-tail diagram. An important advantage of the method is that the answer is expressed immediately in the invariants of the problem. It is also possible to treat in general form the different "degenerate" cases in which the external momenta $q_1, \ldots, q_{n-1}$ are linearly dependent. This is rather important, since for $n \geq 5$, because of the three-dimensionality of space, it is always the degenerate case that is realized. In the last section, the four-tail diagram is analyzed in detail and it is shown that in the general case it is expressed in terms of a single integral, and in any degenerate case it can be calculated analytically to completion.

2. AN INTEGRAL REPRESENTATION

It was pointed out in[3,4], and in more detail in[8], that, in the determination of the energy of a system in the form of a series in the pseudo-potential, for the terms starting from $n = 4$ the standard many-particle theory becomes inapplicable (this result was obtained independently by Hammerberg and Ashcroft[9]). This is connected with the fact that in reality there occurs in the lattice a distortion of the shape of the Fermi surface and it acquires the symmetry of the lattice (even far from the Brillouin zones), while the quantum-mechanical perturbation theory actually takes into account only the corresponding change in the dispersion law, with the symmetry of the ground state remaining as before—spherical. It is therefore necessary to make use of the thermodynamic theory of[10], and then let $T \to 0$.

For closed ring diagrams with "static" external tails (Fig. 1), this procedure can be followed through in general form. In the temperature technique we have

$$I^{(n)}(\mathbf{q}_1, \ldots, \mathbf{q}_n) = 2T \sum_{\omega_l} \int \frac{d\mathbf{p}}{(2\pi)^3} G_0(\mathbf{p}, \omega_l) G_0(\mathbf{p}+\boldsymbol{\varkappa}_1, \omega_l) \ldots G_0(\mathbf{p}+\boldsymbol{\varkappa}_{n-1}, \omega_l), \quad (2.1)$$

$$\boldsymbol{\varkappa}_i = \mathbf{q}_1 + \ldots + \mathbf{q}_i, \quad (2.2)$$

$$G_0(\mathbf{p}, \omega_l) = (i\omega_l - \varepsilon_\mathbf{p} + \mu)^{-1}, \quad \omega_l = (2l+1)\pi T. \quad (2.3)$$

In the limit $T \to 0$ the summation over ω_l, as is easily seen, can be replaced by integration over ω (cf., e.g.,[10]):

$$T \sum_{\omega_l} \to \int_{-\infty}^{\infty} d\omega / 2\pi i.$$

If in the integral obtained we now make the replacement $i\omega + \mu \to \omega$, the integration will be parallel to the imaginary axis from $-i\infty$ to $+i\infty$. Using the fact that the poles of the integrand (cf. (2.3)) lie on the real axis, we turn the integration contour round, placing it along the real axis; for $\omega > \mu$ the contour lies above the real axis, and for $\omega < \mu$ it lies below. It is clear that this

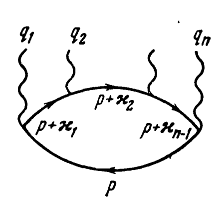


FIG. 1

is equivalent to integrating along the real axis with Green functions of the form

$$G_0(\mathbf{p}, \omega)=[\omega-\varepsilon_p+i\delta\,\mathrm{sign}\,(\omega-\mu)]^{-1}. \tag{2.4}$$

Thus, taking the limit $T \to 0$ in the ring diagrams in the temperature technique in fact turns out to be equivalent to using Green functions of the form (2.4), rather than the usual ones[10] having an imaginary correction in momentum space, in the $T = 0$ technique. Taken separately, the Green functions in the two forms coincide when $\delta \to 0$, but their products, e.g., products of four functions G_0 (see below), differ. (We recall that δ should tend to zero in the final result.)

It was with the use of G-functions of precisely the form (2.4) that, after the introduction of Feynman parametrization and integration over ω and $\mathbf{p}$, the following integral representation for a many-tail diagram was obtained in our earlier paper[7]:

$$J^{(n)}(\mathbf{q}_1,\ldots,\mathbf{q}_n)=\frac{m_e}{\pi^2}(-1)^{n-1}\frac{\partial^{(n-2)}}{\partial\mu^{(n-2)}}\int_0^1 d\alpha_1\int_0^{1-\alpha_1} d\alpha_2\ldots\int_0^{\Sigma} d\alpha_{n-1}\sqrt{\xi_{n-1}}\,\theta(\xi_{n-1}), \tag{2.5}$$

where

$$\xi_m=2m_e\mu+\sum_{i,j=1}^{m}\varkappa_i\varkappa_j\alpha_i\alpha_j-\sum_{i=1}^{m}\alpha_i\varkappa_i^2,$$
$$\Sigma\equiv 1-\sum_{i=1}^{n-2}\alpha_i. \tag{2.6}$$

Several useful properties of the many-tail diagram can be found by means of the representation (2.5). Since the integrand is non-negative, it is easily seen that for any $\mathbf{q}$ we have $J^{(2)} < 0$, and $J^{(3)} > 0$. For $n \geq 4$ it is necessary to leave at least one of the derivatives with respect to μ in explicit form and therefore the many-tail diagrams do not have a well-defined sign. Furthermore, it is easy to find the limiting value for all $q_i = 0$. Finding the volume of the corresponding region in α-space, we obtain

$$J^{(n)}(0,\ldots,0)=\frac{\sqrt{2}\,m_e^{3/2}}{\pi^2}(-1)^{n-1}\frac{1}{(n-1)!}\frac{\partial^{(n-2)}}{\partial\mu^{(n-2)}}\sqrt{\mu} \tag{2.7}$$

(this expression fixes the sign for large k_F).

We now remark that many-point functions symmetrized in the external tails $q_1,\ldots,q_n$ always appear in the energy. For the ring diagrams we have

$$\Lambda_0^{(n)}(\mathbf{q}_1,\ldots,\mathbf{q}_n)=\frac{1}{n}\mathscr{P}J^{(n)}(\mathbf{q}_1,\ldots,\mathbf{q}_n). \tag{2.8}$$

Here the symmetrization operation clearly consists in permutation of the external momenta:

$$\mathscr{P}J^{(n)}(\mathbf{q}_1,\ldots,\mathbf{q}_n)=\frac{1}{n!}\sum_{\mathscr{P}}\mathscr{P}J^{(n)}(\mathbf{q}_1,\ldots,\mathbf{q}_n). \tag{2.9}$$

Clearly, for $n = 2$ and $n = 3$, the integral $J^{(n)}$ is itself automatically symmetric, but symmetrization becomes necessary for the four-tail diagram (cf. Sec. 5). For the symmetrized combinations (2.8), the following identity is valid:

$$\Lambda_0^{(n+1)}(\mathbf{q}_1,\ldots,\mathbf{q}_n,0)=-\frac{1}{n+1}\frac{d\Lambda_0^{(n)}(\mathbf{q}_1,\ldots,\mathbf{q}_n)}{d\mu}. \tag{2.10}$$

This can be proved directly, using (2.5). For this it is necessary to place the momentum $\mathbf{q} = 0$ successively in all positions in the left-hand side of (2.10) and carry out the corresponding replacement of the variables α.

The identities (2.10) are the analog of the "hierarchical" relations obtained in[11] for the complete many-point functions $\Gamma^{(n)}$, but for $n > 2$, generally speaking, do not follow directly from these relations (although they can be proved by the same diagrammatic method).

3. INTEGRATION. THE NONDEGENERATE CASE

The representation (2.5) turns out to be convenient for direct integration, since in it, unlike the standard form, the electron occupation numbers n_p are absent. We introduce a standard integral of the following form:

$$I_{t,m}=\int\limits_S\ldots\int d\alpha_1\ldots d\alpha_m F_t(\xi_m(\alpha_i)), \tag{3.1}$$

where $F_t(\xi) = \xi^t\theta(\xi)$, and ξ_m is defined in (2.6). The region S is specified by the inequalities

$$\alpha_i\geq 0,\quad \alpha_1+\alpha_2+\ldots+\alpha_m\leq 1. \tag{3.2}$$

In this notation, the n-point function (2.5) can be written in one of the following forms:

$$\begin{aligned}J^{(n)}&=\frac{1}{3\pi^2}(-1)^{n-1}\left(\frac{\partial}{\partial\mu}\right)^{n-1}I_{3/2,\,n-1}\\&=\frac{m_e}{\pi^2}(-1)^{n-1}\left(\frac{\partial}{\partial\mu}\right)^{n-2}I_{1/2,\,n-1}\\&=\frac{m_e^2}{\pi^2}(-1)^{n-1}\left(\frac{\partial}{\partial\mu}\right)^{n-3}I_{-1/2,\,n-1}.\end{aligned} \tag{3.3}$$

Since t is half-integer and ξ^t becomes a purely imaginary quantity in the region of negative ξ, the function $F_t(\xi)$ can be written in the form

$$F_t(\xi)=\mathrm{Re}\,\xi^t,$$

and the operation Re can be taken outside the integral in (3.1). Therefore, taking large k_F ($k_F^2 = 2m_e\mu$), we shall omit the θ-function in (3.1) and then operate with Re on the analytic continuation of the answer into the region of small k_F.

The difficulty in calculating (3.1) consists in the fact that, after a few integrations over the α_i, a large number of different regions arise and the cumbersome integrands lose their symmetry, which is re-established only in the final answer. The idea of the following treatment consists precisely in choosing a symmetric method of integration, in which the answer at each stage is expressed in terms of the invariants of the many-tail diagrams. For this it is possible to use the fact that the parameters α_i in the integrand always appear in a product with a corresponding vector κ_i. It is therefore convenient to introduce a basis of the vectors κ_i and regard the parameters α_i as the coordinates of a certain vector γ in this basis:

$$\gamma=\sum_{i=1}^{m}\alpha_i\varkappa_i. \tag{3.4}$$

In this section, we shall consider the nondegenerate case when the vectors $\kappa_1, \kappa_2,\ldots,\kappa_m$ are linearly independent. Then it is not difficult to see that since the parameters α_i range over the region S (3.2), the vector γ ranges over a region whose edges are formed by the vectors κ_i—this region is called an m-dimensional symplex and will be symbolized by its vertices $S_m(0, \kappa_1,\ldots,\kappa_m)$. We note that the integration is in fact performed over the "interior," corresponding to the dual scheme in the sense of Landau[12]. Figure 2 shows an example for the case $m = 3$ (corresponding to the four-tail diagram), in which the three-dimensional symplex is a tetrahedron.

Next, ξ_m (2.6) can be represented in a convenient form if we move the coordinate original to the center

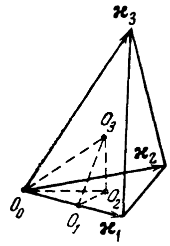
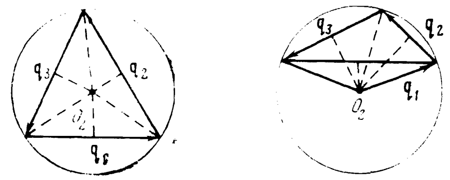


FIG. 2



a b

FIG. 3

O_m of the sphere circumscribing the m-dimensional symplex. Obviously, the coordinates of the center are determined from the system of equations

$$\mathbf{R}_m^2=(\mathbf{R}_m-\varkappa_i)^2,\quad i=1,2,\ldots,m, \tag{3.5}$$

which is soluble for the nondegenerate case. Introducing the vector $\mathbf{X} = \boldsymbol{\gamma} - \mathbf{R}_m$, after simple transformations we obtain

$$\xi_m=k_F^2-\mathbf{R}_m^2+\mathbf{X}^2. \tag{3.6}$$

Thus, the initial integral (3.1) is rewritten in the symmetric form

$$I_{t,m}=\frac{1}{V_m}\int_{S_m} d^mX\,F_t(\xi_m(X^2)) \tag{3.7}$$

$$(S_m\equiv S_m(-\mathbf{R}_m,\ \varkappa_1-\mathbf{R}_m,\ \ldots,\ \varkappa_m-\mathbf{R}_m)).$$

Here we have taken into account the Jacobian of the change of variables and have introduced, accordingly, the Gram determinant:

$$V_m=G^{1/2}(\varkappa_1,\ldots,\varkappa_m),\quad G(\varkappa_1,\ldots,\varkappa_m)=\det(\varkappa_i,\varkappa_j) \tag{3.8}$$

(V_m is the volume of the m-dimensional parallelipiped constructed on the vectors κ_i).

We now divide the volume of integration S_m into a number of smaller "elementary" symplexes, which are more convenient for the integration. For this we carry out the following construction, specifying the vertices and certain edges of the new region. We join the center O_m of the m-dimensional sphere of radius R_m corresponding to the m-dimensional symplex to the center O_{m-1} of the sphere of the (m − 1)-th surface, which is the symplex S_{m-1} with circumscribing sphere of radius R_{m-1}. Extending the analogous construction further, we finally obtain the center O_1 of the one-dimensional sphere circumscribing a surface (i.e., an edge) with radius $R_1 = q_i/2$, and the vertex O_0 of the symplex S_m, which corresponds to $R_0 = 0$. The points $O_m,\ldots,O_1, O_0$ are the vertices of an elementary symplex. For m = 3 the corresponding construction is carried out in Fig. 2, in which one of the resulting elementary symplexes is shown.

From the construction, it is not difficult to understand that the m vectors $\mathbf{h}_i = O_iO_{i-1}$ are mutually orthogonal and therefore form a convenient Cartesian system of coordinates. Using the fact that the function F_t in (3.7) depends only the square $\mathbf{X}^2$ of a vector in the m-dimensional space, for each elementary symplex we shall choose its own orthonormal basis:

$$\mathbf{X}=\sum_{i=1}^{m}X_i\mathbf{e}_i,\quad \mathbf{e}_i=\mathbf{h}_i/h_i. \tag{3.9}$$

With this it is easy to see that

$$\mathbf{R}_i^2=\mathbf{R}_{i-1}^2+\mathbf{h}_i^2=\ldots=\mathbf{h}_1^2+\mathbf{h}_2^2+\ldots+\mathbf{h}_i^2. \tag{3.10}$$

Then the integral over an elementary symplex can be written in the form

$$\tilde{I}_{t,m}=\int_{S_m} d^mXF_t(\xi_m(X^2))=\int_0^{h_m}dX_m\int_0^{B_{m-1}X_m}dX_{m-1}\ldots\int_0^{B_1X_2}dX_1F_t(\xi_m), \tag{3.11}$$

where $S_m \equiv S_m(O_m,\ldots,O_1, O_0)$ and

$$\xi_m=k_F^2-\mathbf{R}_m^2+\mathbf{X}_1^2+\mathbf{X}_2^2+\ldots+\mathbf{X}_m^2, \tag{3.12}$$

$$B_i=h_i/h_{i+1}[(R_i^2-R_{i-1}^2)/(R_{i+1}^2-R_i^2)]^{1/2}. \tag{3.13}$$

Next, the elementary integral (3.11) can be conveniently transformed if we again take into account that ξ_m depends only on the square of the integration vector, and introduce spherical coordinates. For this we introduce r_m, the radius-vector in the m-dimensional space (the distance from O_m), and Ω_m, the corresponding solid angle:

$$d^mX=(r_m)^{m-1}dr_m d\Omega_m.$$

From the definition of $d\Omega_m$ as an element of area on the surface of the unit sphere, we have

$$d\Omega_m=\frac{h_m}{\tilde{r}_m}\frac{d^{m-1}X}{\tilde{r}_m^{m-1}}.$$

Here $\tilde{r}_m$ is the radius-vector of the point on the surface S_{m-1} at which the element $d^{m-1}X$ is situated. Then,

$$\tilde{I}_{t,m}=\int d\Omega_m\int_0^{\tilde{r}_m(\Omega)}(r_m)^{m-1}dr_mF_t(\xi_m(r_m^2))$$

$$=h_m\int_{S_{m-1}}d^{m-1}X\int_0^1\rho_m^{m-1}d\rho_mF_t(\xi_m(\rho_m^2\tilde{r}_m^2)).$$

Here we have introduced the dimensionless quantity $\rho_m = r_m/\tilde{r}_m$ and, using the geometrical properties of the symplex, have replaced the integration over the solid angle by integration over the (m − 1)-dimensional symplex S_{m-1}. Taking into account that $\tilde{r}_m^2 = h_m^2 + r_{m-1}^2$, we now introduce spherical coordinates with the center at O_{m-1}, and so on. After m analogous steps, we finally obtain

$$\tilde{I}_{t,m}=\tilde{V}_m\int_0^1 d\rho_1\int_0^1\rho_2d\rho_2\ldots\int_0^1\rho_m^{m-1}d\rho_mF_t(\xi_m), \tag{3.14}$$

$$\xi_m=(k_F^2-R_m^2)+\rho_m^2[(R_m^2-R_{m-1}^2)+\rho_{m-1}^2[\ldots+\rho_2^2[(R_2^2-R_1^2)+\rho_1^2R_1^2]\ldots]],$$

where $\tilde{V}_m = h_1h_2\ldots h_m$ is the volume of the right parallelipiped corresponding to the elementary symplex. This expression is also easily represented in another form, in which the integrand does not depend on the external parameters:

$$\tilde{I}_{t,m}=(k_F^2-R_m^2)^{t+m/2}\int_0^{B_1}d\lambda_1\int_0^{B_2}\lambda_2d\lambda_2\ldots\int_0^{B_m}\lambda_m^{m-1}d\lambda_mF_t(\xi_m), \tag{3.15}$$

$$\xi_m=1+\lambda_m^2(1+\lambda_{m-1}^2(\ldots+\lambda_2^2(1+\lambda_1^2)\ldots)).$$

Here, supplementing (3.13), we have introduced the further definition

$$B_m=[(R_m^2-R_{m-1}^2)/(k_F^2-R_m^2)]^{1/2}. \tag{3.16}$$

The expression (3.15) is the final answer for the integral over an elementary symplex—an answer whose merit is that it is expressed in terms of the invariants of the problem, i.e., the radii of the circumscribing i-dimensional spheres, where i = 1, 2, ..., m.

The complete integral $I_{t,m}$ (3.7) is composed of the contributions of the separate elementary symplexes:

$$I_{t,m}=\frac{1}{V_m}\Sigma\,\mathrm{sign}(\mathbf{R}_1,\ldots,\mathbf{R}_m)\,\mathcal{I}_{t,m}(\mathbf{R}_1,\ldots,\mathbf{R}_m,k_F). \qquad (3.17)$$

Here it must be kept in mind that in some cases, depending on the arrangement of the centers of the i-dimensional spheres with respect to the surfaces, the elementary integrals should occur with a minus sign (cf., e.g., Fig. 3), and this is taken into account by the sign coefficient.

To conclude this Section, we shall illustrate the method using simple examples.

We shall consider the "two-tail" diagram (n = 2). From (3.3) we have

$$J^{(2)}(\mathbf{q},\,-\mathbf{q})=-(m_e/\pi^2)I_{1/2,\,1}.$$

There are two equivalent one-dimensional symplexes with parameters $R_1 = h_1 = q/2$, $V_1 = q$. From (3.17) and (3.15) we find

$$I_{1/2,1}=2\frac{1}{V_1}\mathcal{I}_{1/2,1}=\frac{k_F-R_1^2}{R_1}\int\limits_0^{B_1}d\lambda_1(1+\lambda_1^2)^{1/2}=k_F\left(\frac{1}{2}+\frac{k_F^2-R_1^2}{4k_FR_1}\ln\frac{k_F+R_1}{k_F-R_1}\right)$$

Carrying out the analytic continuation, we obtain the well-known result $(x = q/2k_F)$

$$J^{(2)}(\mathbf{q},-\mathbf{q})=-\frac{m_ek_F}{\pi^2}\left(\frac{1}{2}+\frac{1-x^2}{4x}\ln\left|\frac{1+x}{1-x}\right|\right). \qquad (3.18)$$

We next consider the case of the three-tail diagram, which, with this method, turns out to be not much more complicated.

The two-dimensional symplex, a triangle (cf. Fig. 3), is broken down into six elementary symplexes—pairwise-equal right-angled triangles. The answer will be expressed in terms of the following invariants:

$$R_2=q_R,\quad R_1^i=\frac{q_i}{2},\quad V_2=\frac{q_1q_2q_3}{2q_R}$$

(R_2 is the radius of the circumscribing circle). From (3.13) and (3.16) we have

$$B_1=\frac{R_1}{(R_2^2-R_1^2)^{1/2}},\quad B_2=\left(\frac{R_2^2-R_1^2}{k_F^2-R_2^2}\right)^{1/2}.$$

The integral (3.15) over an elementary symplex is easily taken:

$$\mathcal{I}_{-1/2,2}=(k_F^2-R_2^2)^{-1/2+1}\int\limits_0^{B_1}d\lambda_1\int\limits_0^{B_2}\lambda_2\,d\lambda_2[1+\lambda_2^2(1+\lambda_1^2)]^{-1/2}$$

$$=(k_F^2-R_2^2)^{1/2}\left(B_2\ln\frac{B_2B_1+[1+B_2^2(1+B_1^2)]^{1/2}}{(1+B_2^2)^{1/2}}+\right. \qquad (3.19)$$

$$\left.+\,\mathrm{arctg}\frac{B_1}{[1+B_2^2(1+B_1^2)]^{1/2}}-\mathrm{arctg}\,B_1\right)=\frac{(R_2^2-R_1^2)^{1/2}}{2}\ln\frac{k_F+R_1}{k_F-R_1}$$

$$-(k_F^2-R_2^2)^{1/2}\left(\mathrm{arctg}\frac{R_1(k_F^2-R_2^2)^{1/2}}{k_F(R_2^2-R_1^2)^{1/2}}-\mathrm{arctg}\frac{R_1}{(R_2^2-R_1^2)^{1/2}}\right).$$

It is clear that

$$h_2^i/R_2=(R_2^2-R_1^{i2})^{1/2}/R_2=|\cos\theta_i|,$$

and the sign with which each elementary integral should be included in (3.17) is equal to sign $(\cos\theta_i)$; here,

$$\cos\theta_i=-(\mathbf{q}_k\mathbf{q}_l)/q_kq_l, \qquad (3.20)$$

where (i, k, l) is a cyclic permutation of (1, 2, 3).

We shall sum the six corresponding contributions to (3.17). It is not difficult to see that the sum of the last terms from (3.19) vanishes exactly, and the sum of the other three arctangents is easily transformed. Finally, recalling (3.3) and performing the analytic continuation, we obtain

$$J^{(3)}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)=\frac{m_e^2}{\pi^2}I_{-1/2,2}=\frac{2m_e^2}{\pi^2}\frac{q_R^2}{q_1q_2q_3}\left[\sum_i\cos\theta_i\ln\left|\frac{2k_F+q_i}{2k_F-q_i}\right|\right.$$

$$\left.-\Delta\begin{pmatrix}2\,\mathrm{Arctg}(A\Delta), & k_F/q_R>1\\ \ln|(1-A\Delta)/(1+A\Delta)|, & k_F/q_R<1\end{pmatrix}\right] \qquad (3.21)$$

Here,

$$A=\frac{q_1q_2q_3}{(2k_F)^3}\left[1-\frac{1}{2}\frac{q_1^2+q_2^2+q_3^2}{(2k_F)^2}\right]^{-1},\quad \Delta=\left(\frac{k_F^2-q_R^2}{q_R^2}\right)^{1/2}. \qquad (3.22)$$

We obtained the expression (3.21) earlier[7] by considerably more cumbersome calculations.

4. INTEGRATION. THE CASE OF DEGENERACY

Before treating the degenerate cases directly, we shall make some useful coordinate transformations, which, incidentally, also enable us to give formal expressions for the sign function in (3.17).

We introduce the index L to denote a definite surface of the initial m-dimensional symplex. Obviously, it is the (m − 1)-dimensional symplex drawn on a set of all but one of the vectors $\{\kappa_i\}$—we shall assume, by definition, that the L-th surface does not contain the vector κ_{L-2}. The vertices of this symplex S^L_{m-1} and the center O_m form the m-dimensional symplex S^L_m. For each such symplex we introduce its own labeling of the vectors κ^L_i (the labeling of the q_i is fixed by the form of the diagram). Namely, we define the labeling in such a way that the only vertex of the initial symplex that does not lie on the given surface S^L_{m-1} always has the label of the last vertex m: κ^L_m. It is easy to see that this is achieved by choosing, in analogy with (2.2), the following definition:

$$\varkappa_i^L\equiv\mathbf{q}_L+\mathbf{q}_{L+1}+\ldots+\mathbf{q}_{L+i-1}=\varkappa_{L+i-1}-\varkappa_{L-1} \qquad (4.1)$$

(here, to make the expression symmetric, the cyclic condition $q_{i+m+1} \equiv q_i$ is understood).

In this local system the center of the m-dimensional sphere is described by the vector $\mathbf{R}^L_m = \mathbf{R}_m - \kappa_{L-1}$, and the (m − 1)-dimensional symplexes have the standard form

$$S^L_{m-1}(-\mathbf{R}_m^L,\varkappa_1^L-\mathbf{R}_m^L,\ldots,\varkappa^L_{m-1}-\mathbf{R}_m^L).$$

In Fig. 4, for illustration, three systems are drawn for the two-dimensional symplex corresponding to the three-tail diagram, m = 2: (a) L = 1, (b) L = 2, (c) L = 3. It is clear that the sign with which the contribution from a given symplex S^L_m appears is determined by whether the center O_m and the odd site κ^L_m lie on the same side of the hyperplane S^L_{m-1} or not. There-

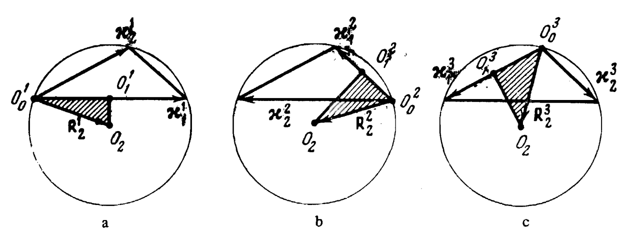


FIG. 4

fore, denoting the normal to S^L_{m-1} by N^L_{m-1}, we obtain

$$I_{t,m}=\frac{1}{V_m}\sum_{L=1}^{m+1}\text{sign}\frac{\mathbf{N}^L_{m-1}\mathbf{R}_m{}^L}{\mathbf{N}^L_{m-1}\varkappa_m{}^L}\int_{S_m{}^L}d^mXF_t(X^2). \quad (4.2)$$

It is clear that the analogous procedure can now be carried out with each of the symplexes S_{m-1}, etc., so that the overall sign in (3.17) will be expressed by the product of the sign factors arising in each step.

We now consider the degenerate situation when the m vectors κ_i are linearly dependent. (We shall assume here that any m − 1 vectors are linearly independent.) We introduce in place of q_m the vector $q'_m = q_m + \epsilon$, where ϵ does not lie in the volume constructed on the $\{\kappa_i\}$, so that the nondegenerate case considered in the preceding Section arises, and then let $\epsilon \to 0$. It is interesting that this limit can be taken in general form.

We shall single out an arbitrary (m − 1)-dimensional surface S^L_{m-1} and obtain an equation for the determination of the center O_m of the m-dimensional sphere pertaining to the given system. It is not difficult to see that, to order ϵ^2,

$$-2\varepsilon\mathbf{h}_m{}^L=(\varkappa_m{}^L-\mathbf{R}^L_{m-1})^2-(\mathbf{R}^L_{m-1})^2. \quad (4.3)$$

(Strictly, this expression is valid for L = 1, and the other L have their own ϵ', but $\epsilon' \to 0$ too.) We recall that here h^L_m is the vector joining the centers O_m and O^L_{m-1}.

It follows from (4.3) that if

$$(\varkappa_m{}^L-\mathbf{R}^L_{m-1})^2\neq(\mathbf{R}^L_{m-1})^2, \quad (4.4)$$

then, for $\epsilon \to 0$, we shall have $h^L_m \to \infty$ and also alone, consequently, $R^L_m \to \infty$ $((R^L_m)^2 = (h^L_m)^2 + (R^L_{m-1})^2)$. The case of equality in (4.4) corresponds to the still more degenerate situation in which the centers of the radii of all (m−1)-dimensional spheres coincide. A corresponding example will be given in the next Section, but here we shall consider the general case in which (4.4) is fulfilled.

For this we shall make use of the representation (3.14), in which it is necessary to let $R_m \to \infty$. In this case, of course, it is impossible to satisfy the condition $k_F \to R_m$, and therefore in the inner integral it is necessary to use for the lower limit the quantity $\bar{\rho}_m$ defined by the equality $\xi_m(\bar{\rho}_m) = 0$ (the action of the θ-function). It is not difficult to convince oneself that

$$\bar{\rho}_m{}^2=1-\frac{\xi_{m-1}}{R_m{}^2}\left[1+O\left(\frac{1}{R_m{}^2}\right)\right]$$

With the same accuracy, $\xi_m \approx R^2_m(\rho^2_m - \bar{\rho}^2_m)$, and for the inner integral we have

$$\int_{\bar{\rho}_m}^1 \rho_m{}^{m-1}(\xi_m)^t d\rho_m=\xi^{t+1}_{m-1}/2(t+1)R_m{}^2.$$

Furthermore, $\tilde{V}_m = \tilde{V}_{m-1}h_m \approx \tilde{V}_{m-1}R_m$, and therefore

$$I^{deg}_{t,m}=\lim\frac{1}{R_m}\frac{1}{2(t+1)}I_{t+1,m-1}, \quad R_m\to\infty. \quad (4.5)$$

We now collect, in the expression (3.17), the contributions of all the elementary symplexes bordering upon the L-th surface, since for these all the transformations carried out above are equivalent, i.e., we shall make use of (4.2). Then, using (4.5), we obtain

$$I^{deg}_{t,m}=\sum_{L=1}^{m+1}P^L_{t,m}I^L_{t+1,m-1}. \quad (4.6)$$

The sign coefficient in (4.2) is simply written down, if we take into account that $N^L_{m-1} = -h^L_m$, and for the case $R^L_m \to \infty$ we shall have sign $(-h^L_m \cdot R^L_m) > 0$. Therefore, for $\epsilon \to 0$,

$$P^L_{t,m}=\lim\frac{\text{sign}(-\mathbf{h}_m{}^L\boldsymbol{\varepsilon})}{2(t+1)}\frac{V^L_{m-1}}{V_mR_m{}^L}=\lim\frac{\text{sign}(-\mathbf{h}_m{}^L\boldsymbol{\varepsilon})}{2(t+1)}\frac{h_m{}^L}{|\boldsymbol{\varepsilon}\mathbf{h}_m{}^L|R_m{}^L}.$$

Hence, making use of (4.3), we find

$$P^L_{t,m}=\frac{1}{(t+1)}\frac{1}{(\varkappa_m{}^L-\mathbf{R}^L_{m-1})^2-(\mathbf{R}^L_{m-1})^2}$$

Now, taking (4.6) and the definitions (3.3) into account, we obtain the final expression for the degenerate case:

$$J^{(n)}_{deg}(\mathbf{q}_1,\ldots,\mathbf{q}_n)=-m_e\sum_{L=1}^{n}P^L_{n-1}J^{(n-1)}(\mathbf{q}_L,\mathbf{q}_{L+1},\ldots,\mathbf{q}_{L+n-3},-(\mathbf{q}_L+\mathbf{q}_{L+1}+\ldots+\mathbf{q}_{L+n-3})). \quad (4.7)$$

Here,

$$P_m{}^L=\frac{2}{(\varkappa_m{}^L-\mathbf{R}^L_{m-1})^2-(\mathbf{R}^L_{m-1})^2}=\frac{2}{\varkappa_m{}^L(\varkappa_m{}^L-2\mathbf{R}_m{}^L)}. \quad (4.8)$$

Thus, in the degenerate case, a many-tail diagram is expressed by a linear combination of many-tail diagrams with one less tail.

We emphasize that, by virtue of the three-dimensionality of the space of the $\{\kappa\}$, many-tail diagrams with $n \geq 5$ will always correspond to the degenerate case ($\kappa_n = 0$, and therefore the case n = 4 remains nondegenerate) and, by successively applying the procedure described above for taking the limit, it is possible to express them in terms of the values of the two-tail, three-tail and four-tail diagrams and their derivatives. It can be assumed, therefore, that the method proposed makes it possible, in principle, to obtain the expression for an arbitrary n-tail diagram.

For illustration, we now consider the degenerate case of the three-tail diagram corresponding to $q_1 \parallel q_2 \parallel q_3$ ($\kappa^1_1 = q_1$, $\kappa^1_2 = q_1 + q_2$). For the coefficients (4.8) we find

$$P_2{}^1=\frac{2}{\varkappa_2{}^1(\varkappa_2{}^1-2\mathbf{R}_1{}^1)}=\frac{2}{(\mathbf{q}_1+\mathbf{q}_2)(\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}_1)}=-\frac{2}{\mathbf{q}_2\mathbf{q}_3}$$

The others are obtained by cyclic permutation, and by means of the general expression (4.7) we find

$$J^{(3)}_{deg}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)=2m_e\left[\frac{J^{(2)}(\mathbf{q}_1,-\mathbf{q}_1)}{\mathbf{q}_2\mathbf{q}_3}+\frac{J^{(2)}(\mathbf{q}_2,-\mathbf{q}_2)}{\mathbf{q}_3\mathbf{q}_1}+\frac{J^{(2)}(\mathbf{q}_3,-\mathbf{q}_3)}{\mathbf{q}_1\mathbf{q}_2}\right] \quad (4.9)$$

(the scalar products are necessary only for choosing the sign). Here $J^{(2)}$ is defined in (3.18). In the particular case $q_2 = -q_1$, $q_3 = 0$, we have an example in which the inequality (4.4) is not fulfilled (the radii of the two-dimensional and one-dimensional spheres coincide). The answer is easily obtained by performing the limiting procedure in (4.9); it is expressed in terms of derivatives of $J^{(2)}$:

$$J^{(3)}_{deg}(\mathbf{q},-\mathbf{q},0)=\frac{1}{2}\frac{m_e{}^2}{\pi^2q}\ln\left|\frac{2k_F+q}{2k_F-q}\right|. \quad (4.10)$$

5. ANALYSIS OF THE FOUR-TAIL DIAGRAM

We turn now to a detailed analysis of the ring diagram with four external momenta. We first consider the general case in which the vectors q_1, q_2, and q_3 are linearly independent.

According to (3.3),

$$J^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = (m_e^3/\pi^2) I_{-5/2,\,3}. \tag{5.1}$$

In the integral (3.15) over an elementary symplex the first two integrations are done without difficulty:

$$\begin{aligned} I_{-5/2,\,3} &= \int_0^{B_1} d\lambda_1 \int_0^{B_2} \lambda_2\, d\lambda_2 \int_0^{B_3} \lambda_3^2\, d\lambda_3 \{1+\lambda_3^2[1+\lambda_2^2(1+\lambda_1^2)]\}^{-5/2} \\ &= \int_0^{B_3} \frac{d\lambda_3}{(1+\lambda_3^2)^{1/2}} \left[\operatorname{arctg} \frac{\{1+\lambda_3^2[1+B_2^2(1+B_1^2)]\}^{1/2}}{B_1(1+\lambda_3^2)^{1/2}} - \operatorname{arctg}\frac{1}{B_1} \right]. \end{aligned} \tag{5.2}$$

(The invariants B_1, B_2, and B_3 are defined in (3.13), (3.16).) The last integral, apparently, cannot be taken in analytic form. We shall represent it in another form, by integrating by parts and carrying out simple transformations. In addition, we shall analytically continue the answer obtained for the case $k_F > R_3$ into the whole region of k_F. As a result, we find

$$I_{-5/2,\,3} = I_{\rm I} + I_{\rm II}, \tag{5.3}$$

where for $R_2 < k_F < \infty$

$$I_{\rm I} = \operatorname{arctg} \frac{R_1(R_2^2-R_1^2)^{1/2}}{k_F^2-R_1^2+k_F(k_F^2-R_2^2)^{1/2}} \ln \frac{(R_3^2-R_2^2)^{1/2}+(k_F^2-R_2^2)^{1/2}}{|k_F^2-R_3^2|^{1/2}}$$

and for $0 < k_F < R_2$

$$I_{\rm I} = \operatorname{arctg}\left(\frac{R_3^2-R_2^2}{R_2^2-k_F^2}\right)^{1/2} \times \ln \frac{\{(k_F^2-R_1^2)^2+[k_F(R_2^2-k_F^2)^{1/2}-R_1(R_2^2-R_1^2)^{1/2}]^2\}^{1/2}}{R_2|k_F^2-R_1^2|^{1/2}},$$

while for $I_{\rm II}$ we have for $\infty > k_F > R_3$

$$I_{\rm II} = \mathscr{R} \int_0^{B_3} \frac{r\,dr\,\operatorname{Arsh} r}{(r^2+1)^{1/2}(r^2+D_0^2)^{1/2}(r^2+D_1^2)},$$

for $R_3 > k_F > R_2$

$$I_{\rm II} = \mathscr{R} \int_0^{\infty} \frac{r\,dr\,\operatorname{Arsh} r}{(r^2+1)^{1/2}(r^2+D_0^2)^{1/2}(r^2+D_1^2)} + \int_{|B_3|}^{\infty} \frac{r\,dr\,\operatorname{Arch} r}{(r^2-1)^{1/2}(r^2-D_0^2)^{1/2}(r^2-D_1^2)}$$

and for $R_2 > k_F > 0$

$$I_{\rm II} = \mathscr{R}\,\mathrm{P} \int_{D_0}^{|B_3|} \frac{r\,dr\,\arcsin r}{(1-r^2)^{1/2}(r^2-D_0^2)^{1/2}(r^2-D_1^2)};$$

here we have used the notation

$$\mathscr{R} = -\frac{R_1}{R_3}\frac{(R_3^2-R_2^2)^{1/2}(R_2^2-R_1^2)^{1/2}}{R_3^2-R_1^2}$$

and also introduced the constants

$$D_0 = \left(\frac{R_3^2-R_2^2}{R_3^2}\right)^{1/2}, \quad D_1 = \left(\frac{R_3^2-R_2^2}{R_3^2-R_1^2}\right)^{1/2}.$$

The representation (5.3) is convenient, in particular, because the strongest singularity that is characteristic for a non-degenerate four-tail diagram has been taken outside the integral and is contained in $I_{\rm I}$.

It is not difficult to see directly that

$$I_{\rm I} \sim \ln|k_F^2 - R_3^2|,$$

in accordance with the general analysis of the singularities carried out in[7]. The complete expression for $J^{(4)}$ is collected, according to (3.17), from the contributions (5.3).

Next, it is necessary to construct the symmetrized combination (2.8), which for the case of the four-tail diagram has the following form:

$$\begin{aligned} \Lambda_0^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = {}^1/_{12}[&J^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \\ &+ J^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_4, \mathbf{q}_3) + J^{(4)}(\mathbf{q}_1, \mathbf{q}_3, \mathbf{q}_2, \mathbf{q}_4)]. \end{aligned} \tag{5.4}$$

It is interesting that, to obtain the asymptotic form for $k_F \ll q_i$, the usual representation of $J^{(4)}$ in terms of the occupation numbers n_p turns out to be useful. This representation is easily found from (2.9) (for $\varkappa_i \neq \varkappa_j$):

$$J^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = \sum_{i=1}^4 f_{\varkappa_i},$$
$$f_{\varkappa_1} \equiv 2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{n_{\mathbf{p}+\varkappa_1}}{(\varepsilon_{\mathbf{p}+\varkappa_1}-\varepsilon_{\mathbf{p}+\varkappa_2})(\varepsilon_{\mathbf{p}+\varkappa_1}-\varepsilon_{\mathbf{p}+\varkappa_3})(\varepsilon_{\mathbf{p}+\varkappa_1}-\varepsilon_{\mathbf{p}+\varkappa_4})}. \tag{5.5}$$

The other $f_{\varkappa_i}$ are obtained by cyclic permutation (we recall that $\varkappa_4 = 0$). By expanding the denominators in series in k_F/q_i and integrating term by term, we obtain the following asymptotic form:

$$\begin{aligned} J^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = {}&\gamma(\mathbf{q}_1, \mathbf{q}_1+\mathbf{q}_2, \mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3) \\ &+\gamma(\mathbf{q}_2, \mathbf{q}_2+\mathbf{q}_3, \mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)+\gamma(\mathbf{q}_3, \mathbf{q}_3+\mathbf{q}_4, \mathbf{q}_3+\mathbf{q}_4+\mathbf{q}_1) \\ &+\gamma(\mathbf{q}_4, \mathbf{q}_4+\mathbf{q}_1, \mathbf{q}_4+\mathbf{q}_1+\mathbf{q}_2), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \gamma(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -\frac{8}{3}\frac{m_e^3 k_F^3}{\pi^2}\frac{1}{a^2b^2c^2}\Big\{1+\frac{2}{5}k_F^2\Big[&\left(\frac{\mathbf{a}}{a^2}+\frac{\mathbf{b}}{b^2}\right)^2 \\ &+\left(\frac{\mathbf{b}}{b^2}+\frac{\mathbf{c}}{c^2}\right)^2+\left(\frac{\mathbf{c}}{c^2}+\frac{\mathbf{a}}{a^2}\right)^2\Big]+\ldots\Big\}. \end{aligned} \tag{5.7}$$

This expression is convenient for estimating the contributions to the energy from distant reciprocal-lattice sites. The opposite case of large k_F is evidently of no great interest, but the corresponding asymptotic form can in principle be obtained directly from (2.5) without difficulty. (Its first term coincides with (5.21).)

We proceed now to examine the degenerate cases. It is worth noting immediately that these are especially important for the four-tail diagram, since they are realized, in particular, for the smallest reciprocal-lattice vectors, which, as a rule, give a large contribution to the energy.

Suppose that the vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ lie in the same plane and that the inequalities (4.4) are fulfilled. Then it follows from (4.7) that

$$\begin{aligned} J_{deg}^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = -m_e[&P_3^1 J^{(3)}(\mathbf{q}_1, \mathbf{q}_2, -(\mathbf{q}_1+\mathbf{q}_2)) + P_3^2 J^{(3)}(\mathbf{q}_2, \mathbf{q}_3, -(\mathbf{q}_2+\mathbf{q}_3)) \\ &+ P_3^3 J^{(3)}(\mathbf{q}_3, \mathbf{q}_4, -(\mathbf{q}_3+\mathbf{q}_4)) + P_3^4 J^{(3)}(\mathbf{q}_4, \mathbf{q}_1, -(\mathbf{q}_4+\mathbf{q}_1))]. \end{aligned} \tag{5.8}$$

For the coefficient P_3^1, it is not difficult, by transforming (4.8) to an invariant form, to obtain:

$$P_3^1 = -2\left[(\mathbf{q}_1\mathbf{q}_2)\frac{(\mathbf{s}_{34}\mathbf{s}_{12})}{s_{12}^2} + (\mathbf{q}_3\mathbf{q}_4)\right]^{-1}, \quad \mathbf{s}_{ij} = [\mathbf{q}_i\mathbf{q}_j]. \tag{5.8a}$$

The other P_3^i are obtained by means of cyclic permutation.

We next consider the particular case of a planar system in which all the vectors lie on one circle and, consequently, (4.4) becomes an equality. The result can be obtained by exact passage to the limit in (5.8):

$$J_{deg}^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = \frac{m_e^3}{\pi^2}\sum_{L=1}^{6} P_L I(k_F, R_2, R_1^L). \tag{5.9}$$

Here the index L labels all six edges $R_1^L = {}^1/_2|\mathbf{q}_L|$ ($\mathbf{q}_5 \equiv \mathbf{q}_1 + \mathbf{q}_2$, $\mathbf{q}_6 \equiv \mathbf{q}_2 + \mathbf{q}_3$);

$$\begin{aligned} I(k_F, R_2, R_1) = {}&\frac{1}{2(R_2^2-R_1^2)}\left[\frac{1}{R_1}\ln\left|\frac{k_F+R_1}{k_F-R_1}\right| - \frac{1}{R_2}\ln\left|\frac{k_F+R_2}{k_F-R_2}\right|\right] \\ &+ \frac{1}{R_1(R_2^2-R_1^2)^{1/2}}\frac{1}{|k_F^2-R_2^2|^{1/2}} \cdot \begin{cases} \mathscr{A} \text{ for } k_F/R_2 > 1 \\ \mathscr{L} \text{ for } k_F/R_2 < 1 \end{cases}, \end{aligned} \tag{5.9a}$$

where

$$\mathscr{A} = \operatorname{arctg}\frac{R_1(R_2^2-R_1^2)^{1/2}}{k_F^2-R_1^2+k_F(k_F^2-R_2^2)^{1/2}};$$

$$\mathscr{L} = \frac{1}{2}\ln\frac{(k_F^2-R_1^2)^2+[k_F(R_2^2-k_F^2)^{1/2}-R_1(R_2^2-R_1^2)^{1/2}]^2}{R_2^2|k_F^2-R_1^2|};$$

the coefficients P_L are found by means of the expression

$$P_L = \frac{[\mathbf{q}_L \mathbf{R}_2{}^L]^2}{[\mathbf{q}_L{}^{in}\mathbf{q}_L][\mathbf{q}_L\mathbf{q}_L{}^{out}]}, \tag{5.9b}$$

where $\mathbf{R}_2^L$ is the vector drawn from the origin of $\mathbf{q}_L$ to the center of the circle, $\mathbf{q}_L^{in}$ is that vector which terminates at the origin of the vector $\mathbf{q}_L$, $\mathbf{q}_L^{out}$ is that vector which starts out from the terminus of $\mathbf{q}_L$, and $\mathbf{q}_L^{in}$ and $\mathbf{q}_L^{out}$ should, not form a triangle (we recall that $\mathbf{q}_5$ and $\mathbf{q}_6$ are included in the vectors $\mathbf{q}_L$). This algorithm leads to the following set of "triads" of vectors:

$$\mathbf{q}_L{}^{in}, \mathbf{q}_L, \mathbf{q}_L{}^{out}; \quad \mathbf{q}_4, \mathbf{q}_1, \mathbf{q}_2; \quad \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3;$$
$$\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4; \quad \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_1; \quad \mathbf{q}_4, \mathbf{q}_5, -\mathbf{q}_2; \quad \mathbf{q}_1, \mathbf{q}_6, -\mathbf{q}_3.$$

We shall examine in more detail the important particular case of a planar system in which $\mathbf{q}_3 = -\mathbf{q}_1$ and $\mathbf{q}_4 = -\mathbf{q}_2$ (see Fig. 5a). In this case it follows from (5.8a) that

$$P_3{}^1 = P_3{}^4 = -P_3{}^2 = -P_3{}^3 = -1/\mathbf{q}_1\mathbf{q}_2.$$

Making use of (5.8) and the expression (3.21), we obtain

$$J^{(4)}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1, -\mathbf{q}_2) = \frac{2m_e{}^3}{\pi^2}\frac{1}{q_1{}^2q_2{}^2-(\mathbf{q}_1\mathbf{q}_2)^2}\Big[q_1 \ln\left(\frac{2k_F+q_1}{2k_F-q_1}\right)$$
$$+q_2\ln\left(\frac{2k_F+q_2}{2k_F-q_2}\right) - \frac{1}{2}q_3{}^+\ln\left(\frac{2k_F+q_3{}^+}{2k_F-q_3{}^+}\right) - \frac{1}{2}q_3{}^-\ln\left(\frac{2k_F+q_3{}^-}{2k_F-q_3{}^-}\right) \tag{5.10}$$
$$-\frac{q_1q_2q_3{}^+}{\mathbf{q}_1\mathbf{q}_2}\Delta^+ \operatorname{arctg}(A^+\Delta^+) + \frac{q_1q_2q_3{}^-}{\mathbf{q}_1\mathbf{q}_2}\Delta^- \operatorname{arctg}(A^-\Delta^-)\Big].$$

Here we have introduced the notation $\mathbf{q}_3^{\pm} = -(\mathbf{q}_1 \pm \mathbf{q}_2)$ and, correspondingly, the radii of the circumscribing circles

$$q_R{}^{\pm} = q_1q_2q_3{}^{\pm}/2[q_1{}^2q_2{}^2-(\mathbf{q}_1\mathbf{q}_2)^2]^{1/2};$$

for the quantities $\Delta^{\pm}$ and $A^{\pm}$, see (3.22). We recall that the expressions (5.10) are directly valid for the region $k_F > q_R^{\pm}$, and for lower k_F it is necessary to make use of the analytic continuation.

The symmetrized many-point function $\Lambda_0^{(4)}$ should be constructed in accordance with (5.4) and, in the case under consideration, has the following form:

$$\Lambda_0{}^{(4)}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1, -\mathbf{q}_2) = \frac{1}{12}[J^{(4)}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1, -\mathbf{q}_2)$$
$$+J^{(4)}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_2, -\mathbf{q}_1) + J^{(4)}(\mathbf{q}_1, -\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_2)]. \tag{5.11}$$

The first integral in this expression is defined above, in (5.10). It corresponds to the diagram of Fig. 5a. But for the next integral (Fig. 5b) we have $\kappa_1 = \mathbf{q}_1$, $\kappa_2 = \mathbf{q}_1 + \mathbf{q}_2$ and $\kappa_3 = \mathbf{q}_1$. Hence it follows that the radii R_3 and R_2 of the three- and two-dimensional spheres respectively coincide and the inequality (4.4) is not fulfilled. We therefore cannot substitute the corresponding arguments into (5.8) directly—some of the denominators P_3^i then vanish. However, it is not difficult, by means of (5.8), to carry out the limit procedure by taking $\mathbf{q}_3 = -\mathbf{q}_2(1+\lambda)$ and then letting $\lambda \to 0$ (an analogous device was applied above in obtaining (4.10)). It is also

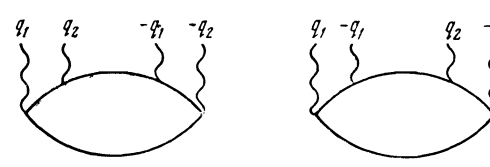


FIG. 5

possible to pass to the limit in (5.9). As a result we find the following expression:

$$J^{(4)}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_2, -\mathbf{q}_1) = \frac{m_e{}^3}{\pi^2}\frac{1}{q_1{}^2q_2{}^2-(\mathbf{q}_1\mathbf{q}_2)^2}$$
$$\times\left\{\frac{\mathbf{q}_1\mathbf{q}_3{}^+}{q_1}\ln\frac{2k_F+q_1}{2k_F-q_1} + \frac{\mathbf{q}_2\mathbf{q}_3{}^+}{q_2}\ln\frac{2k_F+q_2}{2k_F-q_2} + q_3{}^+\ln\frac{2k_F+q_3{}^+}{2k_F-q_3{}^+}\right. \tag{5.12}$$
$$\left.-2\frac{(\mathbf{q}_1\mathbf{q}_2)(q_3{}^+)^2}{q_1q_2q_3{}^+}\frac{1}{\Delta^+}\operatorname{arctg}(A^+\Delta^+)\right\}.$$

The last term in (5.11) is obtained from (5.12) by the replacement $\mathbf{q}_1 \to -\mathbf{q}_1$. For the symmetrized four-point function a considerably simpler expression is obtained, since the logarithmic terms from (5.10) and (5.12) cancel exactly:

$$\Lambda_0{}^{(4)}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1, -\mathbf{q}_2) = -\frac{m_e{}^3}{6\pi^2}\frac{4k_F{}^2-q_3{}^{+2}}{q_1q_2q_3{}^+(\mathbf{q}_1\mathbf{q}_2)}\frac{1}{\Delta^+}\cdot$$
$$\times\begin{cases} -\frac{1}{2}\ln|(1-A^+\Delta^+)/(1+A^+\Delta^+)| & \text{for } k_F/q_R{}^+<1 \\ \operatorname{Arctg} A^+\Delta^+ & \text{for } k_F/q_R{}^+>1 \end{cases}$$
$$+\frac{m_e{}^3}{6\pi^2}\frac{4k_F{}^2-q_3{}^{-2}}{q_1q_2q_3{}^-(\mathbf{q}_1\mathbf{q}_2)}\frac{1}{\Delta^-}\cdot \tag{5.13}$$
$$\times\begin{cases} -\frac{1}{2}\ln|(1-A^-\Delta^-)/(1+A^-\Delta^-)| & \text{for } k_F/q_R{}^-<1 \\ \operatorname{Arctg} A^-\Delta^- & \text{for } k_F/q_R{}^->1 \end{cases}.$$

(Here we have analytically continued into the region of small k_F. We call attention to the choice of branch of the arctangent, $0 < \operatorname{Arctan} x < \pi$, both here and in (3.21).)

By passing to the limit $\mathbf{q}_2 \to 0$ in (5.13), we can obtain the following limit value:

$$\Lambda_0{}^{(4)}(\mathbf{q}, -\mathbf{q}, 0, 0) = \frac{1}{6}\frac{m_e{}^3}{\pi^2 k_F}\frac{1}{4k_F{}^2-q^2}. \tag{5.14}$$

It is easily verified that this satisfies a relation that follows from a general analysis of (2.10):

$$\Lambda_0{}^{(4)}(\mathbf{q}, -\mathbf{q}, 0, 0) = \frac{1}{12}\frac{d^2\Lambda_0{}^{(2)}(\mathbf{q}, -\mathbf{q})}{d\mu^2}$$

The corresponding singularity in (5.14) is the strongest for the four-tail diagram and corresponds to the furthermost degeneracy, down to a one-dimensional system (cf.[7]).

We return again to the diagram of Fig. 5b. It is a typical example of a diagram containing "anomalous" contributions, i.e., of a diagram giving a different answer in the limit $T \to 0$ from that in the technique with $T = 0$ (cf.[3,4]). These extra terms are associated with going round the double pole (the arguments of two G-functions coincide).

It is interesting to analyze separately the "normal" contributions and the "anomalous" corrections that arise only in the temperature technique—these first arose in another problem[13]. (We recall that our initial representation (2.5) corresponds to the limit $T \to 0$ and therefore automatically contains both types of contribution simultaneously). For this we carry out the summation over ω_l in (2.1), taking exact account of the presence of the double pole[14], and then take the limit $T \to 0$:

$$J^{(4)}(\mathbf{q}_1, -\mathbf{q}_1, -\mathbf{q}_2, \mathbf{q}_2) = 2\int\frac{d^3p}{(2\pi)^3}\left\{\frac{n_{\mathbf{p}+\mathbf{q}_1}}{(\varepsilon_{\mathbf{p}+\mathbf{q}_1}-\varepsilon_{\mathbf{p}})^2(\varepsilon_{\mathbf{p}+\mathbf{q}_1}-\varepsilon_{\mathbf{p}-\mathbf{q}_2})}\right.$$
$$+\frac{n_{\mathbf{p}-\mathbf{q}_2}}{(\varepsilon_{\mathbf{p}-\mathbf{q}_2}-\varepsilon_{\mathbf{p}})^2(\varepsilon_{\mathbf{p}-\mathbf{q}_2}-\varepsilon_{\mathbf{p}+\mathbf{q}_1})} - \frac{n}{(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}+\mathbf{q}_1})^2(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{q}_2})} \tag{5.15}$$
$$\left.-\frac{n_{\mathbf{p}}}{(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{q}_2})^2(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}+\mathbf{q}_1})} - \frac{\delta(\mu-\varepsilon_{\mathbf{p}})}{(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}+\mathbf{q}_1})(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{q}_2})}\right\}.$$

In the technique corresponding to $T = 0$ we obtain $J^{(4)}$ without the last term, which thus corresponds to the "anomalous" contribution (i.e., to differentiation of the occupation numbers n_p with respect to μ). For the case $q_1, q_2 > 2k_F$, it can be integrated easily by making use of standard techniques (cf.[15]):

$$J_{\text{anom}}^{(4)} = -\frac{m_e^3}{\pi^2}\frac{1}{q_1 q_2 q_3^+}\frac{1}{\Delta^+}\ln\frac{q_1^2 q_2^2 + 4k_F^2 q_1 q_2 + 2k_F q_1 q_2 q_3^+ \Delta^+}{q_1^2 q_2^2 + 4k_F^2 q_1 q_2 - 2k_F q_1 q_2 q_3^+ \Delta^+} \quad (5.16)$$

($q_1, q_2 > 2k_F$). In the region of small q_i, it is in general impossible to separate the "anomalous" from the normal contribution—only the integral corresponding to their sum converges. We remark that a curious feature of the "anomalous" contribution is the fact that for small k_F it is proportional to k_F whereas the normal contribution is proportional to k_F^3 (cf. (5.7)).

Finally, we shall consider the case of still higher degeneracy, when all the external vectors of the four-tail diagram are parallel (the linear case). We introduce the unit vector $\mathbf{e}$ and let $\mathbf{q}_i = q_i\mathbf{e}$, so that the q_i are numbers with a sign. Treating this case as a limit of the planar case (5.8) and making use of (4.9), we obtain after a number of transformations

$$J_{\text{lin}}^{(4)}(q_1, q_2, q_3, q_4) = (2m_e)^2\left[-\frac{J^{(2)}(q_1, -q_1) + J^{(2)}(q_3, -q_3)}{q_2 q_4 q_5 q_6} + \frac{J^{(2)}(q_2, -q_2) + J^{(2)}(q_4, -q_4)}{q_1 q_3 q_5 q_6} + \frac{J^{(2)}(q_5, -q_5) + J^{(2)}(q_6, -q_6)}{q_1 q_2 q_3 q_4}\right]. \quad (5.17)$$

Here, to make the expression symmetric, we have introduced the notation $q_5 = q_1 + q_2$, $q_6 = q_2 + q_3$. The expression (5.17) is not directly applicable when q_5 or q_6 equals zero. However, this case is easily obtained as a limiting case from (5.17). Let $q_2 = -q_1$ ($q_5 = 0$), and $q_3 \neq -q_1$ ($q_6 \neq 0$). Then,

$$J^{(4)}(q_1, -q_1, q_3, -q_3) = \frac{m_e^3}{4\pi^2 k_F^3 x_1^2 x_3^2}\frac{1}{4(x_1 - x_3)}\left\{(1 - x_1^2 + 2x_1x_3)\ln\left|\frac{1+x_1}{1-x_1}\right| - (1 - x_3^2 + 2x_1x_3)\ln\left|\frac{1+x_3}{1-x_3}\right| + [1 - (x_3 - x_1)^2]\ln\left|\frac{1 + (x_3 - x_1)}{1 - (x_3 - x_1)}\right|\right\}, \quad (5.18)$$

$$x_1 = q_1/2k_F, \qquad x_3 = q_3/2k_F.$$

As an important example of linear degeneracy, we shall consider diagrams with all four external momenta equal in magnitude. (Such a contribution to the energy is given, e.g., by the nearest reciprocal-lattice site.) It follows directly from (5.18) ($x_1 = x = -x_3$) that

$$J^{(4)}(\mathbf{q}, -\mathbf{q}, -\mathbf{q}, \mathbf{q}) = -\frac{m_e^3}{4\pi^2 k_F^3 x^4}\left\{\frac{1-3x^2}{4x}\ln\left|\frac{1+x}{1-x}\right| - \frac{1-4x^2}{8x}\ln\left|\frac{1+2x}{1-2x}\right|\right\}. \quad (5.19)$$

A contribution corresponding to $x_3 = x_1 = x$ also appears in the symmetrized many-point function. It too is not difficult to find from (5.18), by taking one more limit:

$$J^{(4)}(\mathbf{q}, -\mathbf{q}, \mathbf{q}, -\mathbf{q}) = \frac{m_e^3}{4\pi^2 k_F^3 x^4}\left\{\frac{x^2}{1-x^2} - \frac{x}{2}\ln\left|\frac{1+x}{1-x}\right|\right\}. \quad (5.20)$$

(This particular case was treated by another method in[9].)

Finally for $\Lambda^{(4)}$ (cf. (4.4)), we find

$$\Lambda_0^{(4)}(\mathbf{q}, -\mathbf{q}, \mathbf{q}, -\mathbf{q}) = \frac{m_e^3}{48\pi^2 k_F^3 x^4}\left\{\frac{1-4x^2}{2x}\ln\left|\frac{1+x}{1-x}\right| - \frac{1-4x^2}{4x}\ln\left|\frac{1+2x}{1-2x}\right| + \frac{x^2}{1-x^2}\right\}. \quad (5.21)$$

In the limit $q \to 0$ we obtain

$$\Lambda^{(4)}(0,0,0,0) = \frac{1}{24}\frac{m_e^3}{\pi^2 k_F^3}, \quad (5.22)$$

which also follows from (5.14) (cf. also (2.10)).

This completes the mathematical analysis of the four-tail diagrams.

*$[q_i q_j] \equiv q_i \times q_j$.

1)Nuclear Research Institute, Swerk, Poland.

1. W. Harrison, Pseudopotentials in the Theory of Metals, Benjamin, N. Y., 1966 (Russ. transl. Mir, M., 1968).
2. E. G. Brovman and Yu. Kagan, Zh. Eksp. Teor. Fiz. 52, 557 (1967) [Sov. Phys.-JETP 25, 365 (1967)].
3. E. G. Brovman and Yu. Kagan, in Lattice Dynamics, ed. A. A. Maradudin and Horton, North-Holland, Amsterdam, 1974.
4. E. G. Brovman and Yu. Kagan, Usp. Fiz. Nauk 112, 369 (1974) [Sov. Phys.-Uspekhi, 17, 162 (1974)].
5. E. G. Brovman, Yu. Kagan, and A. Kholas, Zh. Eksp. Teor. Fiz. 61, 2429 (1971) [Sov. Phys.-JETP 34, 1300 (1972)].
6. J. Lindhard, K. Danske Vidensk. Selsk. Mat.-Fys. Medd. 28, No. 8 (1954).
7. E. G. Brovman and Yu. Kagan, Zh. Eksp. Teor. Fiz. 63, 1937 (1972) [Sov. Phys.-JETP 36, 1025 (1973)].
8. E. G. Brovman, Doctoral Dissertation, IAE (Atomic Energy Institute), 1973.
9. J. Hammerberg and N. W. Ashcroft, Report 1943, Cornell University, 1973.
10. A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Quantum Field Theoretical Methods in Statistical Physics), Fizmatgiz, M., 1962 (English translation published by Pergamon Press, Oxford, 1965).
11. E. G. Brovman and Yu. Kagan, Zh. Eksp. Teor. Fiz. 57, 1329 (1969) [Sov. Phys.-JETP 30, 721 (1970)].
12. L. D. Landau, Zh. Eksp. Teor. Fiz. 37, 62 (1959) [Sov. Phys.-JETP 10, 45 (1960)].
13. W. Kohn and J. M. Luttinger, Phys. Rev. 118, 41 (1960).
14. J. M. Luttinger and J. C. Ward, Phys. Rev. 118, 1417 (1960).
15. J. Mathews and R. L. Walker, Mathematical Methods of Physics, Benjamin, N. Y., 1964 (Russ. transl. Atomizdat, M., 1972).

Translated by P. J. Shepherd
194