Energy of a Fermi system in an external field with radiation corrections taken into account

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(Submitted June 30, 1973)
Zh. Eksp. Teor. Fiz. **66**, 2065–2077 (June 1974)

A system of interacting relativistic Fermi particles of several kinds situated in an external field is considered when the temperature differs from zero. Expressions have been obtained for the thermodynamic potential and the number of particles in terms of renormalized quantities.

1. INTRODUCTION

Renormalization in the theory of a relativistic Fermi-system at a finite temperature T is carried out in complete analogy with the usual quantum electrodynamics in a vacuum at T = 0[1]. All the subtractive terms in this case are the same as those in quantum electrodynamics in a vacuum. However, there exists a physically observable quantity for a system of Fermi-particles for which there exists no analogous renormalized expression in quantum electrodynamics—this is the energy of the system (or the thermodynamic potential for $T \neq 0$). From a formal point of view in perturbation theory the energy of the system is, apart from a multiplicative factor, the sum of connected vacuum diagrams. These diagrams all diverge strongly in the ultraviolet region. It is not such a simple matter to express the vacuum diagrams in terms of renormalized Green's functions and the vertex parts. Therefore, even in nonrelativistic theory where there are no ultraviolet divergences in order to obtain a convenient expression for the energy of the system taking into account the shift of the single particle energy levels (i.e., in fact a mass renormalization) circuitous methods are utilized and one tries to express the energy of the system in terms of the single particle Green's functions, for example, with the aid of integrating over the charge. In relativistic theory such a reduction of the energy to renormalized quantities is certainly necessary for the elimination of divergences. In this case the standard technique, say, integration over the bare charge, turns out to be inconvenient since it is unavoidably associated with utilizing unrenormalized (and divergent) quantities. Therefore the derivation of an expression for the energy of a Fermi-system taking radiation corrections into account in terms of renormalized quantities is an independent and important problem. It is specifically in this respect that the theory of a Fermi-system essentially differs from the usual quantum electrodynamics in a vacuum.

The energy of a homogeneous plasma at T = 0 taking radiation corrections into account was first studied in a paper by Fradkin[1] where it was shown that in the absence of an external field the energy is expressed in terms of the renormalized compact operator for the polarization. In the paper of Akhiezer and Peletminskii[2] a plasma at a finite temperature was considered, and it was established that in the lowest approximation with respect to e^2 the divergences in the expression for the thermodynamic potential are removed by a renormalization of the mass and charge. Later in our papers[3,4] renormalized expressions were proposed for the energy of the plasma at T = 0 in any arbitrary order of perturbation theory in the presence of an external field. In particular, we have also demonstrated the relationship of the energy to the renormalized operator for the polarization, but in contrast to[1] we deal with the total (noncompact) operator for the polarization taking into account the intermediate one-phonon states.

In the present article we consider the most general case of a Fermi-system of particles of several kinds interacting with a quantized electromagnetic field when the temperature and the external field are both different from zero. We obtain expressions for the thermodynamic potential and the average number of particles of each kind taking into account radiation corrections in terms of renormalized quantities. In the absence of an external field this expression reduces to a sum of vacuum Feynman diagrams with physical values for the mass and the charge of the particles provided with definite subtractive terms which remove the ultraviolet divergences. From our formulas expressions follow for the number of particles and the energy of the plasma at T = 0 which differ from those of[1]. However, this difference appears only in third order perturbation theory and is due to diagrams for the compact polarization operator in which photons are emitted by particles of different kinds.

2. EQUATION FOR THE AVERAGE FIELD AND THE CONDITION OF NEUTRALITY

We consider a system of charged Fermi-particles of several different kinds at a finite temperature T situated in a given stationary external unrenormalized electromagnetic field $A_u^{(0)}$. For the sake of simplicity we restrict ourselves to the study of only electromagnetic interactions so that the complete interaction in the system is assumed to be of the form

$$H_I = \int\limits_{x_0=0} d^3x\, j_{\alpha u}(x)\, (\hat{A}^\alpha(x) + A_u^{(0)\alpha}(x)). \tag{1}$$

Here A is the operator for the quantized electromagnetic field; $j_{\alpha u}$ is the unrenormalized current density for the charged particles

$$j_{\alpha u}(x) = e_0 \sum_{i=1}^{\cdot} \zeta_i \frac{1}{2} [\bar{\psi}_i(x), \gamma_\alpha \psi_i(x)]. \tag{2}$$

The summation is carried out over particles of different kinds; ζ_i are the relative charges; e_0 is the unrenormalized electron charge. For the sake of simplifying the notation we denote by x_0 the variable denoting reciprocal temperature which appears in (1) and (2) and subsequently in the temperature Green's functions. The variable x_0 varies between the limits from zero up to $\beta = 1/kT$ (k is the Boltzmann constant) and it is between these specific limits that in future all integrations over this variable with be extended. We shall be interested in statistical averages over a grand canonical ensemble. Therefore in addition to the temperature T and the external field $A_u^{(0)}$ our system will also be characterized by s chemical potentials μ_i for each kind of particles. For $\mu_i = A_u^{(0)} = T = 0$ the system reduces to the usual quantum electrodynamics in a vacuum.

Renormalization in such a theory has been described in the literature[1,4]. From a formal point of view it does not differ from renormalization in quantum electrodynamics in a vacuum. All the subtractive constants refer to the values $\mu_i = A_u^{(0)} = T = 0$. In future an important role will be played by the renormalized n-photon compact vertex parts for the average field A which we symbolically denote by $\Pi_n(A)$. They are related to the corresponding unrenormalized vertices $\Pi_{nu}(A)$ by the equations: $\Pi_n = z_3^{n/2}\Pi_{nu}$, where z_3 is the usual renormalization constant for the photon wave function, while $\Pi_{nu}(A)$ is the sum of contributions of Feynman graphs with n external photon lines without one-photon intermediate states in the external field A. The final renormalization of Π_2 (i.e., of the polarization operator) is accomplished, as is well known, by means of subtractions. The completely renormalized polarization operator Π_{2r} is defined by the relation

$$\Pi_{2r}(A)=\Pi_2(A)-\Pi_2^{(0)}(k=0)+\Pi_2'^{(0)}(k=0)D^{(0)-1}. \qquad (3)$$

Here $D^{(0)}$ and $\Pi_2^{(0)}$ denote the bare photon propagator and the polarization operator for zero values of μ_i, A and T, i.e., for a vacuum; k is the four-momentum of the photon. We note that in (3) and subsequently symbolic compact notation is used in which all quantities are understood to be matrices in configuration space and with respect to the vector indices of the photons. It is understood that differentiation in (3) refers to the scalar function contained in $\Pi_2^{(0)}$. The final renormalization of Π_1 we shall also carry out with the aid of subtractions by defining

$$\Pi_{1r}(A)=\Pi_1(A)-\Pi_1(0)-A\left(\frac{\partial\Pi_1}{\partial A}\right)\Big|_{A=0}. \qquad (4)$$

The $\Pi_n(A)$ so defined for $n \geq 3$, Π_{1r} and Π_{2r} are finite, contain neither ultraviolet nor infrared divergences and are expressed in perturbation theory in terms of renormalized charges and masses of the particles. For $n \geq 2$ this is well known. For n = 1 this circumstance is a consequence of the identity

$$\frac{\partial\Pi_n(A)}{\partial A}=-i\Pi_{n+1}(A), \qquad (5)$$

from which it follows that in (4) photon vertices starting with n = 3 in fact occur. We note one more important identity similar to (5):

$$\left(\frac{\partial\Pi_n(A)}{\partial\mu_m}\right)_A=\frac{i}{\zeta_m e}\Pi_{n+1(m)}(A). \qquad (6)$$

Here and in subsequent discussion it is implied that the extra vector index on the right-hand side should be set equal to zero, and an integration should be performed over the coordinates of this extra photon. The index (m) on the right-hand side denotes that one should select only those Feynman diagrams for Π_{n+1} in which the additional photon with zero polarization is joined to the line of the particle of kind m.

In terms of the quantities introduced above the average observable (renormalized) field A satisfies the equation[1,4]:

$$D^{-1}(0)A=D^{(0)-1}A^{(0)}+i\Pi_1(0)+i\Pi_{1r}(A). \qquad (7)$$

Here $A^{(0)}$ is the renormalized external field associated with the starting field by the relation $A^{(0)}=z_3^{1/2}A_u^{(0)}$.

The quantity D(A) is the complete photon propagator for the external field A related to $\Pi_{2r}(A)$ by the equation

$$D^{-1}(A)=D^{(0)-1}-\Pi_{2r}(A). \qquad (A)$$

For a given $A^{(0)}$ equation (7) has solutions for A falling off at large spatial distances only under the condition

$$\Pi_1(0)=0, \qquad (8)$$

i.e., only under the condition that the average current density in the system for a given μ_i and T in the absence of an external field is equal to zero. From physical considerations it is natural to assume that this condition of neutrality far from the sources of the field is satisfied for any real system. Condition (8) imposes a relation on the independent parameters μ_i and T which must be taken into account in determining the thermodynamic potential and the average number of particles of the system.

Equation (7) for the observable field A can also be rewritten in the form

$$D^{-1}|_0 A=D^{(0)-1}A^{(0)}+j, \qquad (9)$$

where the symbol $|_0$ implies zero values for the chemical potential, the field and the temperature. $D|_0$ is the total Green's function for the photon in quantum electrodynamics in a vacuum. Correspondingly j can be interpreted as the observable current density without taking external currents into account which give rise to the field $A^{(0)}$. The quantity j is related to Π_1 by the equation

$$j=i\left(\Pi_1(A)-A\frac{\partial\Pi_1}{\partial A}\Big|_0\right). \qquad (10)$$

It differs somewhat from $i\Pi_{1r}$, but it is also finite.

3. DEPENDENCE OF THE NUMBER OF PARTICLES AND OF THE THERMODYNAMIC POTENTIAL ON μ_i and A

The average number of particles of kind a is related to the vertex Π_1 by the equation

$$N_a=\frac{i}{e\zeta_a}\int_{x_0=0} d^3x\,\Pi_{1(a)}(x|A). \qquad (11)$$

Here we have explicitly indicated the dependence of Π_1 on the coordinates and the zero vector index of the photon. Expression (11) is not renormalizable directly. Following the idea of Fradkin[1] we consider instead of N_a the derivative $\partial N_a/\partial\mu_m$. In differentiating (11) with respect to μ_m one should have in mind that $\Pi_{1(a)}$ depends on μ_m both explicitly, and also through the average field A. Therefore, utilizing (5), (6) and (7), we obtain

$$\frac{\partial N_a}{\partial\mu_m}=-e^{-2}\zeta_a^{-1}\zeta_m^{-1}\int_{x_0=0} d^3x\,[\Pi_{2(am)}(A)+\Pi_{2(a)}(A)D(A)\Pi_{2(m)}(A)]. \qquad (12)$$

In expression (12) it is already not difficult to carry out the final renormalization. Indeed, the subtractive, in accordance with (3), terms refer to vacuum for A = 0 and give a zero contribution to the integral (12), since in momentum space they must be taken at zero photon energy and for its three-momentum tending to zero. Therefore in (12) we can simply replace $\Pi_{2(am)}$ and $\Pi_{2(a)}$ correspondingly by $\Pi_{2r(am)}$ and $\Pi_{2r(a)}$. Then in the brackets of (12) we shall have the complete (noncompact) renormalized polarization operator $P_{(am)}(A)$ with the one-photon intermediate states having been taken into account (but without diagrams of the bubble type):

$$P_{(am)}=\Pi_{2r(am)}+\Pi_{2r(a)}D\Pi_{2r(m)}. \tag{B}$$

Finally we have

$$\frac{\partial N_a}{\partial \mu_m}=-e^{-2}\zeta_a^{-1}\zeta_m^{-1}\int\limits_{x_0=0} d^3x\, d^4y\, P^{00}_{(am)}(xy|A). \tag{13}$$

Formula (13) gives a renormalized expression for $\partial N_a/\partial\mu_m$ and can be utilized for the calculation of N_a and of the thermodynamic potential Ω. For a given temperature T we choose in the space of the variables μ_i a certain path $l(T)$ which lies entirely on the neutrality surface (8) and which connects the given point μ_i, T with the point $\mu_i = 0$,T, which, evidently, satisfies (8). We then obtain

$$N_a(\mu_i, A^{(0)}, T)-N_a(0, A^{(0)}, T)=\sum_{m=1}^{s}\int\limits_{l(T)} d\mu_m'\frac{\partial N_a(\mu_i', A^{(0)}, T)}{\partial\mu_m'} \tag{14}$$

and after repeated integration obtain

$$\Omega(\mu_i, A^{(0)}, T)-\Omega(0, A^{(0)}, T)=-\sum_{a=1}^{s}\mu_a N_a(0, A^{(0)}, T) -\sum_{a=1}^{s}\int\limits_{l(T)} d\mu_a' N_a(\mu_i', A^{(0)}, T). \tag{15}$$

Formulas (14) and (15) completely solve the problem of determining Ω and N_a in terms of the renormalized quantities for arbitrary μ_i, if Ω and N_a are known for $\mu_i = 0$. The differentials $d\mu_a'$ along the path $l(T)$ in these formulas are related by the equation which follows from condition (8):

$$\sum_{a=1}^{s}\zeta_a^{-1}\Pi^{00}_{2r(a)}(k=0,\ A=0)\,d\mu_a=0. \tag{16}$$

To obtain finally the values of Ω and N_a evidently it is required that we know their dependence on the external field (for $\mu_i = 0$). To investigate it we shall proceed as follows. We introduce into the theory the numerical parameter λ, multiplying by it $A^{(0)}_u$ in the initial interaction H_I given by (1) and we study the dependence on λ of the quantities in which we are interested.

If the condition of neutrality (8) is satisfied, then as $\lambda \to 0$ Eq. (7) admits the solution $A_\lambda|_{\lambda=0} = 0$, although it is possible that it is not the only one. The cases when for $\lambda \to 0$ the quantity A_λ does not vanish correspond to spontaneous violation of space symmetry (i.e., to a phase transition) and are not considered by us here. For weak fields this is in any case excluded. Thus, we assume that for our system $A_\lambda \to 0$ as $\lambda \to 0$ (condition of weak inhomogeneity).

It is now easy to show that

$$\frac{\partial\Omega}{\partial\lambda}=iZ_3^{-1}\int\limits_{x_0=0} d^3x\, A_a^{(0)}(x)\Pi_1^a(x|A_\lambda). \tag{17}$$

Differentiating this equation once again with respect to λ and taking into account the relation

$$\partial A_\lambda/\partial\lambda=D(A_\lambda)D^{(0)-1}A^{(0)}, \tag{18}$$

which follows from (7) we obtain

$$\partial^2\Omega/\partial\lambda^2=Z_3^{-1}\int\limits_{x_0=0} d^3x\,[A^{(0)}\Pi_2(A_\lambda)D(A_\lambda)D^{(0)-1}A^{(0)}]. \tag{19}$$

We now take into account that

$$\Pi_2 D=D^{(0)-1}(Z_3 D-D^{(0)}). \tag{20}$$

Then integrating (19) twice with respect to λ between the limits from 0 to 1 and introducing the renormalized external current $j_e = D^{(0)-1}A^{(0)}$ we obtain

$$\Omega(\mu_i, A^{(0)}, T)-\Omega(\mu_i, 0, T)=\int_0^1 d\lambda\int_0^\lambda d\lambda'\int\limits_{x_0=0} d^3x\,[j_e(D(A_{\lambda'})-D|_0)j_e] +iZ_3^{-1}\int\limits_{x_0=0} d^3x\,A_a^{(0)}(x)\Pi_1^a(x|A_\lambda)|_{\lambda=0}+\frac{1}{2}\int\limits_{x_0=0} d^3x\,[j_e(D|_0-Z_3^{-1}D^{(0)})j_e]. \tag{21}$$

In this equation only the first term on the right-hand side is renormalized. However, the second term is equal to zero when the neutrality condition (8) and the condition of weak inhomogeneity are satisfied. And the third term depends on neither μ_i, nor on T and, evidently, is related to the renormalization of external charges. It need not be taken into account at all. When this is taken into account formula (21) allows us to obtain $\Omega(\mu_i, A^{(0)}, T)$ in terms of renormalized quantities if we know $\Omega(\mu_i, 0, T)$.

Differentiating (21) with respect to μ_a we obtain the average numbers of particles N_a. Naturally in doing so we must take into account also the contribution of the second term on the right-hand side of (21) since a change in one of the μ_a violates the condition of neutrality (8), and outside the neutrality surface $A_\lambda|_{\lambda=0} \neq 0$. We obtain the sum of two terms $N_a^{(1)} + N_a^{(2)}$, where $N_a^{(2)}$ arises from the first term in (21), is at least quadratic with respect to the external field $A^{(0)}$ and is explicitly renormalized:

$$N_a^{(2)}=-\int_0^1 d\lambda\int_0^\lambda d\lambda'\int\limits_{x_0=0} d^3x\left(j_e\frac{\partial D(A_{\lambda'})}{\partial\mu_a}j_e\right), \tag{22}$$

while $N_a^{(1)}$ is linear with respect to the external field and arises from the second term in (21). On taking into account the dependence of Π_1 on μ_a (both explicit and through the average field A_λ) we obtain:

$$N_a^{(1)}=e^{-1}\zeta_a^{-1}\int d^3x\,[j_e D(0)\Pi_{2(a)}(0)]. \tag{23}$$

In equation (23) we can replace $\Pi_{2(a)}$ by the completely renormalized polarization operator $\Pi_{2r(a)}$ since the subtractive terms give no contribution. Therefore

$$N_a^{(1)}=e^{-1}\zeta_a^{-1}\int\limits_{x_0=0} d^3x\,[j_e D(0)\Pi_{2r(a)}(0)]. \tag{24}$$

Formulas (14), (15), (21), (22) and (24) completely determine the dependence of Ω and N_a on the chemical potentials and the external field and are renormalized. To obtain finally the values of Ω and N_a it is necessary also to know the dependence on the temperature of at least the value of Ω for $A^{(0)} = \mu_i = 0$. Instead of that we obtain in the next section an explicit expression for Ω for arbitrary A, μ_i and T in the form of a sum of vacuum Feynman diagrams renormalized in a definite manner. By means of this we shall not only obtain the dependence on the temperature which we lack, but we shall in fact carry out the integrations over μ_i and λ in formulas (14), (15) and (21).

4. RENORMALIZED EXPRESSION FOR THE THERMODYNAMIC POTENTIAL IN TERMS OF VACUUM FEYNMAN DIAGRAMS

We consider the sum of connected vacuum Feynman diagrams in an external field A without loop insertions which contain as parameters the physical charges and masses of the particles and without the last integration over the temperature which yields the multiplier β. We add to it the subtractive terms for all the internal subdiagrams for the polarization of the vacuum and for all the proper masses of the particles in accordance with the usual prescription for the elimination of infinities in Feynman diagrams[5]. In carrying this out we must only have in mind the following special features of applying the standard technique in our case. First of all, all the subtractive terms contain the values of the proper mass and of the derivative of the polarization operator with respect to the square of the momentum for zero chemical potentials, field and temperature. Secondly, the subdiagrams for the proper mass and the vacuum polarization in vacuum diagrams refer most frequently to the so-called overlapping insertions. The subtractive terms must be introduced separately for each method of separating out a divergent subdiagram[5]. Thirdly, we need not make any subtractions for the subdiagrams of the vertex part and for the renormalization of the wave functions of the charged particles, since these subtractive terms, as is well known, mutually cancel in diagrams containing only closed fermion loops.

Thus we denote a definite sum of vacuum diagrams with eliminated internal divergences by V*. The quantity V* contains also its own divergences. In order to liquidate them we carry out additional subtractions and define as the finally renormalized quantity

$$V=V^*-V^*|_0-\frac{1}{2}A^2\frac{\partial^2V^*}{\partial A^2}\Big|_0, \tag{25}$$

where, as usual, the notation $|_0$ implies $\mu_i = A = \beta^{-1} = 0$. The quantity V is finite. We verify this on the example of a homogeneous plasma in the following section. Now we prove that V is simply related to the thermodynamic potential Ω.

First of all we note that

$$\partial V^*/\partial A=-i\Pi_1. \tag{26}$$

This identity is trivial for unrenormalized quantities. It is then simply a special case of (5) for n = 0. In our case we must also take into account the contribution of the subtractive terms and the renormalization of charge and mass. If differentiation with respect to A in (26) involves a fermion line which does not enter the subdiagram of proper mass or polarization with which the subtractive term under consideration is associated, then after differentiation we obtain exactly the subtractive term for the corresponding diagram Π_1. But if we differentiate a fermion line entering the composition of the divergent subdiagram under consideration, then the contribution from the differentiation of the subtractive term will be equal to zero. But after differentiation with respect to A the subdiagrams for the proper mass and vacuum polarization will go over respectively into the subdiagrams for the vertex and the three-photon vertex entering into the composition of Π_1. According to our prescription no subtractive terms have to be made to correspond to them. Thus, as a result of differentiating the subtractive terms in V* we obtain exactly all the subtractive terms for Π_1 which realize the renormalization of mass and charge. We note that the renormalization of the mass and the charge for Π_1 is carried out in a trivial manner, since in Π_1 there are no overlapping insertions associated with the diagrams for the proper mass and vacuum polarization.

Thus, (26) is satisfied. Taking (10) into account we obtain

$$\partial V/\partial A=-j, \tag{27}$$

$$(\partial V^*/\partial\mu_i)_A=(\partial V/\partial\mu_i)_A=N_i. \tag{28}$$

From (9) we can conclude that

$$\partial A/\partial\mu_i=D|_0\partial j/\partial\mu_i \tag{29}$$

and therefore

$$(\partial V/\partial\mu_i)_A=\frac{\partial}{\partial\mu_i}\left(V+\frac{1}{2}jD|_0j\right). \tag{30}$$

Comparison of (28) and (30) leads to the expression for the thermodynamic potential

$$\Omega=-\left(V+\frac{1}{2}jD|_0j\right)+f(A^{(0)},T), \tag{31}$$

where $f(A^{(0)},T)$ does not depend on the chemical potentials. We note that for $\mu_i = A^{(0)} = \beta^{-1} = 0$ and j vanish and therefore $\Omega|_0 = f(0,0)$. Consequently

$$\Omega-\Omega|_0=-(V+{}^1/_2jD|_0j)+f(A^{(0)},T)-f(0,0). \tag{32}$$

We first show that f in fact is independent of the external field $A^{(0)}$. For this we consider the theory with external field $\lambda A^{(0)}$. Then we have

$$\frac{\partial V}{\partial\lambda}=\frac{\partial V}{\partial A}\frac{\partial A}{\partial\lambda}=-jDj_e. \tag{33}$$

Here we have utilized (18). Differentiating equation (9) containing the external field $\lambda A^{(0)}$ with respect to the parameter λ we obtain

$$\frac{\partial j}{\partial\lambda}=D^{-1}|_0(D-D|_0)j_e \tag{34}$$

and further

$$-\frac{\partial^2}{\partial\lambda^2}\left(V+\frac{1}{2}jD|_0j\right)=j_e(D-D|_0)j_e. \tag{35}$$

According to (21) this is exactly the second derivative with respect to λ of the observable part of the thermodynamic potential. Taking into account the fact that Ω does not contain any terms linear in λ we then find that the function f does not depend on the external field and is a function only of the temperature f(T).

We now state the arguments which prove that the function f(T) is also independent of the temperature. For this we consider the regularized theory in which all the Feynman integrals including those for the vacuum diagrams converge well. For such a theory with finite regularization parameters L we obtain an equation analogous to (32) where all the quantities depend on L. The quantity Ω_L can also be represented in the form of a sum of unrenormalized vacuum graphs. All the graphs, except for those of zero order in the interaction, will vanish for $|\mu_i|\to\infty$, if the regularization is sufficiently strong (for example, the integration over the three-momenta will be limited to a finite sphere of radius L). In (32) we set A= 0 and let $|\mu_i|\to\infty$. Then on the left-hand side there will remain only the vacuum graph without interaction from Ω_L and all the graphs for $-\Omega_L|_0$ which do not depend on the temperature. On the right-hand side we shall have j = 0 and of V_L there will remain only the graph without the interaction from V* and all the graphs for $-V^*|_0$ which again do not depend on the temperature. Thus, in the limit $|\mu_i|\to\infty$ the difference $f_L(T)-f_L(0)$ does not depend on the temperature and is consequently equal to zero. But it does not depend on μ_i and therefore is always equal to zero. Going to the limit for $L\to\infty$, we obtain the desired result.

Thus, we find an explicit and renormalized expression for the thermodynamic potential for an arbitrary Fermi-system of charged particles in the form

$$\Omega(\mu_i, A^{(0)}, T)-\Omega(0, 0, 0)=-(V+{}^1/_2 jD|_0 j), \tag{36}$$

where V is the sum of connected vacuum diagrams without loops renormalized according to the prescription given above, while j is the observable current which is also expressed in terms of V by Eq. (27). Formula (36) enables us to obtain the thermodynamic potential taking into account the radiation corrections utilizing only the physical values of the masses and the charges of the particles. In the next section we illustrate the effectiveness of this formula using the example of a spatially homogeneous plasma.

5. HOMOGENEOUS PLASMA

It is of interest to examine in greater detail the simplest and practically important case when the external field is absent (homogeneous system).

Since for A = 0 the average numbers of particles for $\mu_i = 0$ vanish, in order to calculate N_a one can use formulas (13) and (14). In (13) the integration over x can be carried out in a trivial manner for a homogeneous system. Setting for the sake of simplicity the volume of the system equal to unity we obtain

$$\frac{\partial N_a}{\partial \mu_m}=-e^{-2}\zeta_a^{-1}\zeta_m^{-1}P^{00}_{(am)} \qquad (k=0,\ A=0), \tag{37}$$

where k is the four-momentum of the photon. The quantity $P^{00}_{(am)}$ can be expressed in terms of the components of the compact polarization tensor $\Pi^{00}_{2r(am)}$. Denoting for brevity $\Pi^{00}_{2r(am)}$ (k = 0, A = 0) = q_{am} and taking into account the structure of the photon Green's function for k = 0 and A = 0 (cf.,[1,3]), we obtain

$$P^{00}_{(am)}\ (k=0,\ A=0)=q_{am}-q_a q_m/q, \tag{38}$$

where we have used the notation

$$q_a=\sum_{m=1}^{s} q_{am}, \quad q=\sum_{a=1}^{s} q_a. \tag{39}$$

The neutrality condition (16) can be written in the form

$$\sum_{a=1}^{s}\zeta_a^{-1} q_a\, d\mu_a=0. \tag{40}$$

Taking (40) into account we obtain the expression for the average number of particles N_a (a = 1, . . ., s − 1) in the form

$$N_a=-e^{-2}\zeta_a^{-1}\sum_{m=1}^{s-1}\zeta_m^{-1}\int d\mu_m(q_{am}-q_{as}\,q_m/q_s). \tag{41}$$

In this equation we call attention to the fact that as a result of the neutrality condition the contribution of the noncompact part of the polarization operator (the second term on the right-hand side of (38)) is completely cancelled out. The remaining contribution is determined entirely by the compact part. This is effectively equivalent to the situation as if we did not take into account the dependence of the average field A on the chemical potentials, and in this sense corresponds to the approach in[1] to the analogous problem for one kind of particles. In our theory necessarily several kinds of particles must take part (not fewer than two) and this corresponds to the physical statement of the problem.

For the simplest case of two kinds of particles it follows from (41)

$$N_1=-e^{-2}\zeta_1^{-2}\int d\mu_1(q_{11}q_{22}-q_{12}^2)/(q_{22}+q_{12}), \tag{42}$$

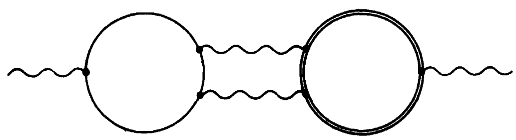

FIG. 1

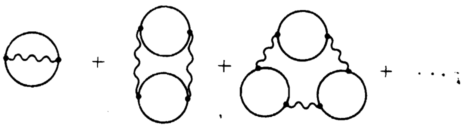

FIG. 2

and at the same time

$$\zeta_1 N_1+\zeta_2 N_2=0. \tag{43}$$

Formula (42) becomes simplified if we take into account that q_{12} is less than q_{11} and q_{22}. In q_{12} diagrams are included of the type shown in Fig. 1 in which photons are emitted by different particles. Therefore q_{12} is of the order of smallness of $(\zeta_1\zeta_2 e^2)^3$. And the diagonal terms q_{11} and q_{22} are respectively of the order of $\zeta_1^2 e^2$ and $\zeta_2^2 e^2$. Therefore in the lowest order in terms of q_{12}/q_{22} we obtain

$$N_1\approx-e^{-2}\zeta_1^{-2}\int d\mu_1\, q_{11}(1-q_{12}/q_{22}). \tag{44}$$

Here the first term already coincides exactly with the result of reference[1] for a homogeneous plasma, while the second term gives a correction of relative order $\zeta_1^3\zeta_2 e^4$.

In conclusion we examine the value of the thermodynamic potential for a homogeneous plasma utilizing expression (36), and we confirm that it is finite and renormalized. In doing this we restrict ourselves to a definite set of graphs shown in Fig. 2. These graphs correspond to the approximation of a high density plasma, and for a nonrelativistic plasma lead to the well known Gell-Mann-Bruckner formula. For the sake of simplicity we further restrict our consideration to the contribution of particles of one kind (electrons).

A direct summation of diagrams shown in Fig. 2 yields the expression

$$-\frac{1}{2\beta}\sum_{\omega_n}\frac{1}{(2\pi)^3}\int d^3k\,\mathrm{Sp}\{\ln(1-D^{(0)}\Pi)-1\}. \tag{45}$$

The operation of taking the trace refers to the vector indices of photons of frequency $\omega_n = 2\pi ni/\beta$. The quantity Π is a relativistic polarization operator in the lowest approximation with respect to e^2 with physical values for the charge and mass of the electron. In order to construct V* we provide in (45) subtractive terms for internal subdiagrams for the polarization and the proper mass of the electron. Addition of subtractive terms for polarization subdiagrams leads to the replacement of each Π by the renormalized polarization operator Π_r defined in accordance with (3).

The subtractive term associated with diagrams for the proper mass of the electron can be easily obtained in explicit form. Taking into account the rules for the removal of divergences in diagrams with overlapping inserts we obtain

$$\delta V_m=-\frac{1}{\beta(2\pi)^3}\sum_{\omega_n}\int d^3p\,\mathrm{Sp}\{\delta m\, G(p)\}. \tag{46}$$

Here δm is the proper mass of the electron on the mass

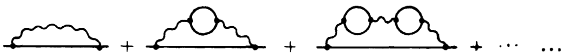

FIG. 3

shell corresponding to the sum of the diagrams shown in Fig. 3. The trace is taken over the spin indices of of the electron of frequency $\omega_n = i(2n+1)\pi\beta^{-1} + \mu$; $G(p)$ is the Green's function for a free electron. Carrying out a further subtraction according to (25) we obtain

$$V=\frac{-1}{2\beta}\sum_{\omega_n}\int d^3k\,\mathrm{Sp}\{\ln(1-D^{(0)}\Pi_r)-1\}\frac{1}{(2\pi)^3}+\frac{1}{4\pi i(2\pi)^3}\int d^4k\,\mathrm{Sp}\{\ln(1-D^{(0)}\Pi_r^{(0)})-1\}+\delta V_m-\delta V_m|_0, \quad (47)$$

with $\Pi_r^{(0)} = \Pi_r|_0$ being the usual renormalized polarization operator in quantum electrodynamics for a vacuum with $T = 0$. With an accuracy up to its sign expression (47) is the expression for the thermodynamic potential of the plasma. It is expressed in terms of the renormalized charge and mass of the electron. We must prove that (47) is finite.

We introduce the notation

$$\delta\int dk_0\,F(k_0)\equiv\frac{1}{\beta}\sum_{\omega_n}F(\omega_n)-\frac{1}{2\pi i}\int dk_0\,F(k_0), \quad (48)$$

where $F(k_0)$ is an arbitrary analytic function of the variable k_0 over the whole plane with cuts for real values of k_0, $|k_0| > b$, while ω_n are frequencies equal to $i2n\pi/\beta$ for a photon and $i(2n+1)\pi\beta^{-1} + \mu$ for an electron. The difference appearing on the right-hand side of (48) can be rewritten in the form of integrals over discontinuities in $F(k_0)$ at the cuts:

$$\delta\int dk_0\,F(k_0)=\frac{1}{4\pi i}\int_b^{\infty} dk_0[\varphi(k_0)-1]\Delta F(k_0)+\frac{1}{4\pi i}\int_{-\infty}^{-b} dk_0[\varphi(k_0)+1]\Delta F(k_0), \quad (49)$$

where for a photon $\varphi(k_0) = \coth 1/2\beta k_0$, and for an electron $\varphi(k_0) = \tanh 1/2\beta(k_0 - \mu)$. For large values of k_0 the expressions in brackets on the right-hand side of (49) fall off exponentially guaranteeing that the integrals converge well. Utilizing the notation in (48) we rewrite (47) in the following form:

$$V=\frac{-1}{2\beta(2\pi)^3}\sum_{\omega_n}\int d^3k\,\mathrm{Sp}\{\ln[1-D^{(0)}(\Pi_r-\Pi_r^{(0)})/(1-D^{(0)}\Pi_r^{(0)})]\}-\frac{1}{2}\delta\int\frac{d^4k}{(2\pi)^3}\mathrm{Sp}\{\ln(1-D^{(0)}\Pi_r^{(0)})-1\}+\delta V_m-\delta V_m|_0. \quad (50)$$

The second term in (50) is finite. Indeed, utilizing (49), we can represent it in the form of an integral over the discontinuities of the function $\mathrm{Sp}\{\ln(1-D^{(0)}\Pi_r^{(0)})-1\}$ situated at $|k_0| > \sqrt{4m^2+k^2}$. After integration over k_0 an expression is obtained which falls off with increasing $|k|^2$ as $\exp[-\beta(4m^2+k^2)^{1/2}]$, so that the whole integral $\delta\int d^4k$ will turn out to be finite. This conclusion applies also to other integrals of the type $\delta\int d^4k$ which we shall encounter later.

Further, as can be easily seen, the difference $\Pi_r - \Pi_r^{(0)}$ falls off as its arguments increase. Therefore in the first term divergences can arise possibly in the first term in the expansion of the logarithm in a series in terms of the ratio $D^{(0)}(\Pi_r - \Pi_r^{(0)})/(1 - D^{(0)}\Pi_r^{(0)})$.

Utilizing the explicit expression for Π_r we represent this term in the form

$$\frac{1}{2\beta}\sum_{\omega_n}\int\frac{d^3k}{(2\pi)^3}D|_0(\Pi_r-\Pi_r^{(0)})=-\frac{1}{2}\delta\int\frac{d^4k}{(2\pi)^3}\mathrm{Sp}\,D|_0\Pi_r^{(0)}+\frac{1}{2\beta^2}\sum_{\omega_{n_1},\omega_{n_2}}\frac{e^2}{(2\pi)^6}\int d^3p_1\,d^3p_2\,\mathrm{Sp}\{\gamma_\alpha G(p_1)\gamma_\beta G(p_2)D^{\alpha\beta}(p_1-p_2)|_0\}$$
$$-\frac{1}{2(2\pi i)^2}\int d^4p_1\,d^4p_2\frac{e^2}{(2\pi)^6}\mathrm{Sp}\{\gamma_\alpha G(p_1)\gamma_\beta G(p_2)D^{\alpha\beta}(p_1-p_2)|_0\}. \quad (51)$$

Here $D|_0 = D^{(0)}/(1 - D^{(0)}\Pi_r^{(0)})$ is the Green's function for the photon taking into account the simplest insertion for the polarization of the vacuum in ordinary quantum electrodynamics. The first term in (51) is finite. It will cancel the contribution of order e^2 from the second term in (50). The second and third terms in (51) give on addition a contribution equal to

$$\delta\int\frac{d^4p}{(2\pi)^3}\mathrm{Sp}\{\Sigma(p)G(p)\}, \quad (52)$$

where $\Sigma(p)$ is the proper mass of the electron corresponding to the diagram of Fig. 3 with renormalized charge and mass of the electron and a renormalized photon propagator but without proper subtractions. We represent in the usual way

$$\Sigma(p)=\Sigma_r(p)+\delta m+(\hat{p}-m)\Sigma'|_0, \quad (53)$$

where $\Sigma_r(p)$ is a finite expression for the proper mass. Then the contribution from the second term in (53) will exactly cancel the terms $\delta V_m - \delta V_m|_0$ in (50), while the contribution from the third term is equal to zero, since the function $(\hat{p}-m)G(p)$ has no singularities at a finite distance and the integral (49) will vanish (for a correct proof one must consider the regularized $G(p)$).

Thus, all the divergences in the expression for V are indeed removed. The final expression for the thermodynamic potential taking into account the chosen set of Feynman diagrams has the form

$$\Omega=\frac{1}{2\beta}\sum_{\omega_n}\int\frac{d^3k}{(2\pi)^3}\mathrm{Sp}\{\ln[1-D|_0(\Pi_r-\Pi_r^{(0)})]+D|_0(\Pi_r-\Pi_r^{(0)})\}-\delta\int\frac{d^4p}{(2\pi)^3}\mathrm{Sp}\{\Sigma_r(p)G(p)\}+\frac{1}{2}\delta\int\frac{d^4k}{(2\pi)^3}\mathrm{Sp}\{\ln(1-D^{(0)}\Pi_r^{(0)})+D|_0\Pi_r^{(0)}\}. \quad (54)$$

The first term in (54) is essentially the usual Gell-Mann-Bruckner expression except that in it a renormalization of the mass and charge have been carried out and in place of the Green's function for the free photon the total Green's function is utilized taking into account the simplest diagram for the polarization of the vacuum for $\mu = T = 0$. The second term in (54) evidently corresponds to a change in the electron mass as a result of its interaction with photons. The third term is of order e^4 and does not depend on μ.

We recall that in the lowest order in e^2 corresponding to the simplest diagram from the series under consideration (the first diagram in Fig. 2) renormalization and elimination of divergences have been carried out previously in[2]. The method proposed by us, as has been shown, enables us to obtain the thermodynamic potential in any arbitrary order of perturbation theory.

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Translated by G. Volkoff.
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