Critical indices for systems with slowly decaying interaction

P. M. Blekher and Ya. G. Sinaĭ

Applied Mathematics Institute
L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences
(Submitted March 7, 1974)
Zh. Eksp. Teor. Fiz. 67, 391–396 (July 1974)

Hierarchical Dyson models, which have properties resembling those of ferromagnetic spin systems with a power interaction potential, are considered. The calculation of the critical indices for such systems is reduced to the determination and investigation of the solutions of certain nonlinear integral equations. The results obtained by this approach are described.

The semi-phenomenological theory of phase transitions starts from the assumption that the thermodynamic potentials, as functions of the dimensionless temperature $\epsilon = (T - T_{cr})/T_{cr}$ or of the external field $h - h_{cr}$, have power singularities in the vicinity of the critical temperature $T = T_{cr}$. The exponents in the corresponding powers are called critical indices. For ferromagnetic systems the most important critical indices are the indices α, β, γ, δ, η and ν, in terms of which the asymptotic forms of the following quantities are expressed:

$$C_H \sim \varepsilon^{-\alpha},\quad M\sim(-\varepsilon)^{\beta},\quad \chi_T\sim\varepsilon^{-\gamma},\quad H\sim|M|^{\delta}\operatorname{sgn}(M),$$
$$\Gamma(r)\sim|r|^{-(d-2+\eta)},\quad \xi\sim\varepsilon^{-\nu}.$$

Here C_H is the specific heat in constant magnetic field, M is the spontaneous magnetization in zero field, H is the critical isotherm, $\Gamma(r)$ is the pair correlation function (at $T = T_{cr}$), and ξ is the correlation length. Scaling arguments are invoked to explain the power singularities, it being assumed that each system possesses certain self-similarity properties in the vicinity of the critical temperature (cf. [1]). The problem of the theory is to determine the values of the critical indices from the form of the interaction Hamiltonian.

It has become clear recently that the study of the critical region for classical lattice models with a slowly decaying potential is considerably simpler than for short-range potentials. The reason lies in the fact that a Hamiltonian constructed with a potential U(r) decaying like $1/r^u$ at infinity ($d < u < 2d$, d is the geometrical dimensionality of the model) possesses the scaling property from the outset; if V' and V'' are two volumes and the distance R between them is large compared with their dimensions, the interaction energy H^{int} between them is equal to $S(V')S(V'')/R^u$ in leading order. Conventionally, we can call the quantity S(V) the total spin in the volume V. For classical lattice models in which an individual variable $s(x)$ ($x \in Z^d$) takes the values ± 1,

$$S(V)=\sum_{x\in V}s(x).$$

Indeed, in general one can also construct other slowly decaying potentials, in which other additive quantities appear. Applying the scaling transformation to H^{int}, one can obtain immediately that, at the critical temperature $T = T_{cr}$, in typical configurations the total spin S(V) should take values of the order of $|V|^{u/2d}$, where $|V|$ is the number of lattice points situated in V. This immediately gives the critical-index value $\eta = 0$, if we take for the dimensionality the anomalous dimension $d_a = 2d/(u - d)$.

A complete mathematical investigation of models with a power potential has not yet been carried out, although particular physically convincing results have been obtained (cf. [2]). The mathematical difficulty lies in the fact that, when the chain of recursion equations for the distribution of the total spin is set up, additional terms arise because of the interaction at the boundary, and it is not entirely simple to take accurate account of these terms.

In this paper we consider the so-called hierarchical models introduced by Dyson [3], which, in many respects, simulate systems with a power potential, but which, because of the absence in them of the above-mentioned boundary terms, are somewhat simpler to analyze. At the same time, there are reasons to suppose that the behavior of hierarchical models and models with a power interaction potential are, in principle, the same in the vicinity of the critical point.

In the hierarchical models the interaction potential is not translationally invariant but, essentially, it also falls off in a power fashion. Because of the special form of the Hamiltonian, the recursion equations for the probability distribution of the total spin take a rather simple form and permit a detailed mathematical investigation. The expressions which then arise for the critical indices can be regarded as a confirmation of the Wilson formulas (cf. [4]). At the end of the article we indicate the analog for the hierarchical models of Wilson's ϵ-expansion, in which, instead of the expansion in the dimensionality parameter ϵ, expansions in a parameter associated with the power exponent in the long-range potential appear.

The Hamiltonians in the hierarchical models are constructed for volumes V_n consisting of k^n points ($k = 2^d$, d is the geometrical dimensionality of the model), each volume V_n being divided into k equal sub-volumes V_{n-1,i_1} ($i_1 = 1, \ldots, k$); each of these subvolumes is divided into k equal sub-volumes V_{n-2,i_1,i_2}, and so on. The set of sub-volumes

$$V_{n-s,\,i_1,\ldots,i_s}$$

forms the hierarchical structure of the volume V_n. The Hamiltonian $H_n(V_n)$ in the volume V_n, for the ferromagnetic case, is determined by means of the recursion relation

$$H_n(V_n)=\sum_{i=1}^{k}H_{n-1}(V_{n-1,i})-|V_n|^{-\zeta}[S(V_n)]^2, \tag{1}$$

where ζ is a parameter of the model and the $S(V_n)$ is the total spin in the volume V_n. It is clear that (1) describes a pair interaction, which, however, is not translationally invariant. The potential corresponding to (1) falls off like $1/r^u$ ($u = d\zeta$) at large distances r. For this reason it makes sense to consider $\zeta > 1$; otherwise, the free energy will increase more rapidly than the first power of the volume. In addition, for $\zeta > 2$, phase transitions are completely absent in the system (cf. [3]). Thus, there remains $1 < \zeta < 2$.

The critical temperature T_{cr} is uniquely determined by the fact that, for it,

$$H^{int} = -|V_n|^{-\zeta}[S(V_n)]^2$$

takes values of order 1 in typical configurations. This means that the critical index $\eta = 0$, if for the anomalous dimension d_a we take the value $2d/(u-d)$.

We introduce the probability distribution for the total spin:

$$f_n(t;\beta) = \text{Prob}\,(S(V_n) = t;\beta)$$

in the volume V_n, for inverse temperature β and zero external field. From the form of the Hamiltonian (1), the chain of recursion relations

$$f_{n+1}(t;\beta) = \frac{\Xi^{k}_{n-1}(\beta)}{\Xi_n(\beta)}\exp\{\beta t^2|V_n|^{-\zeta}\}\sum f_n(t_1;\beta)\dots f_n(t_k;\beta), \qquad (2)$$

easily follows, the summation being performed with the condition $\Sigma t_i = t$. We now turn to the distribution for the normalized average spin, putting

$$h_n(z;\beta)\Delta_n = f_n(z|V_n|^{\zeta/2};\beta),\quad \Delta_n = 2|V_n|^{-\zeta/2}.$$

Then from (2) we have

$$h_{n+1}(z;\beta) = L_n e^{\beta z^2}\sum h_n(z_1;\beta)\dots h_n(z_k;\beta)\Delta_n^{k-1},$$

with

$$\sum z_i = zk^{-\zeta/2}.$$

For $n \to \infty$ the functions $h_n(z;\beta_{cr})$ converge to the function $h(z;\beta_{cr})$, which is the solution of the nonlinear integral equation

$$h(z;\beta_{cr}) = Le^{\beta_{cr}z^2}\int\dots\int\prod_{i=1}^{k} h(z_i;\beta_{cr})\,\delta\left(\sum_{i=1}^{k} z_i - \gamma z\right)\prod_{i=1}^{k} dz_i, \qquad (3)$$

where $\gamma = k^{-\zeta/2}$.

In [7] and [8] the Gaussian solution of Eq. (3):

$$h(z;\beta) = [a_0(\beta)/\pi]^{1/2}\exp(-a_0(\beta)z^2),$$
$$a_0(\beta) = \beta\, k^{2-\zeta}/(k-k^{2-\zeta})$$

was investigated and it was shown that this solution is stable for $\zeta < 3/2$ and gives the indices predicted by the Landau theory. The stability is to be understood in the sense that the convergence to the Gaussian solution is conserved in the presence of a small perturbation of the bare interaction.

For $\zeta > 3/2$ the Gaussian solution is certainly unstable. It turns out that for such ζ non-Gaussian stable solutions of (3) arise.

By means of the replacement

$$h(z;\beta) = \exp(-a_0(\beta)z^2)g(z;\beta)$$

Eq. (3) is brought to the form

$$g(z;\beta) = L\int\dots\int\exp\{-\beta Q(z_1,\dots,z_k)\}\prod_{i=1}^{k} g(z_i;\beta)\,\delta\left(\sum_{i=1}^{k} z_i - \gamma z\right)\prod_{i=1}^{k} dz_i,$$

$$Q(z_1,\dots,z_k) = \left[z_1^2 + \dots + z_k^2 - \frac{1}{k}(z_1 + \dots + z_k)^2\right]\frac{k^{2-\zeta}}{k-k^{2-\zeta}}. \qquad (4)$$

The expression

$$g^{(0)}(z;\beta) \equiv \text{const} = [a_0(\beta)/\pi]^{1/2}.$$

corresponds to the Gaussian solution. The spectrum of the linearized problem has in this case the form k, k/c, k/c^2, ..., where $c = k^{2-\zeta}$. According to the general theory of bifurcations, for nonlinear transformations in a finite-dimensional space new branches of solutions arise for those parameter values for which there is a unity in the spectrum of the linearized problem. We have shown that, in the infinite-dimensional case of the non-linear integral operator appearing in the right-hand side of (4), non-Gaussian solutions also arise (about the points $\zeta_l = 2 - 1/l$, $l = 2, 3, \dots$) which fall off like $\exp(-\epsilon|z|^s)$, where $s = 2/(2-\zeta)$ and is close to $2l$, $\epsilon = (c_1 - c)a$, and

$$a = \frac{k(k-1)}{2}\int e^{-\varphi(\beta)z^2}e_l(z;\beta)\int\dots\int\exp\{-\beta Q(z_1,\dots,z_k)\}$$
$$\times\, e_l(z_1;\beta)e_l(z_2;\beta)\,\delta\left(\sum_{i=1}^{k} z_i - \gamma z\right)\prod_{i=1}^{k} dz_i\,dz,$$

where $e_l(z;\beta)$ is the eigenfunction of the linearized problem for the Gaussian solution, corresponding to the eigenvalue

$$\lambda_l = k/c^l \approx 1,\quad \varphi(\beta) = \beta\frac{k^{2-\zeta}-1}{k-k^{2-\zeta}}.$$

It is found that $a \neq 0$ always.

In order to explain the appearance of the non-Gaussian solutions we shall fix $\beta = 1$ in (4) and expand the arbitrary function $g(z;1) = g(z)$ in a series in the eigenfunctions $e_l(z;1)$:

$$g(z) = 1 + \sum_{l=0}^{\infty}\alpha_l e_l(z;1).$$

Substituting this expansion into (4), we can rewrite (4) as a system of equations for the coefficients α_l:

$$\alpha_l = \lambda_l\alpha_l + \sum_{m,n=1}^{\infty} d^{(l)}_{mn}\,\alpha_m\alpha_n.$$

$\alpha_l \equiv 0$ corresponds to the Gaussian solution. If any λ_{l_0} is close to 1, another solution can be written in the form

$$\alpha_{l_0} = (1-\lambda_{l_0})/d^{(l_0)}_{l_0 l_0} + O((1-\lambda_{l_0})^2),\quad \alpha_l = O((1-\lambda_{l_0})^2)$$

for $l \neq l_0$. Moreover, for the non-Gaussian solution we can write a formal expansion in powers of the small parameter $1 - \lambda_{l_0}$. However, the convergence of such an expansion is completely unclear.

The mathematically more accurate derivation of the non-Gaussian solutions is carried out in two stages and, in its principal features, follows the general theory of bifurcations and invariant manifolds (cf. [5,6]). We shall regard the right-hand side of (4) as the result of applying a nonlinear transformation T to the function g. Then the solution of Eq. (4) is the stationary point for T: g = Tg. Since all the stationary points that we are considering are unstable, to determine them we must start by deriving the "stable separatrix," i.e., by deriving those functions h for which $T^n h$ converges to the required solution g as $n \to \infty$. More precisely, we shall construct first those functions h which under the action of iterations T^n remain all the time close to h. It turns out that the form of such functions can be described more or less explicitly. Namely, we shall consider an l_0-parameter family of functions, of the form

$$h(z;a) = \exp\left[-\varepsilon e_{l_0}(z;1) + \sum_{i=0}^{l_0} a_i e_i(z;1)\right].$$

It is shown by the method of contractive mappings that a uniquely defined set of parameters $\bar{a}_0, \dots, \bar{a}_{l_0}$ will be found for which the function $h(z;\bar{a})$ lies on the separatrix, and all the $\bar{a}_i$ are proportional (in leading order) to ϵ^2.

The second stage consists in checking that, for the functions $h(z; \bar{a})$ found, we do in fact have the convergence $T^n h(z; \bar{a}) \to g$ as $n \to \infty$. A complete description of all the calculations pertaining to this will be published elsewhere.

Only the branch passing through the Gaussian branch for $\zeta_2 = 3/2$ is stable for $\zeta < 3/2$, in the same sense that the Gaussian solution was stable for $\zeta > 3/2$. For the solutions from this branch, in the spectrum of the linearized problem there are two eigenvalues greater than 1, one of which, equal to k, is trivial and is eliminated if we consider only solutions of (4) that are normalized to 1. The second eigenvalue λ, on the other hand, plays a fundamental role, and the remaining indices are expressed in terms of it. Namely, using arguments similar to those in[7,8], we can show that

$$\alpha = 2 - \frac{1}{\log_2 \lambda}, \quad \beta = \frac{\log_2 c}{2 \log_2 \lambda}, \quad \gamma = \frac{1 - \log_2 c}{\log_2 \lambda}, \quad \delta = \frac{2 - \log_2 c}{\log_2 c}, \quad \eta = 0, \quad \nu = \frac{1 - \log_2 c}{2 \log_2 \lambda}. \tag{5}$$

The formulas (5) enable us to write down an analog of the Wilson ϵ-expansion (cf.[4]) in which the parameter $\epsilon = c_2 - c$ appears in place of the dimensionality parameter ϵ. The situation is obviously reduced to expanding λ in ϵ. There are reasons to suppose that λ is only an infinitely differentiable, and not an analytic function of ϵ. We have found the first two terms of its expansion for $k = 2$:

$$\lambda = \sqrt{2} + \frac{\varepsilon}{3} + \frac{65 + 20\sqrt{2}}{27} \varepsilon^2 + O(\varepsilon^3).$$

The non-Gaussian branch passing through ζ_2 has been investigated recently by one of us (P. M. Blekher) on a computer. The results of the computation indicate that this branch does not experience other bifurcations and reaches $\zeta = 2$ without singularities. The detailed results of the calculations will be published elsewhere.

The authors thank I. M. Lifshitz for useful discussions.

1. H. E. Stanley, Introduction to Phase Transitions and Critical Phenomena, Oxford University Press, 1971 (Russ. Transl., Mir, M., 1973).
2. M. E. Fisher, S. Ma and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).
3. F. J. Dyson, Commun. Math. Phys. **12**, 91 (1969).
4. K. G. Wilson, Phys. Rev. **B4**, 3184 (1971).
5. V. I. Arnold, Usp. Mat. Nauk **27**, 119 (1972).
6. M. W. Hirsch, C. C. Pugh and M. Shub, Bull. Am. Math. Soc. **76**, 1015 (1970).
7. P. M. Bleher and Ya. G. Sinai, Commun. Math. Phys. **33**, 23 (1973).
8. P. M. Blekher, IPM (Applied Mathematics Institute) Preprint No. 45 (1973); Trudy MMO (in press).

Translated by P. J. Shepherd
45