Oscillations in films of superfluid solutions

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The effect of the surface normal component on oscillations of superfluid-solution films is investigated.
It is shown that two types of oscillations exist, third sound and second surface sound. The dispersion
and damping of third sound due to the surface normal component are determined.

Distinct oscillations, called third sound, exist in superfluid films[1]. In third sound, the superfluid component oscillates under the influence of a Van der Waals force, while the normal component is immobile. These oscillations, at relatively high temperatures, experience a damping connected with evaporation and condensation of vapor, since third sound is accompanied also by temperature oscillations. The process of establishment of equilibrium with the vapor is slow, and its time increases rapidly with decreasing temperature, owing to the exponential decrease of the vapor density. Therefore, at not too low frequencies, the influence of the vapor can be neglected. The third sound then attenuates weakly, and its velocity u_3 is given by

$$u_3^2 = \frac{\rho_s}{\rho} fd(1+\gamma),$$

where ρ_s and ρ are respectively the superfluid and total density of the liquid, f is the Van der Waals constant, d is the equilibrium thickness of the film, and γ is a correction connected with thermal effects. At a temperature T = 1°K we have $\gamma = 10^{-2}$ and it decreases with temperature. Thus, thermal effects do not play an essential role in third sound.

On the free surface of a solution of He^3 in He^4 there are surface impurity levels of He^3[2], the presence of which leads in the hydrodynamics of superfluid solutions to the existence of a surface normal density[3]. In weak solutions at low temperatures, a situation is possible when the volume normal component can generally be neglected and one can consider a superfluid liquid in which the normal component exists only on the surface.

In this paper we study the influence of a surface normal component on the oscillations of superfluid films. We consider a film at temperatures when the contribution of the phonons and rotons is inessential. If the concentration of the solution is low and the mean free path of the volume impurity excitation is $l \gg d$, then the impurities are not transported over the volume of the film. In the case of low-frequency oscillations, there is time for equilibrium with respect to the momentum and the number of impurity particles to be established between the surface and volume excitations. These times were determined experimentally in[4], and it was found that the momentum relaxation time is shorter by three orders of magnitude than the particle-number relaxation time. At high frequencies, the surface impurities oscillate independently of the immobile volume impurities.

Let us find the system of equations of motion of a film with allowance for the presence of the surface normal component. In the derivation of the equations we shall assume, for simplicity, that there are no impurities at all in the volume. Since we are interested in the problem of small oscillations, we seek the equations in an approximation linear in the velocities.

Let the liquid occupy a layer of thickness d + ζ on the (x, y) plane, where $\zeta = \zeta(x, y, t)$ is the deviation of the surface from the equilibrium value and is of the same order of smallness as the velocity. We define the surface of the film (just as in[3]) by the condition that there be no total surface mass, i.e., by the condition that the total mass M of the liquid in the film be equal to

$$M = \int \int_0^{d+\zeta} \rho \, dz \, ds$$

where ds is the area element of the (x, y) surface. We assume the liquid to be incompressible, and therefore

$$\int \rho \frac{\partial \zeta}{\partial t} ds = 0.$$

In differential form, this expression can be rewritten as

$$\rho \frac{\partial \zeta}{\partial t} + \frac{\partial J_\alpha}{\partial x_\alpha} = 0, \tag{1}$$

where x_α is a component of the two-dimensional vector **r**, and J_α is a two-dimensional vector with the meaning of the mass flux density over the film. In the view of Galilean invariance, this quantity is equal to the momentum per unit area of film, which consists of two terms, the momentum per unit volume, equal to $d\rho v_{s\alpha}$, and the momentum per unit surface i, which according to our definition of ζ is equal to

$$i_\alpha = v_n(v_{n\alpha} - v_{s\alpha}),$$

where v_n is the surface normal density while $v_{n\alpha}$ and $v_{s\alpha}$ are the normal and superfluid velocities.

We can write down analogously an equation expressing the conservation of the momentum

$$\frac{\partial i_\alpha}{\partial t} + d\rho \frac{\partial v_{s\alpha}}{\partial t} + \frac{\partial}{\partial x_\beta} \pi_{\alpha\beta} = 0, \tag{2}$$

where $\pi_{\alpha\beta}$ is the tensor of the momentum flux density, the value of which is to be determined.

The transport equations for the heat and for the surface impurities take the same form as in the case of an infinite volume[3]

$$\frac{\partial v}{\partial t} + v_0 \frac{\partial v_{n\alpha}}{\partial x_\alpha} = 0, \tag{3}$$

$$\frac{\partial \sigma}{\partial t} + \sigma_0 \frac{\partial v_{n\alpha}}{\partial x_\alpha} = 0, \tag{4}$$

where v and σ are the deviations of the number of surface impurities and of the entropy from their equilibrium values v_0 and σ_0.

It is necessary to add to these equations the equations for the superfluid velocity

$$\frac{\partial v_{s\alpha}}{\partial t} + \frac{\partial \mu}{\partial x_\alpha} = 0, \tag{5}$$

where μ is the chemical potential of He^4.

The deviation δp of the pressure from equilibrium is connected with the deviation of the surface by the condition $\delta p = \rho f \zeta$. Since the liquid is incompressible, Eq. (5), with allowance for the foregoing, takes the form

$$\frac{\partial v_{sa}}{\partial t} + f \frac{\partial \zeta}{\partial x_\alpha} = 0. \qquad (6)$$

The momentum flux density can be obtained from the condition that the hydrodynamic equations should identically satisfy the energy conservation law, which in the presence of Van der Waals forces in differential form is given by

$$\frac{\partial E}{\partial t} + \frac{\partial \theta_\alpha}{\partial x_\alpha} + \rho f \frac{\partial \zeta}{\partial t} = 0, \qquad (7)$$

where θ_α is the flux of the energy over the film and its explicit form is determined together with $\pi_{\alpha\beta}$, while E is the energy of the element of film area and is equal to the sum of the volume energy $\rho v_S^2 / 2$ and the surface energy ϵ.

The thermodynamic identity for the differential $d\epsilon_0$ (ϵ_0 is the energy in the rest system v_S, which is connected with ϵ by the relation $\epsilon = \epsilon_0 + 1 \cdot v_S$), takes the following form:

$$d\varepsilon_0 = T d\sigma + \mu_3 dv + (v_{na} - v_{sa}) di_\alpha,$$

where μ_3 is the chemical potential of the impurities. By obtaining the derivative of this expression with respect to time and using Eqs. (1)—(4), (6), and (7), we get

$$\pi_{\alpha\beta} = -(\alpha + \rho f \zeta d) \delta_{\alpha\beta},$$

where $\alpha = \epsilon_0 - Ts - \mu_3 \nu$ is the surface tension of the film.

Thus, Eq. (2) assumes the simple form

$$\frac{\partial i_\alpha}{\partial t} - \frac{\partial \alpha}{\partial x_\alpha} = 0. \qquad (8)$$

We note that the obtained equations differ from the equations of surface hydrodynamics [3]. In the case of infinite helium thickness, the surface flow enter in the equations in a spatial derivative higher by one degree than the volume currents. In Eq. (1), they are contained in derivatives of equal degree. Therefore allowance for the surface normal component in the absence of momentum and impurity exchange with the volume leads not to a dispersion or damping of the sound, but to a decrease of its velocity and to the addition of a new solution.

The presence of impurities in the volume causes μ to start to depend on the impurity concentration. Since there is no impurity transport over the volume, it is easy to obtain the equation for the deviation of the concentration n in the volume from the equilibrium value n_0:

$$\frac{\partial n}{\partial t} + \frac{n_0}{d} \frac{\partial \zeta}{\partial t} = 0. \qquad (9)$$

Equation (6) is rewritten with allowance for the volume impurities in the form

$$\frac{\partial v_{sa}}{\partial t} + f \frac{\partial \zeta}{\partial x_\alpha} + \frac{\partial \mu}{\partial n} \frac{\partial n}{\partial x_\alpha} = 0. \qquad (10)$$

We note that owing to the absence of equilibrium with the vapor, the surface tension of the film cannot be considered on the equilibrium curve, and it must be regarded as a function of the pressure. However, since actually $\alpha = \alpha\{\rho(p)\}$, in the case of an incompressible liquid this dependence is inessential. The surface tension does not depend on the impurity concentration in the volume because there is no exchange of impurity particles.

Let us find the spectrum of the oscillations of the film with the aid of the obtained equations. Let the x axis coincide with the direction of the wave vector **k**. The system of equations is written in the form

$$-i\omega\rho\zeta + ik(\rho d - v_n) v_s + ikv_n v_n = 0, \quad -i\omega\sigma + ik\sigma_0 v_n = 0,$$

$$-i\omega v_n (v_n - v_s) - ik \left(\frac{\partial \alpha}{\partial v} v + \frac{\partial \alpha}{\partial \sigma} \sigma \right) = 0, \quad -i\omega v + ikv_0 v_n = 0,$$

$$-i\omega v_s + ikf\zeta + ik \frac{\partial \mu}{\partial n} n = 0, \quad i\omega n + i\omega n_0 \frac{\zeta}{d} = 0,$$

where ω is the oscillation frequency.

Equating the determinant of this system to zero, we obtain

$$[c^2 - u_3^2(1-\chi)](c^2 - u_2^2) - \frac{fv_n}{\rho}(1-\chi)u_2^2 = 0, \qquad (11)$$

where $c^2 = \omega^2/k^2$, $\chi = n_0/fd \, (\partial\mu/\partial n)$, and u_2 is the velocity of the second surface sound [3], equal to

$$u_2^2 = -\frac{1}{v_n} \left\{ \left(\frac{\partial \alpha}{\partial v} \right)_\sigma v_0 + \left(\frac{\partial \alpha}{\partial \sigma} \right)_v \sigma_0 \right\}.$$

According to [6], the derivative of the chemical potential of a weak solution with respect to the impurity concentration is equal to $-kT/m_4$, where m_4 is the mass of the He^4 atom and k is the Boltzmann constant.

The second term in (11) is of relative order a_0/d, where a_0 is the interatomic distance. Therefore it can be neglected at $|u_2^2 - u_3^2|/u_2^2 \ll 1$, and Eq. (11) has then the following two solutions: one is third sound, and the other is second surface sound, in which oscillations of the surface normal component take place while the surface remains immobile.

Let us examine the dependence of the sound velocities on the film thickness. The velocity of the second surface sound at a given surface density does not depend on the thickness. The third-sound velocity is $u_3 \sim d^{-3}$ [1]. There exists therefore a critical thickness d_0 at which $u_3 = u_2 = u$. Under these conditions, the second term in (11) cannot be neglected. The solution in the vicinity of the intersection point is of the form

$$c^2 = u^2 \pm u \sqrt{fv_n/\rho}.$$

This means that in a film of variable thickness, for example in a gravitational field, a third-sound wave of wavelength small enough to be able to neglect the change of the film thickness over a distance on the order of the wavelength goes over into a second-surface-sound wave, and vice versa at $d \approx d_0$. When measuring the dependence of the third-sound velocity on the thickness of the solution film at this point, a jump in the velocity of relative order of $(a_0/d)^{1/2}$ should be observed. At $T \sim 0.5°K$ and $n_0 \sim 10^{-3}$ we have $u_2 \approx 10^4$ m/sec [3] and d_0 is of the order of several dozen Angstrom units.

At lower frequencies, account must be taken of the momentum relaxation between the surface and the volume. This means that it is necessary to substitute in the right-hand side of (8) the force of friction between the surface impurity excitations and the volume excitations. At low velocities, this force is obviously proportional to v_n. We denote it by λv_n. The condition for the existence of second surface sound is $\omega \gg \lambda/v_n$. In the opposite limiting case $\omega \ll \lambda/v_n$, the determinant of the system is given by

$$\omega^2 - \tilde{u}_3^2(1-\chi)k^2 + \frac{v_n^2 f}{\lambda\rho} i\omega k^2 = 0,$$

where $\tilde{\mu}_3^2 = u_3^2 - v_n f/\rho$.

Thus, the friction between the impurities and the surface leads to the appearance of damping proportional to the power of the wave vector of the same order as the "slipping" of the volume normal component in the region of applicability of the hydrodynamic theory for volume excitations[5].

Let us estimate the value of λ. It is proportional to the number of collisions between the volume impurities and the surface, to their mass, and to the surface diffuseness coefficient f*. This coefficient was measured in[4] by means of the integral effect of flow of He^3 over a surface of a superfluid film. They obtained the value f* $\approx T/0.8k$ in the temperature range from 0.01 to 0.2°K. Since the number of collisions is $N \approx n_0 \sqrt{T/m_3}$ [6], then

$$\lambda \approx n_0 m_3^{1/2} T^{1/2} (1°\,\mathrm{K})^{-1},$$

where m_3 is the effective mass of the impurities in the volume.

On going to a region of still lower frequencies, account must be taken of the possible exchange of particles between the volume and the surface. Inasmuch as total momentum equilibrium sets in at these frequencies, we have $v_n = 0$. Let us rewrite Eqs. (3) and (9) with the foregoing taken into account. The point of a surface with impurity concentration $\nu + \nu_0(n_0)$ relaxes to the equilibrium state with volume $\nu_0(n_0 + n)$. Therefore Eq. (3) will now take the form

$$\frac{\partial \nu}{\partial t} + \frac{\nu}{\tau} - \frac{\partial \nu_0}{\partial n}\frac{n}{\tau} = 0. \tag{12}$$

Since the number of particles that leave the surface is equal to the number of particles that enter the volume, we have

$$\frac{\partial n}{\partial t} + \frac{n_0}{d}\frac{\partial \zeta}{\partial t} - \frac{\nu}{\tau} + \frac{\partial \nu_0}{\partial n}\frac{n}{\tau} = 0. \tag{13}$$

Equating to zero the determinant of the system of equations (1), (4), (8), (10), (12), and (13), we obtain

$$\omega^2 - \tilde{u}_3^2 k^2 + \frac{\chi\omega^2}{1-\chi-\varkappa/(i\omega\tau-1)} = 0, \tag{14}$$

where $\varkappa = (\partial\nu_0/\partial n)d^{-1}$. This quantity is of the order of the ratio of the number of impurity particles on the surface to the number of impurities in the volume, and is of the order of unity in thin films.

In the region $\omega\tau \gg 1$, expression (14) leads to the following equation for the spectrum:

$$\omega^2 - \tilde{u}_3^2(1-\chi)k^2 - i\omega\frac{\chi\varkappa}{1-\chi}\frac{1}{\tau} = 0,$$

i.e., the imaginary increment to the frequency is $\sim 1/\tau$.

At $\omega\tau \ll 1$ we have

$$\omega^2 - \frac{1-\chi+\varkappa}{1+\varkappa}\tilde{u}_3^2 k^2 - i\omega k^2\tau\frac{\varkappa\chi\tilde{u}_3^2}{(1-\chi+\varkappa)(1+\chi)} = 0,$$

i.e., Im $\omega \sim k^2$. In the region $\omega\tau \sim 1$, a strong dispersion and damping of the third sound should be observed.

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[1] K. R. Atkins, Phys. Rev. 113, 962, 1959.
[2] A. F. Andreev, Zh. Eksp. Teor. Fiz. 50, 1415 (1966) [Sov. Phys.-JETP 23, 939 (1966)].
[3] A. F. Andreev and D. A. Kompaneets, Zh. Eksp. Teor. Fiz. 61, 2459 (1971) [Sov. Phys.-JETP 34, 1316 (1972)].
[4] D. B. Crum, D. O. Edwards, and R. E. Sarwinski, The Ohio State University, preprint.
[5] T. A. Karchava and D. G. Sanikidze, Zh. Eksp. Teor. Fiz. 62, 369 (1972) [Sov. Phys.-JETP 35, 198 (1972)].
[6] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Nauka (1964) [Pergamon, 1971].