Operator approach to quantum electrodynamics in an external field: The mass operator

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An operator diagram technique for the investigation of processes in a homogeneous (constant both in space and in time) external electromagnetic field is formulated for particles of spin 0 and 1/2. The method is based on an operator representation of the Green's function for a charged particle in a field. The special features of the method of calculation are elucidated on the example of finding the mass operator. This problem is solved for the general case of a homogeneous field, and it is shown that in order to determine an average value on the mass shell it is sufficient to know only the spectrum of a certain set of operators.

I. INTRODUCTION

In recent years there has been a considerable increase in the degree of interest in electromagnetic processes occurring in external fields. This is associated on the one hand with the production of intense electromagnetic fields in lasers (the maximum intensities of the magnetic field that have been attained amount to 10^9 Oe) and with the availability of beams of electrons and photons of ultra high energies, and on the other hand with possible astrophysical applications (according to estimates the intensities of the magnetic field of pulsars can attain values of 10^{12} Oe). In the processes indicated above a check can be carried out of quantum electrodynamics in the domain of high energies and intense fields. Such a check is certainly of considerable interest since in contrast to the case of electromagnetic interactions between free particles here there is a requirement in principle of going outside the framework of perturbation theory.

A large number of articles has been devoted to an investigation of a number of processes in the Born approximation in terms of the interaction with the radiation field. In the domain of high energies when the quasiclassical approximation, or the investigation of crossed fields equivalent to it, is applicable a considerable portion of these results is quoted in[1-3]. Starting with the results of the pioneering work of Schwinger[4] based on the proper time method Minguzzi[5], having utilized the technique of expanding the vacuum current in an external field (constant field plus a plane wave field) in powers of the interaction with the field of the plane wave, has calculated the vacuum current to e^2-order, and this makes it possible to find to the same order the polarization operator on the mass shell. However, Minguzzi's paper[5] contained errors which have been corrected by Adler[6], who also investigated the process of the splitting of a real photon into two photons in a magnetic field (it is assumed that this process is significant in producing radiation from pulsars). Making use of the explicit form of the Green's function for an electron in a homogeneous external field obtained by Schwinger[4], Narozhnyĭ[7] obtained the polarization operator to e^2-order for the case of a constant crossed field ($E \perp H$, $E = H$), and then Batalin and Shabad[8] obtained the same result for the case of an arbitrary homogeneous external field. Quite recently Schwinger[9] has formulated a method of calculating the mass operator in an arbitrary homogeneous external field using the example of scalar particles. Within the framework of this approach the case of particles of spin 0 and 1/2 in a magnetic field has been investigated in the papers of Tsai and Yildiz[10, 11]. Batalin and Fradkin[12, 13] have studied the problem of the radiation corrections within the framework of the method of functional integration basing themselves on earlier papers by Fradkin[14]. Ritus[15, 16] has determined the mass and the polarization operators to e^2-order in the special case of a constant crossed field utilizing the well-known solutions of the Dirac equation in such a field. Since in this case the field invariants

$$\mathscr{F}=-{}^1/_4F_{\alpha\beta}F^{\alpha\beta}={}^1/_2(\mathbf{E}^2-\mathbf{H}^2),\qquad \mathscr{G}=-{}^1/_4F_{\alpha\beta}{}^*F^{\alpha\beta}=\mathbf{EH},\tag{1.1}$$

where the dual tensor $F^*_{\alpha\beta} = 1/2\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$ has been introduced, vanish, then all the characteristics depend only on a variable of the type $e^2(F^{\mu\nu}p_\nu)^2/m^2$. For the reason indicated above the results in this case coincide with the mass and the polarization operators (the self-energy parts of the electron and the photon) obtained by the authors with the aid of an operator quasiclassical method[17, 2].

In all the papers quoted above, with the exception of the papers by the present authors, variants of the functional formulation of field theory due to Schwinger or an explicit form of the Green's functions for a particle in a field were utilized. However, quite frequently, just as in the case of free particles, the Feynman approach and the diagram technique associated with it have considerable advantages in view of their physical obviousness and simplicity. It turns out to be possible to formulate a generalized diagram technique for particles of spin 0 and 1/2 situated in an arbitrary homogeneous external field without utilizing an explicit form of the Green's function for a charged particle in the field, but representing it in operator form. Its essential advantage turns out to be the possibility of a unified approach to the calculation of any radiation correction, and also the obviousness and the relative simplicity of the calculations.

We start with the S-matrix in the Furry representation:

$$S=T\exp\left\{-i\int \mathscr{H}_I^F(x)\,d^4x\right\},\tag{1.2}$$

where the Hamiltonian density of the interaction $\mathscr{H}_I^F(x)$ for particles of spin 1/2 is

$$\mathscr{H}_I^F(x)={}^1/_2e[\bar\psi_F(x)\gamma_\mu,\ \psi_F(x)]A^\mu(x)=J_{\mu F}(x)A^\mu(x).\tag{1.3}$$

Here $\psi_F(x)$ is the solution of the Dirac equation in a given external electromagnetic field $A_\mu^F(x)$, $J_{\mu F}(x)$ is the flux of spinor particles in the field, $A_\mu(x)$ is the radiation field. Formula (1.2) follows easily from the standard expression for the S-matrix in the field

$A_\mu(x) + A_\mu^F(x)$ with the aid of an appropriate unitary transformation (cf., for example,[18]). In reducing the S-matrix (1.2) to the normal form contributions arise which can be represented by two sets of diagrams. One of them coincides in the manner of recording it with the set of diagrams for free particles (when $A_\mu^F(x) = 0$), only the charged particle lines now represent particles (Green's functions) in a field. The other set of diagrams owes its origin to the fact that the average vacuum value of the current, generally speaking, differs from zero $\langle 0|J_{\mu F}(x)|0\rangle \neq 0$, since the external field induces a current in the vacuum. For this reason in expanding the T-product into a sum of normal products it is necessary to carry out pairing of operators appearing in the current $J_{\mu F} = 1/2e[\bar\psi_F\gamma_\mu, \psi_F]$, and this leads to a new set of diagrams which contain the so-called tadpoles. The same circumstance is also the cause of the fact that in the representation $\langle 0|S|0\rangle\ e^{iL}$ the phase L is not real, as in the case of free particles, but acquires an imaginary part, which describes creation of pairs of particles, by the external field, i.e., there appears still another type of diagrams—photonless vacuum loops.

In a homogeneous external field the four-momentum is conserved, as a result of which the contributions of the tadpole diagrams disappear. For this reason it is possible in going over to relative amplitudes (cf., for example,[18], p. 441), to separate out the set of diagrams with photonless vacuum loops and to consider only the first set of diagrams.

It is convenient to construct the diagram technique in an external field for squares of amplitudes (an approach widely utilized at present in quantum field theory), when the external lines of the diagram occur in the same physical state. The internal lines for charged particles correspond to Green's functions $G(x,x')$, which in the coordinate representation can be written in operator form ($\epsilon \to +0$):

$$G(x,x')=\frac{1}{P^2-m^2+i\varepsilon}\delta(x-x') \tag{1.4}$$

for particles of spin 0 and

$$G(x,x')=\frac{1}{\hat P-m+i\varepsilon}\delta(x-x') \tag{1.5}$$

for particles of spin 1/2, where the operator[1)] $P_\mu = i\partial/\partial x^\mu - eA_\mu(x)$, $\hat P = P_\mu\gamma^\mu$. On utilizing such a form of writing the formula the problem of calculating the contribution of a definite diagram reduces to finding the average of a certain operator.

We consider as an example the mass operator for a spinor particle (Fig. 1a), where the double line represents a particle in a field. To the lowest order in terms of the interaction with the radiation field we have

$$\langle M\rangle=-e^2\int d^4x\, d^4x'\bar\psi(x)\gamma^\mu G(x,x')\gamma_\mu D_F(x-x')\psi(x'). \tag{1.6}$$

Utilizing (1.5) the explicit form of the Fourier-representation for the photon propagator

$$D_F(x-x')=\frac{i}{(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}e^{ik(x-x')}, \tag{1.7}$$

the self-adjoint property of the operator P_μ, and the commutator $[P_\mu, e^{ikx}] = -k_\mu e^{ikx}$, one can carry out the integration over x in (1.6). As a result we have the following representation for the mass operator:

$$\langle M\rangle=\int d^4x\bar\psi(x)M\psi(x), \tag{1.8}$$

where the operator is

$$M=-\frac{ie^2}{(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}\gamma^\mu\frac{1}{\hat P-\hat k-m+i\varepsilon}\gamma_\mu. \tag{1.9}$$

We note that in calculating the average value of the operator M(1.8) it is sufficient to know the spectrum of the eigenvalues of the complete set of operators determining the physical state ψ, with the value of $\langle M\rangle$ being independent of the explicit form of the representation. This question is investigated in greater detail below.

In the case of particles of spin 0 an analogous analysis yields the following expression for the contribution of the diagram of Fig. 1a to the mass operator:

$$M=-\frac{ie^2}{(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}(2P-k)^\mu\frac{1}{(P-k)^2-m^2+i\varepsilon}(2P-k)_\mu. \tag{1.10}$$

As another example we consider the polarization operator (Fig. 2a). The amplitude for the scattering of a photon by an external field has the form

$$T=-\frac{e^2}{(2\pi)^4}\int d^4x\, d^4x'\, \mathrm{Sp}[G(x,x')\hat e_1 e^{-ik_1x'}G(x',x)\hat e_2 e^{ik_2x}], \tag{1.11}$$

where e_1 (e_2) is the polarization of the initial (final) photon. We represent the Green's function for the electron in the form (equivalent to (1.5))

$$G(x,x')=\left\langle x\left|\frac{1}{\hat P-m+i\varepsilon}\right|x'\right\rangle, \tag{1.12}$$

where $|x\rangle$ is the eigenvector of the coordinate operator $X|x\rangle = x|x\rangle$, normalized in such a way that

$$\langle x|x'\rangle=\delta(x-x'),\qquad \int d^4x|x\rangle\langle x|=1.$$

Taking this into account one can rewrite (1.11) in the form

$$T=-\frac{e^2}{(2\pi)^4}\int d^4x\, d^4x'\, \mathrm{Sp}\left[\langle x|\frac{1}{\hat P-m+i\varepsilon}\hat e_1 e^{-ik_1x}|x'\rangle \times\langle x'|\frac{1}{\hat P-m+i\varepsilon}|x\rangle\hat e_2 e^{ik_2x}\right]. \tag{1.13}$$

Utilizing the completeness theorem one can carry out the integration over x'. As a result we obtain

$$T=-\frac{e^2}{(2\pi)^4}\int d^4x\, \mathrm{Sp}\left[\langle x|\frac{1}{\hat P-m+i\varepsilon}\hat e_1\frac{1}{\hat P-\hat k_1-m+i\varepsilon}\hat e_2|x\rangle\right]e^{i(k_2-k_1)x}. \tag{1.14}$$

In virtue of translational invariance which holds in a homogeneous external field, the average $\langle x|\ldots|x\rangle$ appearing in (1.14) does not depend on the coordinate. Now defining the polarization operator $\Pi_{\mu\nu}(T = i(2\pi)^4\delta(k_1-k_2)e_1^\mu e_2^\nu\Pi_{\mu\nu})$, we have

$$\Pi_{\mu\nu}=\frac{ie^2}{(2\pi)^4}\mathrm{Sp}\langle 0|\frac{1}{\hat P-m+i\varepsilon}\gamma_\mu\frac{1}{\hat P-\hat k-m+i\varepsilon}\gamma_\nu|0\rangle. \tag{1.15}$$

An analogous investigation leads to the following contribution of the diagram of Fig. 2b to the polarization operator for particles of spin 0:

$$\Pi^{(a)}_{\mu\nu}=\frac{-ie^2}{(2\pi)^4}\langle 0|\frac{1}{P^2-m^2+i\varepsilon}(2P-k)_\mu\frac{1}{(P-k)^2-m^2+i\varepsilon}(2P-k)_\nu|0\rangle. \tag{1.16}$$

Moreover, for these particles it is necessary to take

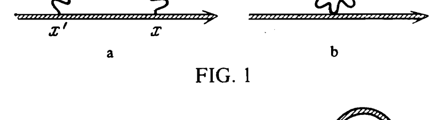

FIG. 1

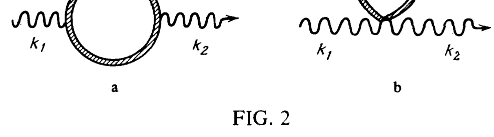

FIG. 2

into account the contribution of the diagram of Fig. 2b:

$$\Pi^{(b)}_{\mu\nu}=\frac{2ie^2}{(2\pi)^4}g_{\mu\nu}\langle 0|\frac{1}{P^2-m^2+i\varepsilon}|0\rangle. \quad (1.17)$$

The imaginary part of the mass operator is associated with the total probability of radiation by a particle in a field, the imaginary part of the polarization operator is associated with the total probability of the production by a photon of a pair in an external field (for the quasi-classical approximation cf., for example[2]). However, within the framework of the technique developed here one can obtain also other characteristics of physical processes, for which it is necessary to use the appropriate projection operators.

An investigation of more complex diagrams can also be carried out in an analogous manner. However the presentation of a concrete set of prescriptions for calculations can be conveniently carried out after we have elucidated the characteristic features of the technique on the example of evaluating the contributions of diagrams of lowest order.

In Sec. II we obtain the mass operator for a particle of spin 0 in an arbitrary homogeneous electromagnetic field, while in section III the same problem is solved for a spinor particle.

II. THE MASS OPERATOR FOR A SCALAR PARTICLE

We consider the mass operator (Fig. 1a) for a particle of spin 0 (1.10):

$$M^{(0)}=-\frac{ie^2}{(2\pi)^4}\int d^4k\,(2P-k)^\mu\frac{1}{k^2+i\varepsilon}\,\frac{1}{(P-k)^2-m^2+i\varepsilon}(2P-k)_\mu. \quad (2.1)$$

The form of (2.1) is the same as for a mass operator for a scalar particle in the absence of a field, but an essentially new element is the presence of noncommuting operators in the integral of (2.1), so that prior to the evaluation of the integral over k it is necessary to carry out the appropriate transformations of the integrand. It is convenient to utilize the exponential parametrization of the propagators appearing in (2.1):

$$\frac{1}{k^2+i\varepsilon}\,\frac{1}{(P-k)^2-m^2+i\varepsilon}=-\int_0^\infty ds\,s\int_0^1 du\,e^{-isu(m^2-i\varepsilon)}e^{is\mathscr{H}}, \quad (2.2)$$

where the operator is

$$\mathscr{H}=(P^2-2Pk)u+k^2=(P-k)^2u+(1-u)k^2. \quad (2.3)$$

Then (2.1) can be rewritten in the form

$$M^{(0)}=\frac{ie^2}{(2\pi)^4}\int_0^\infty ds\,s\int_0^1 du\,e^{-isum^2}\tilde{M}^{(0)}, \quad (2.4)$$

and the problem is reduced to evaluating the integral

$$\tilde{M}^{(0)}=\int d^4k\,(2P-k)^\mu e^{is\mathscr{H}}(2P-k)_\mu. \quad (2.5)$$

We here move the operator $e^{is\mathscr{H}}$ to the right:

$$e^{is\mathscr{H}}(2P-k)_\mu=[2(e^{-2eFsu})_{\mu\nu}(P-k)^\nu+k_\mu]e^{is\mathscr{H}}. \quad (2.6)$$

In deriving this relation we utilize the equations

$$[P_\mu, P_\nu]=-ieF_{\mu\nu}, \quad (2.7)$$

$$e^{isP^2}P_\mu e^{-isP^2}=\sum_{n=0}^{\infty}\frac{(is)^n}{n!}\underbrace{[P^2,[P^2,\dots[P^2}_{n},P_\mu]\dots]], \quad (2.8)$$

where for a homogeneous field[2] $[P^2, P_\mu]=2ieF_{\mu\nu}P^\nu$, so that the sum (2.8) can be evaluated in closed form:

$$e^{isP^2}P_\mu e^{-isP^2}=\left(\sum_{n=0}^{\infty}\frac{(is)^n}{n!}(2ieF)^nP\right)_\mu=(e^{-2eFs}P)_\mu. \quad (2.9)$$

We must now find integrals of the type

$$\int d^4k\,(1;\,k_\mu;\,k_\mu k_\nu)\,e^{is\mathscr{H}}. \quad (2.10)$$

A direct evaluation of them is quite awkward. The investigation is simplified if we utilize the equation

$$\int d^4k\left(\frac{\partial}{\partial k_\mu};\,\frac{\partial^2}{\partial k_\mu\partial k_\nu}\right)e^{is\mathscr{H}}=0. \quad (2.11)$$

The derivatives appearing in (2.11) can be obtained with the aid of the following procedure[3]

$$\frac{\partial}{\partial k}e^{is\mathscr{H}}=\varphi(s)\,e^{is\mathscr{H}}; \quad (2.12)$$

differentiating this relation with respect to s, we obtain

$$\frac{d\varphi(s)}{ds}=ie^{is\mathscr{H}}\frac{\partial\mathscr{H}}{\partial k}e^{-is\mathscr{H}}, \quad (2.13)$$

where it has been taken into account that for commuting operators A and B we have $[\partial A/\partial k, B]=[\partial B/\partial k, A]$. Substituting (2.3) into this and utilizing (2.9) we obtain

$$\frac{d\varphi(s)}{\partial s}=2i[(1-u)k-ue^{-2eFsu}(P-k)]. \quad (2.14)$$

A solution of this differential equation with the obvious initial condition $\varphi(0)=0$ is the expression

$$\varphi(s)=2i\left[s(1-u)k+\frac{e^{-2eFsu}-1}{2eF}(P-k)\right]. \quad (2.15)$$

Taking (2.12) into account the second derivative has the form

$$\frac{\partial^2}{\partial k^\nu\partial k^\mu}e^{is\mathscr{H}}=\left[\frac{\partial\varphi_\mu(s)}{\partial k^\nu}+\varphi_\mu(s)\varphi_\nu(s)\right]e^{is\mathscr{H}}. \quad (2.16)$$

Substituting the expressions obtained above into (2.11), we have

$$\int d^4k\,k\,e^{is\mathscr{H}}=D^{-1}AP\int d^4k\,e^{is\mathscr{H}}, \quad (2.17)$$

where the following notation has been introduced:

$$A=e^{-2eFsu}-1,\quad D=A-2seF(1-u). \quad (2.18)$$

Similarly

$$\int d^4k\,k_\mu k_\nu e^{is\mathscr{H}}=\left\{(D^{-1}AP)_\mu(D^{-1}AP)_\nu-ie\left(\frac{F}{D}\right)_{\mu\nu}\right\}\int d^4k\,e^{is\mathscr{H}}, \quad (2.19)$$

where the quantity on the right hand side is symmetric with respect to the replacement $\mu\leftrightarrow\nu$.

Now the problem has been reduced to the evaluation of the integral

$$L(s)=\int d^4k\,e^{is\mathscr{H}}. \quad (2.20)$$

The function L(S) satisfies the differential equation

$$\frac{dL(s)}{ds}=i\int d^4k[(P^2-2Pk)u+k^2]e^{is\mathscr{H}}. \quad (2.21)$$

Utilizing formulas (2.17), (2.19), one can rewrite (2.21) in the form

$$dL(s)/ds=iB(s)L(s), \quad (2.22)$$

where

$$B(s)=P^2u-2uP\frac{A}{D}P+P\frac{A^TA}{D^TD}P-ie\,\mathrm{Sp}\left(\frac{F}{D}\right). \quad (2.23)$$

In the expression for B(s) the matrix form of recording has been utilized. The calculation of the explicit form of B(s) in a specific field presents a fairly awkward algebraic problem (cf. the Appendix). In the general case when one or both field invariants $\mathscr{F}$ and $\mathscr{G}$ (1.1) differ from zero we have

$$B(s)=\frac{d}{ds}\left(\beta(s)+\frac{i}{2}\ln\beta_0(s)-\frac{I_H^{(0)}}{|e|H}\frac{\pi}{2}\,\mathrm{sign}\,\xi_2\right), \quad (2.24)$$

where

$$\beta(s)=P^2us+\frac{I_H^{(0)}}{|e|H}a(x)-\frac{I_E^{(0)}}{|e|E}l(y),\quad \beta_0(s)=\frac{\eta_1\eta_2}{e^4E^2H^2},$$

$$\eta_1=\operatorname{ch} y-1+\left(\frac{1-u}{u}\right)^2\frac{y^2}{2}+\frac{1-u}{u}y\operatorname{sh} y,\quad y=2|e|Esu,$$
$$\eta_2=1-\cos x+\left(\frac{1-u}{u}\right)^2\frac{x^2}{2}+\frac{1-u}{u}x\sin x,\quad x=2|e|Hsu. \tag{2.25}$$

Here

$$E,\ H=\sqrt{(\mathscr{F}^2+\mathscr{G}^2)^{1/2}\pm\mathscr{F}},$$
$$a(x)=\operatorname{arctg} f_0,\quad f_0=\frac{\xi_1}{\xi_2},\quad \xi_1=1-\cos x,\quad \xi_2=\sin x+x\frac{1-u}{u},$$
$$l(y)=\frac{1}{2}\ln\frac{1+\varphi_0}{1-\varphi_0},\quad \varphi_0=\frac{\tau_1}{\tau_2},\quad \tau_1=\operatorname{ch} y-1,\quad \tau_2=\operatorname{sh} y+y\frac{1-u}{u}.$$
$$I_H^{(0)}=\frac{PF^2P-E^2P^2}{E^2+H^2},\quad I_E^{(0)}=\frac{PF^2P+H^2P^2}{E^2+H^2} \tag{2.26}$$

with $a(iy) = il(y)$. In the special case of a crossed field $\mathscr{F} = \mathscr{G} = 0$ the expression for B(s) can be obtained from (2.24) by a limiting transition, $\mathscr{F}, \mathscr{G} \to 0$:

$$\beta_c(s)=u(1-u)\left[P^2s-\frac{(su)^3}{3}(1-u)e^2PF^2P\right]. \tag{2.27}$$

In virtue of

$$[P^2, PF^2P]=0 \tag{2.28}$$

the operators P^2, $I_H^{(0)}$, $I_E^{(0)}$, appearing in B(s) commute with each other. Therefore the commutator $[B(s_1), B(s_2)] = 0$ and the solution of (2.22) can be represented in the form

$$L(s)=\frac{C}{\sqrt{\beta_0(s)}}\exp\left\{i\left[\beta(s)-\frac{I_H^{(0)}}{|e|H}\frac{\pi}{2}\operatorname{sign}\xi_2\right]\right\}. \tag{2.29}$$

In order to evaluate the constant C we consider the limit $s \to 0$, for which $\beta(s) \to 0$, $\sqrt{\beta_0(s)} \to 2s^2$, so that

$$L(s)=\frac{C}{2s^2}\exp\left(-i\frac{I_H^{(0)}}{|e|H}\frac{\pi}{2}\right). \tag{2.30}$$

But a direct evaluation of the integral (2.20) in this limit yields

$$L(s)=\int d^4k\, e^{isk^2}=-\frac{i\pi^2}{s^2}, \tag{2.31}$$

i.e.,

$$C=-2i\pi^2\exp\left(i\frac{I_H^{(0)}}{|e|H}\frac{\pi}{2}\right). \tag{2.32}$$

Now substituting (2.5), (2.17), (2.19), (2.29), and (2.32) into (2.4), we obtain the mass operator for a scalar particle

$$M^{(0)}=\frac{\alpha}{4\pi}\int_0^\infty ds\, s\int_0^1 du\, e^{-isum^2}Z\frac{2}{\sqrt{\beta_0(s)}}e^{i\tilde{\beta}(s)}, \tag{2.33}$$

where

$$\tilde{\beta}(s)=\beta(s)+i\frac{I_H^{(0)}}{|e|H}\pi\vartheta(-\xi_2),$$
$$Z=4P(1+A)P-2P(2+2A+A^T)\frac{A}{D}P$$
$$+P\frac{A^T}{D^T}(1+2A)\frac{A}{D}P-ie\operatorname{Sp}\left[(1+2A)\frac{F}{D}\right]. \tag{2.34}$$

The function Z represents a matrix form which can be transformed to the form (cf. the Appendix)

$$Z=\frac{1}{\eta_1}\left(\eta_3I_E^{(0)}+\frac{i\eta_5}{2su}\right)-\frac{1}{\eta_2}\left(\eta_4I_H^{(0)}-\frac{i\eta_6}{2su}\right), \tag{2.35}$$

where

$$\eta_3=\operatorname{ch} y-1+2\frac{1-u}{u}y\operatorname{sh} y+2\left(\frac{1-u}{u}\right)^2y^2\operatorname{ch} y,$$
$$\eta_4=1-\cos x+2\frac{1-u}{u}x\sin x+2\left(\frac{1-u}{u}\right)^2x^2\cos x,$$
$$\eta_5=\frac{1-u}{u}y^2(3-2\operatorname{ch} y)+y\operatorname{sh} y\left(1-2\frac{(1-u)^2}{u^2}y^2\right),$$
$$\eta_6=\frac{1-u}{u}x^2(3-2\cos x)+x\sin x\left(1+\frac{2(1-u)^2}{u^2}x^2\right); \tag{2.36}$$

where $\eta_{1,2}$ are defined in (2.25), with $\eta_{2n-1}(ix) = -\eta_{2n}(x)$. The case of the crossed field is obtained from (2.35) by limiting transition $\mathscr{G}, \mathscr{F} \to 0$:

$$\frac{2Z_c}{\sqrt{\beta_{0c}(s)}}=\frac{1}{s^2}\left\{P^2(2-u)^2+\frac{2i}{s}+PF^2Pu^2s^2e^2(1-u)\left(8-\frac{32}{3}u+\frac{13}{3}u^2-u^3\right)\right\}. \tag{2.37}$$

The mass operator $M^{(0)}$ (2.33) obtained above must be renormalized in the standard manner:

$$M_R^{(0)}=M^{(0)}-M^{(0)}(P^2=m^2, F=0)-(P^2-m^2)\,dM^{(0)}/dP^2(P^2=m^2, F=0). \tag{2.38}$$

As a result we obtain (cf., (2.25), (2.35))

$$M_R^{(0)}=\frac{\alpha}{4\pi}\int_0^\infty s\,ds\int_0^1 du\left\{\frac{2}{\sqrt{\beta_0(s)}}Ze^{-isum^2+i\tilde{\beta}(s)}-\frac{Z_0}{s^2}e^{-im^2u^2s}\right.$$
$$\left.-\frac{1}{s^2}(P^2-m^2)\left[(2-u)^2+isu(1-u)Z_0\right]e^{-im^2u^2s}\right\}, \tag{2.39}$$

where $z_0 = m^2(2-u)^2 + 2i/s$. The expression obtained for $M_R^{(0)}$ (2.38) represents a renormalized operator for a scalar particle with the external electromagnetic field taken into account exactly. We note that in the course of regularization the contact term drops out (cf., Fig. 1b).

The mass operator for a scalar particle has also been investigated by Schwinger[9] who started from the variational principle formulated by himself. In evaluating the integral (2.5) he went over to the operators k_μ and introduced the space conjugate to k_μ. The results of both approaches: (2.23), (2.34) and (2.27), (2.30), (2.35)[9] agree among themselves.

For physical applications of great interest is the average value of the operator $M_R^{(0)}$ on the mass shell ($P^2 = m^2$). In finding it one should have in mind that the operator P enters expression (2.39) only in the form of combinations $P^2 - m^2$ and $I_E^{(0)}$, $I_H^{(0)}$ (2.26) which commute with each other. Therefore one can always choose solutions Φ of the Klein-Gordon equation in an external field in such a manner that they would be eigenfunctions of these operators. It is very essential that the operators $I_H^{(0)}$ and $I_E^{(0)}$ turn out to be linearly dependent over the class of solutions Φ of the Klein-Gordon equation: $I_E^{(0)} = m^2 + I_H^{(0)}$. Therefore in order to evaluate the average value of the mass operator on the mass shell $\langle M_R^{(0)}\rangle$, it is sufficient to know the spectrum of the operator $I_H^{(0)}$, which can be taken from the well-known problem for the magnetic field. In the special system in which E ∥ H the following holds $I_H^{(0)} = P_\perp^2$, i.e., the spectrum of $I_H^{(0)}$ is determined by the commutation relations between the components of $P_\perp$ in this system which depends only on the magnetic field. As is well known, the finding of the spectrum is reduced to the problem of the harmonic oscillator[4)], i.e.,

$$I_H^{(0)}\Phi=P_\perp^2\Phi=(2n+1)|e|H\Phi.$$

Then for the evaluation of the average value $\langle M_R^{(0)}\rangle$ over these states it is sufficient in formula (2.39) to replace the operators $P^2 - m^2$, $I_H^{(0)}$, $I_E^{(0)}$ by their eigenvalues:

$$P^2-m^2\to 0,\quad I_H^{(0)}\to(2n+1)|e|H,\quad I_E^{(0)}\to m^2+(2n+1)|e|H. \tag{2.40}$$

By passing in the expression obtained for $\langle M_R^0\rangle$ to the limit $E \to 0$ ($y \to 0$), we obtain the average value of the mass operator in a purely magnetic field[5)] (this question was also considered in[11]):

$$\langle M_{RH}^{(0)}\rangle=\frac{\alpha}{4\pi}\int_0^\infty\frac{dx}{x}\int_0^1 du\, e^{-iH_0xu/2H}\cdot\times\left\{\frac{Z_H\delta^*}{\Delta}\exp\left(i2n\left[a(x)-\frac{ux}{2}\right]-\frac{iux}{2}\right)-Z_0\right\}, \quad (2.41)$$

where

$$H_0=\frac{m^2}{|e|},\quad \Delta=\frac{2u^2}{x^2}\eta_2,\quad \delta=1-u+\frac{iu}{x}(e^{-ix}-1),$$

$$Z_H=m^2(2-u)^2+(2n+1)|e|H\left\{(2-u)^2-1-\frac{1}{\Delta}\left[2u(1-u)\frac{\sin x}{x}+(1-u)^2(4\cos x-1)\right]\right\}+i\frac{2u|e|H}{x}\left\{1+\frac{1}{\Delta}\left[(1-u)(3-2\cos x)+u\frac{\sin x}{x}\left(1+\frac{2(1-u)^2x^2}{u^2}\right)\right]\right\}. \quad (2.42)$$

The quantity z_0 was defined in (2.39), a(x) was defined in (2.26). The imaginary part of this expression determines the total probability of radiation by a particle occupying a state n in a magnetic field of arbitrary intensity (cf.,[2]):

$$W^{(0)}=-\frac{1}{\bar\varepsilon}\operatorname{Im}\langle M_{RH}^{(0)}\rangle, \quad (2.43)$$

where $\bar\epsilon$ is the energy of the particle $\bar\epsilon^2 = m^2 + p + (2n + 1)|e|H$.

Of some interest is the average value of the mass operator in the ground state (n = 0) which characterizes the change in the energy of the ground state as a result of radiation corrections (cr., (2.41)):

$$\langle M_{RH}^{(0)}\rangle|_{n=0}=\frac{\alpha}{4\pi}\int_0^\infty\frac{dx}{x}\int_0^1 du\exp\left(-\frac{iH_0ux}{2H}\right)\left[\frac{1}{\Delta}Z_H(n=0)\delta^* e^{-iux/2}-Z_0\right]. \quad (2.44)$$

We make use of the fact that

$$\delta(x)=1-u+\frac{iu}{x}(e^{-ix}-1) \quad (2.45)$$

has no zeros in the region $\alpha > 0$, $\beta > 0$ of the complex variable $x = \alpha - i\beta$, since the condition $\delta = 0$ leads to the equation $\beta(1-u) + u(1-\cos\alpha e^{-\beta}) = 0$, which is not satisfied in this region. The following relation holds

$$\delta^*/\Delta=1/\delta, \quad (2.46)$$

where Δ is given by expression (2.42). It is also not difficult to verify that in terms containing the additional power of $\Delta = \delta\delta^*$ in the denominator, the factor δ^* cancels with the numerator, so that in the denominator of the mass operator only the terms δ and δ^2 occur. Therefore the contour of integration over x can be rotated through an angle $-\pi/2$. As a result we have

$$\langle M_{RH}^{(0)}\rangle|_{n=0}=\frac{\alpha}{4\pi}m^2\int_0^1 du\int_0^\infty\frac{dz}{z}e^{-u\lambda z}\left\{\left(\frac{1}{\delta_1}e^{-uz/2}-1\right)\left[(2-u)^2-\frac{2u}{\lambda z}\right]+\frac{1}{\lambda\delta_1}e^{-uz/2}\left[2(1-u)\left(1-\frac{e^{-z}}{\delta_1}\right)+\frac{u}{z}\left(1+\frac{uz}{2}-\frac{1}{\delta_1}\right)\right]\right\}, \quad (2.47)$$

where

$$\lambda=H_0/2H,\quad \delta_1=\delta(x=-iz)=1-u+(1-e^{-z})u/z.$$

This expression is real, as it ought to be, since there is no radiation from the ground state. As $H\to 0$ ($\lambda\to\infty$) we have (cf.,[11])

$$\langle M_{RH}^{(0)}\rangle|_{n=0,\lambda\to\infty}=\frac{\alpha}{3\pi}\frac{m^2}{\lambda^2}\left(\ln\lambda-\frac{7}{96}\right)+O\left(\frac{1}{\lambda^3}\right). \quad (2.48)$$

As $H\to\infty$ ($\lambda\to 0$) the principal contribution comes from the term in (2.47) which is proportional to $1/\lambda$. If one denotes the coefficient of $\alpha m^2/4\pi\lambda$ by J, then for J one can obtain a lower bound estimate: $J > 2$, with the energy of the ground state as $H\to\infty$ being $m^2 + |e|H(1 + \alpha J/2\pi)$.

We now also consider the crossed field $\mathcal{G} = \mathcal{F} = 0$, for which the only nonvanishing invariant is

$$\chi^2=\frac{e^2}{m^6}\langle PF^2P\rangle=-\frac{e^2}{m^6}\langle(F_{\mu\nu}P^\nu)^2\rangle. \quad (2.49)$$

The explicit form of $\langle M_{Rc}^{(0)}\rangle$ is obtained on substituting (2.27), (2.37) into (2.39):

$$\langle M_{Rc}^{(0)}\rangle=\frac{\alpha}{4\pi}m^2\int_0^\infty\frac{dz}{z}\int_0^1 du\left\{\exp\left(-\frac{iu}{\chi}\left[z+\frac{z^3}{3}(1-u)^2\right]\right)\left[(2-u)^2+z^2(1-u)\left(8-\frac{32}{3}u+\frac{13}{3}u^2-u^3\right)+\frac{2iu\chi}{z}\right]-\left[(2-u)^2+\frac{2iu\chi}{z}\right]e^{-iuz/\chi}, \quad (2.50)$$

where the replacement $sm^2u\chi = z$ has been made.

For the case when the dimensionless invariant χ^2 significantly exceeds the dimensionless field invariants

$$g^2=\frac{e^2}{m^4}\mathcal{G},\quad f^2=\frac{e^2}{m^4}\mathcal{F}, \quad (2.51)$$

in the zero order approximation one can also in the case of an arbitrary field retain in the mass operator terms with χ^2 having omitted terms containing f^2 and g^2. A procedure of this kind represents a transition to the quasiclassical approximation. As a result the mass operator in the quasiclassical approximation coincides with the mass operator for a crossed field.

The integrals over z in expression (2.50) can be evaluated directly if one utilizes the well-known integral representations for the Bessel functions of imaginary argument $K_\nu(\xi)$ (cf.,[19], p. 412, 984). For the total probability of radiation (2.43) we obtain as a result

$$W^{(0)}=\frac{\alpha}{\pi\sqrt3}\frac{m^2}{\bar\varepsilon}\int_0^1 du\left\{\frac{u}{1-u}\left(\frac{4}{3}-\frac{23}{12}u+\frac{3}{4}u^2\right)K_{2/3}(\xi)+\left(1-u+\frac{3}{4}u^2\right)\int_\xi^\infty K_{1/3}(z)\,dz\right\}, \quad (2.52)$$

where $\xi = 2u/3\chi(1-u)$, $\bar\epsilon$ is the energy of the particle.

Carrying out the transformation of the integrand utilizing the recurrence relations for the K-functions we obtain from (2.52) the well-known expression for the total probability for radiation by a scalar particle in a crossed field (in the quasiclassical approximation cf.,[2], p. 158)

$$W^{(0)}=\frac{\alpha}{\pi\sqrt3}\frac{m^2}{\bar\varepsilon}\int_0^\infty\frac{dy}{(1+y)^2}\int_{2y/3\chi}^\infty K_{5/3}(z)\,dz. \quad (2.53)$$

III. THE MASS OPERATOR FOR A SPINOR PARTICLE

We proceed to investigate the mass operator (Fig. 1a) for a particle of spin 1/2 (1.9) which we write in the form

$$M^{(1/2)}=-\frac{ie^2}{(2\pi)^4}\int d^4k\gamma^\mu\frac{1}{k^2+i\varepsilon}\frac{\hat P-\hat k+m}{(P-k)^2-m^2+{}^1/_2e\sigma F+i\varepsilon}\gamma_\mu, \quad (3.1)$$

where the notation has been introduced: $\sigma F = \sigma^{\nu\mu}F_{\mu\nu}$, $\sigma^{\mu\nu} = 1/2i[\gamma^\mu, \gamma^\nu]$. Carrying out the parametrization of the propagators, as in (2.2), we have

$$M^{(1/2)}=\frac{ie^2}{(2\pi)^4}\int_0^\infty s\,ds\int_0^1 du\exp\{-isu(m^2-i\varepsilon)\}\tilde M^{(1/2)}, \quad (3.2)$$

where

$$\tilde M^{(1/2)}=\int d^4k\,\gamma^\mu(\hat P-\hat k+m)\exp\left\{is\mathcal{H}+\frac{isu}{2}e\sigma F\right\}\gamma_\mu. \quad (3.3)$$

Since the addition to the index of the exponential in the integrand compared to the case of scalar particles (2.5) does not contain k_μ (and commutes with $\mathcal{H}$), then the integrals over k in (3.3) coincide with those calculated earlier for the case of scalar particles (2.17), (2.29). Utilizing this and also the expression

$$e^{isue\sigma F/2}\gamma_\mu e^{-isue\sigma F/2}=(1+A)_{\mu\nu}\gamma^\nu, \tag{3.4}$$

we obtain

$$M^{(1/2)}=-Y_1 e^{isue\sigma F/2}\frac{2\pi^2 i}{\sqrt{\beta_0(s)}}e^{\tilde\beta(s)}, \tag{3.5}$$

where $\beta_0(s)$, $\tilde\beta(s)$ are defined by formulas (2.25), (2.34), and

$$Y_1=\gamma^\mu\left[\hat P-\left(\frac{A}{D}P\right)_\lambda\gamma^\lambda+m\right](1+A)_{\mu\nu}\gamma^\nu. \tag{3.6}$$

As in the case of scalar particles, in future we shall be interested, in particular, in average values over the solutions of the Dirac equation in a field. Having this in mind it is convenient to represent the index of the exponential in formula (3.5) in a form which contains only operators commuting with $\hat P$ and among themselves (cf. the Appendix):

$$e^{isue\sigma F/2}e^{i\tilde\beta(s)}=Y_2 e^{i\rho(s)}, \tag{3.7}$$

where

$$\rho(s)=us\hat P^2+\frac{I_H^{(1/2)}}{|e|H}[a(x)+\pi\vartheta(-\xi_2)]-\frac{I_E^{(1/2)}}{|e|E}l(y),$$
$$Y_2=\frac{\operatorname{sign}\xi_2}{(1+f_0^2)^{1/2}}\frac{1}{(1-\varphi_0^2)^{1/2}}\left[1+\frac{ie\varphi_0}{2|e|(E^2+H^2)}(E\sigma F+H\sigma F^*)\right] \tag{3.8}$$
$$\times\left[1+\frac{1}{|e|}ief_0(H\sigma F-E\sigma F^*)/2(E^2+H^2)\right],$$

the quantities $a(x)$, $l(y)$, f_0, φ_0 are defined in (2.26). In the argument of the exponential in (3.7) we introduce the operators:

$$I_H^{(1/2)}=\frac{R^2-H^2\hat P^2}{E^2+H^2},\quad I_E^{(1/2)}=\frac{R^2+E^2\hat P^2}{E^2+H^2},$$
$$R=\frac{e}{|e|}\gamma^5 PF^*\gamma=\frac{e}{|e|}\gamma^5P^\mu F^*_{\mu\nu}\gamma^\nu, \tag{3.9}$$

where the operators $I_H^{(1/2)}$, $I_E^{(1/2)}$, R^2 commute among themselves and with the operator $\hat P$ and are expressed in terms of any one of them and $\hat P^2$, so that over the class of solutions ψ of the Dirac equation ($\hat P\psi = m\psi$) in order to obtain the eigenvalues of these operators it is sufficient to know the spectrum of one of them, for example $I_H^{(1/2)}$. The operator R represents the contraction of the tensor polarization operator $R_{\mu\nu}$ (introduced for free particles in[20]) with the tensor $eF^{\mu\nu}/2|e|$. In virtue of the fact that the average value in the states ψ is $\langle R\rangle = em\langle \delta F\rangle/2|e|$, $\langle R\rangle$ characterizes the spin portion of the interaction with the external field. Since $[\hat P, R] = 0$, the eigenvalues of the operator R can be utilized to classify the spin states of the electron in a field[6]. We introduce the solutions $\psi_\pm$ such that

$$R\psi_\pm=\pm\sqrt{R^2}\psi_\pm. \tag{3.10}$$

In evaluating the factor Y_1Y_2 in front of the exponential (cf., (3.6), (3.8)) it is useful to make use of the special system in which $\mathbf{E} \parallel \mathbf{H}$ (cf. the Appendix), and in doing so it is convenient to separate out terms of the type $\{\hat P-m, \Gamma\}$ (Γ is a combination of γ-matrices) and to utilize the relations

$$\left\{\frac{e}{|e|}\sigma F,\hat P\right\}=4R,\quad \left\{\frac{e}{|e|}\sigma F^*,\hat P\right\}=-4R_1, \tag{3.11}$$

where $R_1 = e\gamma^5 PF\gamma/|e|$, R has been defined in (3.9), $\{,\}$ denotes an anticommutator. After fairly awkward algebraic transformations we obtain

$$Y=Y_1Y_2=\frac{1}{(E^2+H^2)\sqrt{\eta_1\eta_2}}\Big\{\gamma F^2P\frac{1-u}{u}[x\tau_3-y\xi_3]+\hat P[H^2(\xi_3\operatorname{sh}y+\tau_3\xi_2)$$
$$+E^2(\tau_3\sin x+\xi_3\tau_2)]+iR_1\frac{1-u}{u}(Ey\xi_1\operatorname{ch}y-Hx\tau_1\cos x)$$
$$+iR\frac{1-u}{u}[Hy(\operatorname{ch}y-\cos x)-(xH+yE)\sin x\operatorname{sh}y]+\frac{i}{2}\{\hat P-m,V\}\Big\}, \tag{3.12}$$

where

$$V=\frac{e}{|e|}\sigma F\frac{1-u}{2u}[Ey\xi_2\operatorname{sh}y+Hx\tau_2\sin x]+\frac{e}{|e|}\sigma F^*\frac{1-u}{2u}\Big[\xi_1\tau_1(xH+yE)$$
$$+\frac{1-u}{u}xy(H\operatorname{sh}y-E\sin x)\Big]-\gamma^5(H^2+E^2)\Big[2\tau_1\xi_1$$
$$+\frac{1-u}{u}(x\tau_1\sin x+y\xi_1\operatorname{sh}y)\Big]+i(\xi_3\tau_2+\tau_3\xi_2)(E^2+H^2), \tag{3.13}$$
$$\xi_3=\sin x+\frac{1-u}{u}x\cos x,\quad \tau_3=\operatorname{sh}y+\frac{1-u}{u}y\operatorname{ch}y,$$

$\tau_{1,2}$ and $\xi_{1,2}$ have been defined in (2.26). Substituting (3.5), (3.7), (3.12) into (3.2) we obtain the quantity $M^{(1/2)}$ which must be renormalized by the standard method (cf., (2.38)):

$$M_R^{(1/2)}=M^{(1/2)}-M^{(1/2)}(\hat P=m,F=0)-(\hat P-m)\frac{dM^{(1/2)}}{d\hat P}(\hat P=m,F=0). \tag{3.14}$$

As a result we obtain the mass operator for a spinor particle in the general case of a homogeneous electromagnetic field represented as an explicit function of the field invariants E and H (2.26):

$$M_R^{(1/2)}=\frac{\alpha}{4\pi}\int_0^\infty s\,ds\int_0^1 du\Big\{\frac{2}{\sqrt{\beta_0}}Ye^{i(\rho(s)-usm^2)}-\frac{2m}{s^2}(1+u)e^{-im^2u^2s}$$
$$+(\hat P-m)\left[\frac{2(1-u)}{s^2}-4i\frac{m^2}{s}u(1-u^2)\right]e^{-im^2u^2s}\Big\}. \tag{3.15}$$

Of great interest for physical applications is the average value of $\langle M_R^{(1/2)}\rangle$ in states ψ which we choose so that they are eigenfunctions of the operators $(\hat P-m)$ and R (3.10). For evaluating averages of the operators R_1 and γF^2P appearing in (3.12) we utilize the relations

$$\{\gamma F^2\hat P,\hat P\}=2(R^2+2\mathscr{F}\hat P^2),\quad \{R_1,R\}=2\mathscr{G}\hat P^2, \tag{3.16}$$

where $\mathscr{F}$ and $\mathscr{G}$ are given in (1.1). Then the evaluation of $\langle M_R^{(1/2)}\rangle$ reduces (taking into account the relations (3.9) given above) to the determination of the spectrum of the eigenvalues of the operator[7] $I_H^{(1/2)}$. In the special reference system $I_H^{(1/2)} = -(\gamma P_\perp)^2 = P_\perp^2 - e\Sigma_3H$, the eigenvalues of this operator (depending only on the magnetic field) are $I_H^{(1/2)}\psi = 2n|e|H\psi$ (n = 0, 1 . . .). All the states with $n \neq 0$ are twofold degenerate. The ground state (n = 0) is nondegenerate, and for that state

$$\langle I_H^{(1/2)}\rangle_0=\langle P_\perp^2-e\Sigma_3H\rangle_0=0,\ \text{i.e.,}\ eH\langle\Sigma_3\rangle_0=\langle P_\perp^2\rangle>0.$$

On the other hand for this state $[R,\Sigma_3]\psi_0 = 0$, since $[iR,\Sigma_3]$ is a selfconjugate operator and $([iR,\Sigma_3])^2 \propto I_H^{(1/2)}$. As a result of this the state $\psi(n = 0)$ is an eigenfunction of the operator $e\Sigma_3$ with a positive eigenvalue. Since (cf., (3.10)) $\langle R\rangle = em\langle \delta F\rangle/2|e|$ or in the special system $\langle R\rangle = em\langle\Sigma_3H - i\alpha_3E\rangle/|e|$ and $\langle\alpha_3\rangle_0 = 0$ in the ground state, then $\langle R\rangle_0 = emH\langle\Sigma_3\rangle/|e|$ ($\langle\Sigma_3\rangle < 0$ for an electron, $\langle\Sigma_3\rangle > 0$ for a positron), i.e., in the ground state the eigenvalue of the operator R is positive.

The quantity $\langle M_R^{(1/2)}\rangle$ is given by expression (3.15), if in the latter we carry out the following replacements:

$$\hat P-m\to 0,\quad I_H^{(1/2)}\to 2n|e|H,\quad I_E^{(1/2)}\to m^2+2n|e|H,$$
$$\gamma F^2P\to\frac{1}{m}[m^2E^2+2n|e|H(E^2+H^2)], \tag{3.17}$$
$$R\to\pm[m^2H^2+2n|e|H(E^2+H^2)]^{1/2},\quad R_1\to\pm\frac{\mathscr{G}m^2}{[m^2H^2+2n|e|H(E^2+H^2)]^{1/2}}.$$

Passing in the expression obtained for $\langle M_R^{(1/2)}\rangle$ to the limit $E\to 0$ ($y\to 0$), we obtain the average value of the mass operator in a purely magnetic field[8] (this special case was also discussed in[10]):

$$\langle M_{RH}^{(1/2)}\rangle=\frac{\alpha}{2\pi}m\int_0^\infty\frac{dx}{x}\int_0^1 du\,e^{-iH_0ux/2H}\left\{\frac{1}{\Delta}\exp\left[2in\left(a(x)-\frac{xu}{2}\right)\right]\right.$$
$$\times\left[\left(2n\frac{H}{H_0}(1-u)-u\right)\left(1-\cos x+u\left(\cos x-\frac{\sin x}{x}\right)\right)+\left(1+u\frac{\sin x}{x}\right)\right.$$
$$\left.\left.+i\zeta u(1-u)\left(\frac{1-\cos x}{x}-\sin x\right)\left(1+2n\frac{H}{H_0}\right)^{1/2}\right]-(1+u)\right\},\quad (3.18)$$

where

$$\zeta=\frac{R}{\sqrt{R^2}}=\begin{cases}\pm1, & n\geqslant1,\\ +1, & n=0.\end{cases}$$

The imaginary part of this expression determines the total probability of radiation $W^{(1/2)}$ by a particle occupying the state n (cf., (2.43)):

$$W^{(1/2)}=-\frac{2m}{\bar{\varepsilon}}\operatorname{Im}\langle M_R^{(1/2)}\rangle,\quad (3.19)$$

where $\bar{\varepsilon}^2 = m^2 + p_{\|}^2 + 2n|e|H$.

The transition to the case of a purely electric field is carried out in the same manner as for scalar particles. The transition to the special case of a crossed field (the quasiclassical approximation) for which the well-known results are obtained (cf.,[10, 16, 2]) is carried out in a similar manner.

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APPENDIX

The transformation of matrix expressions of the type (2.23) can be conveniently carried out in the special reference system in which E ∥ H. We choose the common direction of these vectors as the third axis of Cartesian coordinates. Then the tensor F_μ can be represented in the following manner

$$F_{\mu\nu}=C_{\mu\nu}E+B_{\mu\nu}H,\quad (A.1)$$

where

$$C_{\mu\nu}=g_\mu^0 g_\nu^3-g_\mu^3 g_\nu^0\qquad B_{\mu\nu}=g_\mu^2 g_\nu^1-g_\mu^1 g_\nu^2.\quad (A.2)$$

Here $g^{\mu\nu}$ is the metric tensor. The tensors $C_{\mu\nu}$, $B_{\mu\nu}$ satisfy the following relations

$$(CB)_{\mu\nu}=0,\quad C_{\mu\nu}^2-B_{\mu\nu}^2=g_{\mu\nu},$$
$$(C^{2k+1})_{\mu\nu}=C_{\mu\nu},\quad (B^{2k+1})_{\mu\nu}=(-1)^k B_{\mu\nu},\quad (A.3)$$
$$(C^{2k})_{\mu\nu}=(C^2)_{\mu\nu},\quad (B^{2k})_{\mu\nu}=(-1)^{k+1}(B^2)_{\mu\nu};\ k\geqslant1.$$

The formulas (A.1)–(A.3) enable us to expand an arbitrary function of $F_{\mu\nu}$ in terms of the tensors $C_{\mu\nu}$, $(C^2)_{\mu\nu}$, $B_{\mu\nu}$, $(B^2)_{\mu\nu}$. We illustrate this on the example of the function e^{sF}:

$$(e^{sF})_{\mu\nu}=\left(\sum_{k=0}^\infty\frac{(sF)^k}{k!}\right)_{\mu\nu}=\sum_{k=0}^\infty\frac{s^{2k}}{(2k)!}(CE+BH)_{\mu\nu}^{2k}$$
$$+\sum_{k=0}^\infty\frac{s^{2k+1}}{(2k+1)!}(CE+BH)_{\mu\nu}^{2k+1}=g_{\mu\nu}+(C^2)_{\mu\nu}\sum_{k=1}^\infty\frac{(sE)^{2k}}{(2k)!}\quad (A.4)$$
$$-(B^2)_{\mu\nu}\sum_{k=1}^\infty\frac{(sH)^{2k}}{(2k)!}(-1)^k+C_{\mu\nu}\sum_{k=0}^\infty\frac{(sE)^{2k+1}}{(2k+1)!}+B_{\mu\nu}\sum_{k=0}^\infty\frac{(sH)^{2k+1}}{(2k+1)!}(-1)^k$$
$$=(C^2)_{\mu\nu}\operatorname{ch}(sE)-(B^2)_{\mu\nu}\cos(sH)+C_{\mu\nu}\operatorname{sh}(sE)+B_{\mu\nu}\sin(sH).$$

The division of the functions depending on the tensor $F_{\mu\nu}$ can be carried out simply if we take into account that the system of tensors $C_{\mu\nu}$, $(C^2)_{\mu\nu}$, $B_{\mu\nu}$, $(B^2)_{\mu\nu}$ is linearly independent and complete (in the special system). We consider, for example, the relation (cf., (A.1), (A.4))

$$\frac{e^{sF}-1}{sF}=\frac{C^2(\operatorname{ch}sF-1)+B^2(1-\cos sH)+C\operatorname{sh}sE+B\sin sH}{CsE+BsH},\quad (A.5)$$

which we seek in the form

$$\frac{e^{sF}-1}{sF}=\lambda_1C^2+\lambda_3C+\lambda_2B^2+\lambda_4B.\quad (A.6)$$

Multiplying both parts of (A.6) by the denominator sF and equating the coefficients in front of each of the tensors in both parts of (A.6), (A.5), we obtain

$$\lambda_1=\frac{\operatorname{sh}sE}{sE},\quad \lambda_2=-\frac{\sin sH}{sH},\quad \lambda_3=\frac{\operatorname{ch}sE-1}{sE},\quad \lambda_4=\frac{1-\cos sH}{sH},\quad (A.7)$$

which is the solution of the problem.

After carrying out the operations indicated above (A.4), (A.6), (A.7) we can return to the form of recording the results which is independent of the choice of the system of coordinates, by expression the tensors C, C^2, B, B^2 in terms of powers of F, F*:

$$C_{\mu\nu}=(EF_{\mu\nu}+HF_{\mu\nu}^*)/(H^2+E^2),\quad B_{\mu\nu}=(HF_{\mu\nu}-EF_{\mu\nu}^*)/(H^2+E^2),$$
$$(C^2)_{\mu\nu}=(F_{\mu\nu}^2+H^2g_{\mu\nu})/(H^2+E^2),\quad B_{\mu\nu}^2=(F_{\mu\nu}^2-E^2g_{\mu\nu})/(H^2+E^2),\quad (A.8)$$

where E, H are expressed in terms of $\mathcal{F}$, $\mathcal{G}$ in accordance with (2.26). We note that all the relations for functions of the dual tensor F* are obtained by the replacement E → H, H → −E from the corresponding expressions for F (for example, from (A.1), (A.4), (A.6), (A.7)).

We investigate the transformation of expressions containing γ^μ-matrices on the example of deriving formula (3.7). We start from (3.5), where the index of the exponential has the form

$$i\left[\left(P^2+\frac{1}{2}e\sigma F\right)su-\frac{I_E^{(0)}}{|e|E}l(y)+\frac{I_H^{(0)}}{|e|H}\tilde{a}(x)\right],\quad \tilde{a}(x)=a(x)+\pi\vartheta(-\xi_2).\quad (A.9)$$

The operators $I_H^{(0)}$, $I_E^{(0)}$ do not commute with $\hat{P}$. In the special reference system $I_E = PC^2P$, $I_H = PB^2P$, while their commutators with $\hat{P}$ are

$I_E = PC^2P$, $I_H = PB^2P$, and their commutators with P are

$$[PB^2P,\hat{P}]=-2ieH(\gamma BP),\quad [PC^2P,\hat{P}]=2ieE(\gamma CP);\quad (A.10)$$

and from this it is clear that for the formation of operators commuting with $\hat{P}$ it is necessary to add to $I_H^{(0)}$, $I_E^{(0)}$ appropriate spin terms. We evaluate the following commutators:

$$[\sigma C,\hat{P}]=-4i(\gamma CP),\quad [\sigma B,\hat{P}]=-4i(\gamma BP).\quad (A.11)$$

From formulas (A.10), (A.11) it can be seen that the operators $PC^2P + 1/2eE\sigma C$, $PB^2P - 1/2eH\sigma B$ commute with $\hat{P}$, i.e., are the desired operators. Utilizing (A.8), we obtain in an arbitrary reference system:

$$PC^2P+\frac{1}{2}eE\sigma C=\frac{1}{E^2+H^2}(R^2+E^2\hat{P}^2)=I_E^{(1/2)},$$
$$PB^2P-\frac{1}{2}eH\sigma B=\frac{1}{E^2+H^2}(R^2-H^2\hat{P}^2)=I_H^{(1/2)}.\quad (A.12)$$

Now formula (A.9) assumes the form

$$\exp\left\{i\left(\tilde{\beta}(s)+\frac{1}{2}sue\sigma F\right)\right\}=\exp\left(i\frac{e}{|e|}\Sigma_3\tilde{a}(x)\right)\exp\left(\frac{e}{|e|}\alpha_3 l(y)\right)$$
$$\times\exp\left\{i\left[\hat{P}^2us+\frac{I_H^{(1/2)}\tilde{a}(x)}{|e|H}-\frac{I_E^{(1/2)}l(y)}{|e|E}\right]\right\}.\quad (A.13)$$

The exponential factors occurring on the right hand side can be easily evaluated, and the answer can be written with the aid of (A.8) in an arbitrary reference system:

$$\exp\left(i\frac{e}{|e|}\Sigma_3\tilde{a}(x)\right)=\cos\tilde{a}(x)+i\frac{e}{|e|}\Sigma_3\sin\tilde{a}(x)$$

$$= \frac{\operatorname{sign}\xi_2}{\sqrt{1+f_0^2}}\left(I + i\frac{e}{|e|}f_0\Sigma_3\right) = \frac{\operatorname{sign}\xi_2}{\sqrt{1+f_0^2}}\left\{1 + \frac{ief_0[H\sigma F - E\sigma F^*]}{2|e|(E^2+H^2)}\right\},$$

$$\exp\left(\frac{e}{|e|}\alpha_3 l(y)\right) = \operatorname{ch} l(y) + \frac{e}{|e|}\alpha_3 \operatorname{sh} l(y) \qquad (A.14)$$

$$= \frac{1}{\sqrt{1-\varphi_0^2}}\left(I + \frac{e}{|e|}\varphi_0\alpha_3\right) = \frac{1}{\sqrt{1-\varphi_0^2}}\left\{1 + \frac{ie\varphi_0[E\sigma F + H\sigma F^*]}{2|e|(E^2+H^2)}\right\}.$$

On substituting formula (A.14) into (A.13) we obtain (3.7).

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1)The system of units $\hbar = c = 1$, and the metric $ab = a^0b^0 - \mathbf{ab}$ have been utilized.

2)In this case $[P^\alpha, F^{\mu\nu}] = 0$.

3)Both here and frequently in subsequent discussion we shall omit the vector indices, and this corresponds to a transition to a matrix notation, for example $PAk = P^\mu A_{\mu\nu}k^\nu$.

4)One should keep in mind that the functions Φ by no means reduce to the solutions in a purely magnetic field, but, as has been noted already, in the present approach we need not know the explicit form of the solutions.

5)For transition to the case of a purely electric field it is necessary to fix $(2n + 1)\,|e|H$ which is the special system is the eigenvalue of $P^2_\perp$, and after that to let $H \to 0$ $(x \to 0)$. In this case the spectra of $I_E^{(0)}$ and $I_H^{(0)}$ become continuous.

6)For example, in the case of a purely magnetic field, when $p_{||} = 0$, $R \to e\epsilon\gamma^0(\Sigma H)/|e|$.

7)A similar situation existed in the case of particles of spin 0.

8)The ground state (n = 0) for (3.18) was discussed in a number of paper [21–23].

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