Wave transmittance for a thick layer of a randomly inhomogeneous medium

L. A. Pastur and É. P. Fel'dman

Institute of Radiophysics and Electronics, Ukrainian Academy of Sciences
(Submitted February 18, 1974)
Zh. Eksp. Teor. Fiz. **67**, 487–493 (August 1974)

The one-dimensional problem of calculating the transmission coefficient for a wave traversing a nonabsorbing, random-inhomogeneous medium is considered. Upper and lower bounds are obtained for the coefficient, and it is shown that it decreases exponentially with the layer thickness.

The propagation of waves in random-inhomogeneous media is a problem of considerable interest, and a rather large number of papers have been devoted to various aspects of it (see, for example, [1,2]). The difficulties that arise in the solution of this problem are necessary to study the one-dimensional case. The present paper also tends in this direction. It considers the average of the wave transmittance through a layer of a random-inhomogeneous medium over several realizations, which characterizes the fraction of the energy of the wave which passes through such a layer. We shall obtain upper and lower bounds for this quantity and show that it falls off exponentially with the thickness of the layer. The exponential dependence of the transmittance D on the layer thickness L has already appeared in papers by several authors (for example, [2]). However, some small parameter ϵ was necessarily present in all of these papers (for example, the dimensionless amplitude of the fluctuations) and the expressions obtained for D were in fact approximations in terms of this parameter. This led to the result that the exponential dependence of D on L was actually obtained not for all sufficiently large L, but only for $L \leq L(\epsilon)$, where $L(\epsilon) \to \infty$ as $\epsilon \to 0$ (for example, $L(\epsilon) \sim \epsilon^{-2}$). It is clear that such formulas cannot give an answer to the question of the behavior of D as $L \to \infty$ at fixed ϵ.

We also note that our result is connected in definite measure with the currently discussed problem [3,4] of the disappearance of the static conductivity of one-dimensional systems.

As a model of the random-inhomogeneous medium, we used a random process which is a sequence of rectangular wells and barriers of the same height V_0, with lengths which are independent random quantities, the density distributions of which have the forms $n_0 \exp(-n_0 x)$ and $n_1 \exp(-n_1 x)$. We immediately note that the considerations used below are not specific for this case and also allow us to establish the exponential decay of the transmittance for other models (see the end of the article for more details on this).

Thus, we consider the equation

$$u''+(k^2-V(x))u=0. \tag{1}$$

Here $V(x)=0$ at $x<0$ and $x>L$, and the intermediate region $0 \leq x \leq L$ it is a random real function of the coordinates. We are interested in the solution of this equation which has the form

$$e^{ikx}+Be^{-ikx} \text{ at } x<0, \quad Ae^{ikx} \text{ at } x>L;$$

more precisely, the quantity $D=\overline{|A|^2}$, where the bar indicates averaging over the realizations of the random function V(x). We denote by $s(x)(c(x))$ a solution of Eq. (1) for $x>0$ which satisfies the conditions $s(0)=0$, $s'(0)=k$ at the point $x=0$ ($c(0)=1$, $c'(0)=0$). Requiring continuity of the logarithmic derivative of the function u at the points 0 and L, we find that

$$|A|^2=4(\rho_c^2+\rho_s^2+2)^{-1},$$

where

$$\rho_c^2=c^2+k^{-2}c'^2, \quad \rho_s^2=s^2+k^{-2}s'^2.$$

Therefore

$$D=4\int_{-\infty}^{\infty}\frac{p_L(x,y)\,dx\,dy}{e^{2Lx}+e^{2Ly}+2},$$

where $p_L(x, y)$ is the joint probability density of the random quantities

$$\xi_L=\frac{1}{L}\ln\rho_c(L), \quad \eta_L=\frac{1}{L}\ln\rho_s(L).$$

It is not possible to find an explicit expression for the function $p_L(x, y)$. Therefore, we shall bound D above and below by values which are amenable to analysis.

Obviously,

$$D\leqslant 4\int_{-\infty}^{\infty}\frac{p_L(x)\,dx}{e^{2Lx}+2}, \quad p_L(x)\equiv\int_{-\infty}^{\infty}p_L(x,y)\,dy; \tag{2}$$

$p_L(x)$ is the distribution density of the random quantity ξ_L. As will be shown below, we have $\xi_L \to I>0$ as $L\to\infty$. It is therefore natural to estimate the right-hand side of (2) thus:

$$\int_{-\infty}^{\infty}\frac{p_L(x)\,dx}{e^{2Lx}+2}=\int_{-\infty}^{I-\varepsilon}+\int_{I-\varepsilon}^{\infty}\leqslant\frac{1}{2}\int_{-\infty}^{I-\varepsilon}p_L(x)\,dx$$
$$+e^{-2L(I-\varepsilon)}\int_{I-\varepsilon}^{\infty}p_L(x)\,dx\leqslant Pr\{\xi_L\leqslant I-\varepsilon\}+e^{-2L(I-\varepsilon)}.$$

If we now show that the first term in this inequality, like the second, is bounded by a quantity of the form $\exp(-\alpha(\epsilon)L)$, where $\alpha(\epsilon)>0$ for any $0<\epsilon<I$, and we then so choose ϵ that we guarantee identical rates of decay of these exponents, then the required bound will be obtained.

Thus the problem reduces to proof of the relations

$$\varlimsup_{L\to\infty}\xi_L=I>0, \quad Pr\{\xi_L\leqslant I-\varepsilon\}\leqslant e^{-\alpha(\varepsilon)L}, \quad L\to\infty.$$

We note that the positiveness of the quantity I is closely connected with the localization of the eigenfunctions of Eq. (1), which has been discussed in a number of papers (see, for example, [3]).

We introduce the phase of the function u(x), which is a real solution of Eq. (1) for $x>0$, with the help of the relation $u'/ku=\tan\theta$. From (1), we then obtain an equation for θ:

$$\frac{d\theta}{dx}=-k\left(1-\frac{V(x)}{k^2}\cos^2\theta(x)\right) \tag{3}$$

and the relation

$$\frac{d\rho}{dx}=\frac{V(x)}{2k}\rho\sin 2\theta(x),$$

whence

$$\xi_L=\frac{1}{2kL}\int_0^L V(x)\sin 2\theta(x)\,dx.$$

Therefore, as $L\to\infty$

$$\bar{\xi}_L=\frac{1}{2k}\int V\sin 2\theta p(V,\theta)\,dV\,d\theta, \tag{4}$$

where $p(V, \theta)$ is the joint density distribution of the random quantities $V(x), \theta(x)$ as $x\to\infty$.

As has already been shown in the Introduction, we shall assume that $V(x)=V_0r(x)$, where $r(x)$ takes on two values alternately: 0 and 1, on intervals whose lengths are independent random variables with given distribution densities. Such a random function represents a Markov process, i.e., its probability properties for all $y>x$ are uniquely determined by the value that it has at the point x, and do not depend on the values at the points previous to x. Since θ is a solution of the first-order equation (3), the pair $\theta(x)$, $V(x)$ will also be a Markov process. We denote by $p(x, V, \theta|V_0, \theta_0)$ the joint probability density of the random quantities $V(x)$ and $\theta(x)$ under the condition that at the point $x=0$ they are respectively equal to V_0 and θ_0. This function is a π-periodic solution of the Fokker-Planck equation [5]

$$\frac{\partial p(x,V,\theta|V_0,\theta_0)}{\partial x}=k\frac{\partial}{\partial\theta}\left[\left(1-\frac{V}{k^2}\cos^2\theta\right)p\right]+A_Vp \tag{5}$$

with the initial condition

$$p(0,V,\theta|V_0,\theta_0)=\delta_{VV_0}\delta(\theta-\theta_0).$$

The quantity A_V denotes an operator which acts only on the variable V; its specific form will be given below.

The function $p(V, \theta)$ is the limit of the solution of this equation as $x\to\infty$; it was calculated previously.[5] From the formulas obtained in [5], it is difficult to perceive directly the strict positiveness of the quantities (4). It turns out, however, that this fact can be proved even without finding the explicit solution of the Fokker-Planck equation (5).

We note that by virtue of (3) we have the identity

$$\frac{V}{2k}\sin 2\theta=k\operatorname{tg}\theta-\frac{d}{dx}\ln\cos\theta,$$

and can therefore write the following expression for $\bar{\xi}_L$ in place of (4):

$$\bar{\xi}_L=k\int\operatorname{tg}\theta p(V,\theta)\,dV\,d\theta.$$

For what follows, it is convenient to introduce the variable $z=\tan\theta$. Then

$$\bar{\xi}_L=k\int zp(z,V)\,dz\,dV=k\int_{-\infty}^{\infty}zp(z)\,dz.$$

Thus, $\bar{\xi}_L$ is simply the mean value of the quantity z multiplied by k. But then

$$\bar{\xi}_L=k\operatorname{Im}\int\psi_s'(0,V)\,dV,\qquad \psi(s,V)=\int_{-\infty}^{\infty}e^{isz}p(z,V)\,dz.$$

The stationary probability density $p(V, \theta)$ is a solution of Eq. (5), in which the left side is zero instead of $\partial p/\partial x$. Therefore, $p(z, V)$ satisfies the equation

$$k\frac{\partial}{\partial z}\left[\left(1+z^2-\frac{V}{k^2}\right)p(z,V)\right]A_Vp=0. \tag{6}$$

Performing a Fourier transformation in z on (6), we obtain an equation for $\psi(s, V)$:

$$-k\frac{\partial^2\psi}{\partial s^2}+\left(k-\frac{V}{k}\right)\psi-\frac{1}{is}A_V\psi=0, \tag{7}$$

and we now need to show that

$$\operatorname{Im}\int\psi_s'(0,V)\,dV>0.$$

For this purpose, we multiply (7) by ψ^* and take the imaginary part of the result. We then obtain the relation

$$k\frac{\partial}{\partial s}\left(\frac{\partial\psi}{\partial s}\psi^*-\psi\frac{\partial\psi^*}{\partial s}\right)+\frac{2\operatorname{Re}(\psi A_V\psi^*)}{is}=0,$$

which, after integration over s from 0 to ∞, gives

$$k\psi(0,V)\operatorname{Im}\psi_s'(0,V)=-\int_0^\infty\frac{ds}{s}\operatorname{Re}(\psi A_V\psi^*).$$

Since $\psi(0, V)$ is $p(V)$—the probability that $V(x)=V$, then the latter relation, after division by $p(V)$ and summation over V, leads to the expression

$$\bar{\xi}_L=-\int_0^\infty\frac{ds}{s}\sum_V\frac{\operatorname{Re}(\psi A_V\psi^*)}{p(V)}. \tag{8}$$

If we now use the explicit expressions for A_V and $p(V)$ [5]

$$V=V_0r(r=0,1),\quad A_r\psi=-n_r\psi_r+n_{1-r}\psi_{1-r},$$
$$p_r=n_{1-r}/(n_0+n_1),$$

it turns out that the right-hand side of (8) is

$$\frac{n_0+n_1}{n_0n_1}\int_0^\infty\frac{ds}{s}|n_0\psi(s,0)-n_1\psi(s,1)|^2\geqslant0,$$

and the equality is possible only at $V_0=0$. Thus, the positiveness of $\bar{\xi}_L$ $(L\to\infty)$ is proved.

Modifying somewhat the discussion given above, we can show that all the eigenvalues of the operator in the right-hand side of (5) have negative real parts, namely:

$$\operatorname{Re}\lambda=-\int_0^\infty\frac{ds}{s}|n_0\psi_\lambda(s,0)-n_1\psi_\lambda(s,1)|^2\times\left[\int_0^\infty\frac{ds}{s}(n_0|\psi_\lambda(s,0)|^2+n_1|\psi_\lambda(s,1)|^2)\right]^{-1}$$

(ψ_λ is the eigenfunction which corresponds to the eigenvalue λ). An exception is the eigenvalue that is equal to zero. The eigenfunction corresponding to it is none other than the stationary probability density $p(V, \theta)$. Consequently, the solution $p(x, V, \theta|V_0, \theta_0)$ of Eq. (5) will behave as

$$p(V,\theta)+O(\exp(-|\operatorname{Re}\lambda_*|x))$$

as $x\to\infty$, where λ_* is the eigenvalue whose real part is smallest in absolute value. Therefore, the correlation function of the process $V(x)\sin 2\theta(x)$, which is equal to

$$\sum_{V,V_0}\iint_{-\pi/2}^{\pi/2}V\sin 2\theta V_0\sin 2\theta_0 p(x,V,\theta|V_0,\theta_0)p(V_0,\theta_0)\,d\theta\,d\theta_0-\left(\sum_V\int_{-\pi/2}^{\pi/2}V\sin 2\theta p(V,\theta)\,d\theta\right)^2,$$

will decay at large x as $\exp(-|\operatorname{Re}\lambda_*|x)$. In other words, the correlation of the values of the process $V(x)\sin 2\theta(x)$ at widely spaced points will be exponentially small. We shall make use of this fact in what follows.

We now proceed to an estimate of the probability of large deviations, i.e., values of $\Pr\{\xi_L\leq I-\epsilon\}$. For this purpose, we consider the function

$$F_L(\beta)=\langle\exp(-\beta L\xi_L)\rangle,\quad \beta>0$$

(the angle brackets, like the bar over a letter, indicate averaging). We have the following chain of inequalities:

$$F_L(\beta)=\int_{-\infty}^{\infty}e^{-\alpha Lx}p_L(x)\,dx\geqslant\int_{-\infty}^{I-\varepsilon}e^{-\beta Lx}p_L(x)\,dx\geqslant e^{-\beta L(I-\varepsilon)}Pr\{\xi_L\leqslant I-\varepsilon\}.$$

It then follows that

$$Pr\{\xi_L \leqslant I-\varepsilon\} \leqslant \min_\beta e^{\beta L(I-\varepsilon)} F_L(\beta).$$

But, as $L \to \infty$, $F_L(\beta)$ behaves as $\exp(Lf(\beta))$, where $f(\beta)$ is a sufficiently smooth function of the parameter β. Actually, $F_L(\beta)$ is equal to

$$\left\langle \exp\left(-\frac{\beta}{2k}\int_0^L V(x)\sin 2\theta(x)\,dx\right)\right\rangle,$$

and is analogous to the partition function of a classical one-dimensional system, while β plays the role of the reciprocal temperature and the integral the role of the configurational energy. But, as is well known, in one-dimensional systems with sufficiently rapid decay (at large distances) of the potential of pair interaction, phase transitions are absent. In our case, this should mean that the function $f(\beta)$ has at least two continuous derivatives, inasmuch as the correlations of the process values $V(x)\sin 2\theta(x)$ are exponentially small at distant points (the value of these correlations is the exponent of the "short-range action" in our system).

The discussions developed here are not simply discussions by analogy. Actually, we can convert them to a rigorous proof, which employs the apparatus of the theory of Markov processes and which in its basic points is similar to the proof of the absence of phase transitions in one-dimensional systems (for the latter, see, for example, [6]).

Thus, we can write down the statements that

$$Pr\{\xi_L \leqslant I-\varepsilon\} \leqslant e^{-L\alpha(\varepsilon)}, \quad \alpha(\varepsilon) = -\min_\beta[\beta(I-\varepsilon)+f(\beta)].$$

It follows from the definition of the function $f(\beta)$ that

$$f(0)=0, \quad f'(0)=-I<0, \quad f''(0)=L(\overline{\xi_L^2}-\overline{\xi}_L^2)>0$$

(the last inequality corresponds to the thermodynamic stability inequalities). Therefore $-\alpha(\epsilon)$ is strictly negative. For

$$\varepsilon \ll 3f''^2(0)/f'''(0) \tag{9}$$

this quantity is of the order of $-\epsilon^2/2f''(0)$, and

$$f''(0)=\int_0^\infty B(x)\,dx,$$

where $B(x)$ is the correlation function of the process

$$V(x)\sin 2\theta(x)/2k.$$

Thus, we finally obtain the estimate for the transmission coefficient:

$$D \leqslant e^{-L\alpha(\varepsilon)}+e^{-2L(I-\varepsilon)} \leqslant \exp(-L\min\{\alpha(\varepsilon), 2(I-\varepsilon)\}), \tag{10}$$

which also indicates its exponential decay as $L \to \infty$. In order to obtain an idea of the order of magnitude of the exponent, we consider the case of large energies of the incident wave ($k \to \infty$). In this case, the function $(\theta(x))$ can be found from Eq. (3) with the help of perturbation theory in the quantity V/k^2, and this in turn allows us to calculate the quantities I and $f''(0)$ in the first nonvanishing order. It turns out that

$$I \approx \frac{1}{4k^2}\int_0^\infty B_0(x)\cos kx\,dx = \frac{V_0^2 n_0 n_1}{16k^4(n_0+n_1)}, \quad f''(0)=I,$$

where

$$B_0(x)=\frac{V_0^2 n_0 n_1}{(n_0+n_1)^2}\exp[-(n_0+n_1)|x|]$$

is the correlation function of the process $V = V_0 r(x)$. If we now choose ϵ in (9) so that the equality $\alpha(\epsilon) = 2(I-\epsilon)$ is satisfied, it then turns out that as $k \to \infty$ the exponent in this formula is approximately equal to $-0.36IL$ (here $\epsilon \approx 0.82I$). It must be noted that the criterion (9) transforms in this case to the form $f'''(0) \ll I$ and is definitely satisfied, inasmuch as $I \sim (V_0/k^2)^2$, and $f'''(0)$ in the first nonvanishing order in V_0/k^2 is $(V_0/k^2)^4$.

The upper bound for the transmission coefficient can be obtained in the following fashion. For each realization [7]

$$\lim_{x\to\infty}\frac{\rho(x)}{x}=I,$$

where $\rho(x)$ is either of the two quantities $\rho_c(x)$, $\rho_s(x)$. Since for an arbitrary realization

$$\ln|A(L)|^2=\ln 4-\ln(\rho_c^2(L)+\rho_s^2(L)+2),$$

we have

$$\lim_{L\to\infty} d_L=2I, \quad d_L \equiv -\frac{1}{L}\ln|A(L)|^2. \tag{11}$$

Now, taking into account the inequality between the geometric and arithmetic means, we can write down the result that

$$D_L=\langle|A(L)|^2\rangle=\langle\exp(-Ld_L)\rangle \geqslant \exp(-L\bar{d}_L).$$

Further, in accord with (11), $d_L \to 2I$ as $L \to \infty$ for each realization, and therefore $\bar{d}_L \to 2I$ as $L \to \infty$. We have finally, therefore, $D_L \gtrsim \exp(-2LI)$.

In conclusion, we note that the method of estimation of D that we have used is applicable over a wide range of cases. As possible examples, we note the following: $V(x) = V_0\Sigma\delta(x-x_j)$, where the distances between neighboring points x_j are independent random quantities with sufficiently rapidly decaying probability distribution densities; $V(x)$ is an Ornstein-Uhlenbeck process, which is a Gaussian random process with the correlation function $de^{-\beta x}/2\beta$; $V(x) = V_0\tilde{r}(x)$, where $\tilde{r}(x)$, as before, takes on the alternating values 0 and 1 on intervals whose lengths are independent random quantities with a sufficiently rapidly decaying probability density, which is no longer necessarily an exponential.

Finally, we point out that a similar method can be used for the study of wave processes in one-dimensional lattices.

[1]Yu. N. Barabanenkov, Yu. A. Kravtsov, S. M. Rytov and V. I. Tatarskiĭ, Usp. Fiz. Nauk **102**, 3 (1970) [Sov. Phys.-Usp. **13**, 551 (1971)].

[2]J. B. Keller, Wave Propagation in Random Media, Proc. of Symp. on Appl. Math., Amer. Math Soc., Providence, R.I., p. 227-246.

[3]N. Mott, Electrons in disordered structures (Russian translation, Mir Press, 1969).

[4]V. L. Berezinskiĭ, Zh. Eksp. Teor. Fiz. 65, 1251 (1973) [Sov. Phys.-JETP **38**, 620 (1974).

[5]M. M. Benderskiĭ and L. A. Pastur, ibid. 57, 284 (1969) [**30**, 158 (1970)].

[6]D. Ruelle, Statistical Mechanics, Benjamin, 1969 (Russian translation, Mir Press, 1971).

[7]M. M. Benderskiĭ and L. A. Pastur, in Collection: Teoriya funktsiĭ, funktsional'nyĭ analiz i ikh prilozheniya (Theory of Functions and Functional Analysis, and their Applications), No. 21, Kharkov State University Press (in press).

Translated by R. T. Beyer
55