Coalescence of wave pulses or beams in explosive instability

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A theoretical analysis is given of the coalescence of pulses or beams of parametrically coupled waves in a nonequilibrium medium exhibiting explosive instability. The effect consists essentially of the formation of an overlap region for the wave fields in which the fields increase in explosive fashion. An analytic solution is obtained for the two-wave process, whilst the three-wave interaction is investigated numerically. The coalescence of pulses is characterized qualitatively by the fact that the explosion time may exceed the time necessary for the transit of noninteracting pulses through one another.

It is well known that explosive instability is one of the most striking effects which appear during the interaction between parametrically coupled waves in nonlinear nonequilibrium media. It is manifested by the simultaneous growth in the amplitude of all the interacting waves and, under the usual simplifying assumptions, is found to make all these amplitudes infinite within a finite time. This instability appears, for example, during the interaction between waves with different signs of energy in transparent nonequilibrium media, [1,2] or between waves with positive energy in media for which the imaginary part of the permittivity is proportional to the field perturbations. [3,4] Explosive instability has recently been discussed in the literature where analyses were given of both particular realizations, including nonequilibrium plasmas, and general properties, using different model equations (see, for example, [1,2,5,6] and [7]). However, until quite recently, all the published papers were devoted to the explosive interaction between spatially homogeneous wave fields (modes). From the practical standpoint, the most interesting is the explosive instability of spatially localized fields. Some exact solutions of this problem in the case of one-dimensional resonance interaction between three waves were found for the first time in [8]. The physical discussion was essentially confined to noting the existence of the explosion (collapse) for spatially localized solutions. It is, however, difficult to follow the evolution of the explosion within the framework of this approach as a function of changes in the prescribed initial disturbances.

In this paper, we abandon the one-dimensional solution and consider the explosive instability in the case of a degenerate three-wave process. We use a numerical analysis to investigate the explosive interaction during the collision of two wave pulses, the narrow spectra of which form part of a resonance triplet. It is shown that the interaction between wave packets or beams of parametrically coupled waves in a nonequilibrium medium (the solutions of these problems follow one from the other because of the space-time analogy) differs fundamentally from the parametric interaction usually investigated in nonlinear optics. [9-11] Here, we have a qualitatively new effect, namely, the coalescence of pulses of interacting waves. During the interaction of beams propagating at an angle to one another, this effect involves the localization of the fields of the resonance-coupled waves along one beam[1)].

We note that, during the realization of the explosive interaction between parametrically coupled waves in a nonequilibrium medium, the existence of the coalescence effect in fact removes the restrictions on the group velocities of the waves, which are necessary in the equilibrium medium in order to ensure that the change in the integrated intensities is substantial. [9,10]

1. The equation for the normalized amplitudes of parametrically coupled waves in a medium with explosive instability, in which dispersion within the narrow spectral packets is neglected, can be written in the form

$$\partial a_{1,2}/\partial t+\mathbf{v}_{1,2}\nabla a_{1,2}=a_{2,1}a_3,\quad \partial a_3/\partial t+\mathbf{v}_3\nabla a_3=a_1a_2 \tag{1}$$

where $a_j = A_j e^{i\varphi_j}$ and the initial conditions are

$$a_j(x, y, z, 0)=a_{j0}(x, y, z). \tag{2}$$

The problem of the stationary interaction of beams specified on a plane boundary can be reduced to the one-dimensional variant of (1) and (2) because of the space-time analogy. [9]

It is readily seen that the solution of the problem defined by (1) and (2) has a number of simple properties: (1) if $a_j = a_j(\mathbf{r}, t)$ is a solution of (1) and (2), then $\bar{a}_j = \beta a_j(\beta\mathbf{r}, \beta t)$ is a solution of (1) with initial conditions $\bar{a}_j(\mathbf{r}, 0) = \beta a_{j0}(\beta\mathbf{r})$ (β = const). Hence follows, in particular, that the behavior of short large-amplitude pulses is similar to the behavior of certain long pulses of small amplitude; (2) problems on the interaction of fast pulses of large amplitudes and slow waves with small amplitude are also similar: if $a_j = a_j(\mathbf{r}, t)$ is the solution of (1) and (2), then $\bar{a}_j = \beta a_j(\mathbf{r}, \beta t)$ is a solution of (1), where $\mathbf{v}_j \to \beta\mathbf{v}_j$, with initial conditions $a_j(\mathbf{r}, 0) = \beta a_{j0}(\mathbf{r})$; (3) for spatially localized fields, we have from (1) the following integrals:

$$N_1-N_2=C_1,\quad N_1-N_3=C_2,\quad N_2-N_3=C_3 \tag{3}$$

($N_j = \int |a_j|^2 d\mathbf{r}$), which indicate that the numbers of quanta in the individual pulses increase or decrease simultaneously, just as in the case of spatially homogeneous fields. We must also note that spatially homogeneous solutions can be used in the analysis of localized perturbations if $\mathbf{v}_j = \mathbf{v}$, and also in the case of initially partially overlapping perturbations, if in the region of overlap $a_{j0} \approx$ const and the size of this region is $L \gg \tau_{nl}|\Delta\mathbf{v}|$, where $\Delta\mathbf{v}$ is the group desynchronization and τ_{nl} is the time of the nonlinear interaction.

2. Consider the explosive instability in the case of a degenerate three-wave process, namely, the interaction of a wave with the second harmonic.

This process can be realized in particular in the case of electromagnetic waves when the medium exhibits an inversion of populated two-level particles. If the population difference ΔN and the transverse relaxation time T_2

are large enough, the energy of the electromagnetic wave of frequency ω close to the transition frequency ω_0 may be negative.[12] In the case of a quadratic reactive nonlinearity in the medium (polarization proportional to the square of the field) and the corresponding dispersion, this negative—energy wave will be synchronized with the second harmonic wave, the energy of which is positive. So long as the influence of the change in the population inversion on the interaction can be neglected, we have an explosive situation.

A similar process can also be realized in a medium with quadratic dependence of leakage current on field perturbation.[3,7]

Assuming that $\Phi = 2\varphi_1 - \varphi_2 = 0$, we obtain the solutions for the real amplitudes in the form[2)]

$$\partial A_{1,2}/\partial t' + \mathbf{v}_{1,2}\nabla A_{1,2} = A_1 A_{2,1}, \tag{4}$$

where $A_{1,2}(0, \mathbf{r}) = A_{1,20}(\mathbf{r})$.

If we now change the variables so that $\mathbf{r}' = \mathbf{r} - \mathbf{v}_1 t'$, $t = t'$, and take the x axis in the direction of $\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$, we have instead of (4)

$$\partial A_1/\partial t = A_1 A_2, \quad \partial A_2/\partial t + v\partial A_2/\partial x' = A_1^2 \quad (v = |\mathbf{v}|) \tag{5}$$

with initial profiles $A_{j0}(x', y', z')$. Thus, in the case of the system describing the degenerate interaction in an equilibrium medium,[9,13] we can transform from (5) to equations in terms of ordinary derivatives:

$$\partial A_2/\partial t - A_2^2 = f(x' - vt, y', z'), \tag{6a}$$

$$A_1 = A_{10}(x', y', z')\exp\left(\int_0^t A_2\,dt'\right), \tag{6b}$$

where the function f is determined by the initial conditions

$$f(\eta, y', z') = A_{10}^2(\eta, y', z') - A_{20}^2(\eta, y', z') - v\partial A_{20}/\partial\eta. \tag{7}$$

In the one-dimensional case (A_{10}, A_{20} are constants on any plane parallel to a given plane M), the quantity x' in (5) and (6) is interpreted as the coordinate along the normal to M, and $v = v_{2x} - v_{1x}$.

Integrating (6), in which x', y', z' play the role of parameters, we can readily deduce the field pulse profile at any time.

We now consider solutions of (6) which become infinite in a finite time and correspond to the explosive instability. When $m = A_2 - A_{20}$, we have from (6) (y', z' are omitted)

$$\partial m/\partial t = m^2 + 2A_{20}(x' - vt)m + A_{10}^2(x' - vt), \quad m(0, x') = 0. \tag{8}$$

For given x', this equation initially takes the form

$$m = \int_0^t A_{10}^2(x' - vt')\,dt',$$

whilst for large m with $A_{10}(x' - vt) \not\equiv 0$, $A_{20} \geq 0$, it is described by the explosion equation $\partial m/\partial t = m^2$. Hence, it follows that, at points x' at which the singularity appears, the field varies in accordance with the law $A_2 \propto 1/[t_\infty(x') - t]$ as $t \to t_\infty(x')$.

A relatively simple and useful analytic solution of the Ricatti equation (6a) can be obtained by assuming that $A_{20}(x)$ and $A_{10}(x)$ are piecewise-constant functions. In that case, f(t) and, consequently, the solution of (6a), have singularities which can be excluded by introducing a boundary condition. Suppose that $A_{10}(x' - vt)$, $A_{20}(x' - vt)$ vary discontinuously near $t_{dis}(x')$. Using (8), we can readily show that the change in the solution m(t) across the discontinuity tends to zero as $\Delta t_{dis} \to 0$, and the result is the boundary condition for $A_2(t, x' = \text{const})$ at the point $t_{dis}(x')$ in the form

$$A_2^{(2)} - A_2^{(1)} = A_{20}^{(2)} - A_{20}^{(1)}. \tag{9}$$

Consider the evolution of the one-dimensional initial perturbation in the form of the rectangular pulse

$$A_{10}(x) = \begin{cases} 0, & x<0 \\ a, & 0<x<l; \\ 0, & x>l \end{cases} \quad A_{20}(x) \equiv 0. \tag{10}$$

From (6) and (10) we have the solution for $A_{1,2}(x', t)$. The corresponding expressions for $t > l/v$ ($2al/\pi v < 1$) are

$$A_2 = \begin{cases} \left(\frac{1}{a}\operatorname{ctg}\frac{ax}{v} + \frac{x}{v} - t\right)^{-1}, & 0<x<l \\ \left(\frac{1}{a}\operatorname{ctg}\frac{al}{v} + \frac{x}{v} - t\right)^{-1}, & l<x<vt \\ a\operatorname{tg}\left[a\left(t - \frac{x-l}{v}\right)\right], & vt<x<vt+l \\ 0, & x<0 \text{ и } x>vt+l \end{cases},$$

$$A_1 = \begin{cases} \left(\frac{1}{a}\operatorname{ctg}\frac{ax}{v} + \frac{x}{v} - t\right)^{-1}\sin^{-1}\frac{ax}{v}, & 0<x<l \\ 0, & x<0 \text{ и } x>l \end{cases} \tag{11}$$

The explosion time is given by

$$t_\infty = \begin{cases} \frac{\pi}{2a} \equiv t_*, & \xi = \frac{2al}{\pi v} \geqslant 1 \\ \frac{\pi}{2a}\left(\xi + \frac{2}{\pi}\operatorname{stg}\frac{\pi}{2}\xi\right), & \xi < 1 \end{cases}. \tag{12}$$

The shape of the profile $A_{1,2}$ for $t < l/v$ and $t > l/v$ is shown in Figs. 1a and b, respectively. When the parameter ξ which characterizes A_{10} is greater than unity, the explosion occurs for $t < l/v$ (Fig. 1a), and when $\xi < 1$ it occurs for $t > l/v$ (Fig. 1b).[3)] Figure 2 shows the function $t_\infty(\xi)/t_*$. It is clear that when $\xi > 1$ (long pulses or strong nonlinearity) the explosion time is independent of ξ and is equal to t_*, just as for the spatially homogeneous fields [$A_{10} = a$, $A_{20}(x) = 0$]. As ξ decreases, the ratio t_∞/t_* rapidly increases.

In the case of a non-one-dimensional perturbation of the form given by (10), the expressions given by (12) will include the maximum size $l = l_{max}$ of the region occupied by A_{10} in the direction of $\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$. In the laboratory frame, the field A_2 will "leak out" of this region in the direction of $\mathbf{v}$, and the region will move with velocity $\mathbf{v}_1$.

It follows from the above description that the explosive instability is accompanied by the coalescence of the interacting pulses with the formation of an overlap region between A_1 and the resulting pulse A_2, within which the fields rise to infinity in a finite time. Typical results

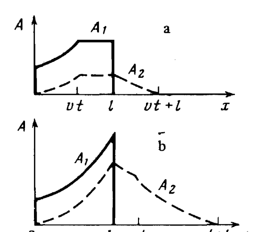

FIG. 1

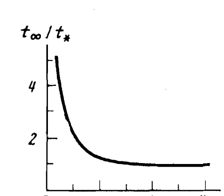

FIG. 2

FIG. 1. Amplitude profiles at different times during the coalescence of pulses of the first and second harmonic [initial profile $A_1(0, x)$ rectangular, $A_2(0, x) \equiv 0$]: (a) $t < l/v$, (b) $t > l/v$.

FIG. 2. Normalized explosion time as a function of the nonlinearity parameter.

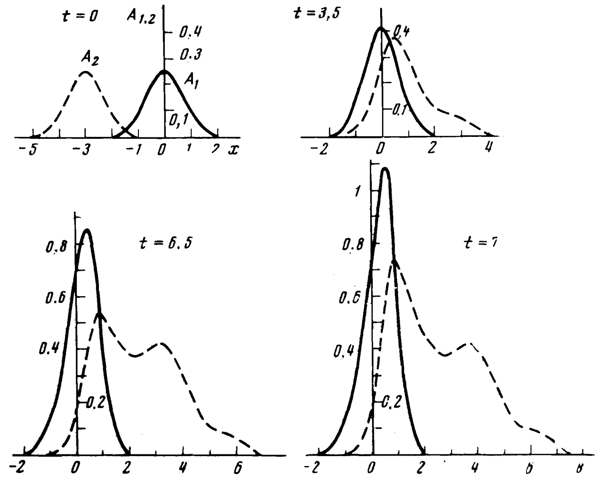

FIG. 3. Illustrating the collision and coalescence of wave pulses in the case of degenerate interaction (v = 1).

of the numerical integration of (6) for the case where the initial shape of A_{10} and A_{20} is bell-shaped are shown in Fig. 3.[4)] It is clear that the interaction between the pulses leads to their coalescence: at times close to t_∞ the pulse tops approach one another and form a sharp structure.[5)]

The physical interpretation of the process shown in Figs. 1 and 3 is relatively simple: in the region occupied by A_1, we have the generation of A_2, and even for $A_{20} = 0$ and despite the fact that A_2 tends to leave the generation region, there is a finite level of A_2 inside it; at the same time, A_1 continues to increase as a result of which there is a further increase in A_2, and so on. This process, in fact, leads to the explosion.

3. We now consider the three-wave process. We shall again suppose that $\varphi_3 - \varphi_1 - \varphi_2 = 0$. The equations for the one-dimensional interaction in the coordinate frame of the second wave take the form

$$\frac{\partial A_1}{\partial t}+v_1\frac{\partial A_1}{\partial x}=A_2A_3,\quad \frac{\partial A_2}{\partial t}=A_1A_3,\quad \frac{\partial A_3}{\partial t}+v_3\frac{\partial A_3}{\partial x}=A_1A_2. \quad (13)$$

All the quantities in these equations can be looked upon as dimensionless variables obtained by the substitutions indicated above.

The set of equations given by (13) was integrated on a computer with $v_1 = -1$, $v_2 = 1$ and initial conditions in the form of two identical bell-shaped pulses A_1 and A_3 located so that the initial time is the collision time for them. The integration was carried out for different values of the amplitudes and the same shape and length of the initial pulses. Typical results are shown in Figs. 4 and 5. We may conclude from the numerical analysis that there

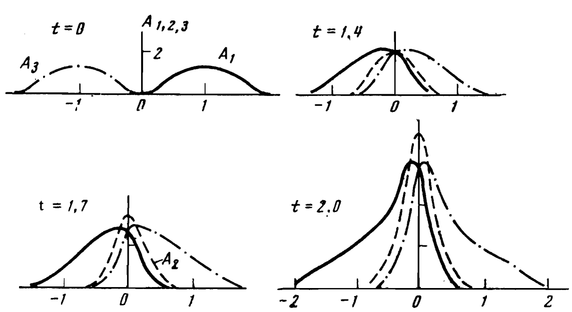

FIG. 4. Coalescence of pulses in the case of three-wave explosive interaction with $A > A_{crit}$.

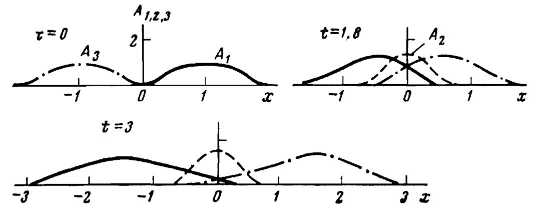

FIG. 5. Separation of pulses in the case of three-wave interaction with $A < A_{crit}$.

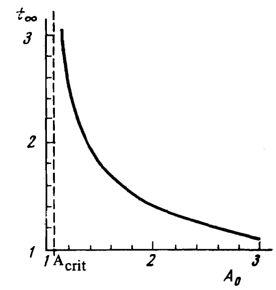

FIG. 6. Explosion time for the collision of two identical pulses as a function of initial amplitudes (three-wave process).

is a threshold amplitude $A_0 = A_{crit}$, above which the generation of A_2 leads to the coalescence of pulses, and we have the development of explosive instability (Fig. 4), whereas for $A_0 < A_{crit}$ there is no explosion and the pulses separate asymptotically (Fig. 5).[6)]

Figure 6 shows a graph of the explosion time as a function of the pulse amplitude. It is clear that as A_0 approaches A_{crit}, the explosion time may substantially exceed the time of linear interpenetration of noninteracting pulses ($\Delta t \approx 2$), which is an approximate measure of the coalescence effect. Moreover, for the chosen values of $v_{1,3}$ right up to amplitudes approaching A_{crit}, the time t_∞ remains of the order of the explosion time in the homogeneous problem $A_{1,3}(0) = A_0$, $A_2(0) = 0$, which is $t_* = \pi/2A_0$.

Some information about the conditions under which the pulses separate can be obtained by investigating the stability of the state when the field $A_{1,3}(x)$ in the region occupied by $A_2(x)$ tends to zero. Equation (13) linearized in the given field $A_2(x)$ has the form

$$\frac{\partial A_1}{\partial t}+v_1\frac{\partial A_1}{\partial x}=A_2(x)A_3,\quad \frac{\partial A_3}{\partial t}+v_3\frac{\partial A_3}{\partial x}=A_2(x)A_1. \quad (14)$$

Using the results reported by Sukhorukov,[14] we can write down the asymptotic solution of (14) for $v_1 = -u$, $v_3 = u$, $A_2(x) = a/\cosh(x/l)$ ($t \gg T = l/u$) in the form[7)]

$$A_{1,3}=\frac{C}{\operatorname{ch}^n(x/l)}\exp\left[\mp\frac{x}{2l}+\left(a-\frac{1}{2T}\right)t\right], \quad (15)$$

where $n = aT$ and C is a constant determined by the initial conditions. From (15) it is clear that for $\xi_1 = 2la/u > 1$ the field $A_{1,3}$ grows, and for $\xi_1 < 1$ the pulses separate. Hence, it follows that an initial perturbation $A_{20}(x) \gg A_{1,30}(x)$ leads to an explosion for $\xi_1 > 1$ and, moreover, the state with $\xi_1 > 1$ cannot be the final product of the separation of pulses.

In the situation where the explosive instability develops in the presence of an initially strong pulse of one wave and two other weak waves, its development is possible, clearly, only when the conditions for the trapping of weak waves by the pump pulse[10,11] are satisfied, i.e., when the weak waves travel in different directions relative to the pump pulse, and the latter is sufficiently strong.

Therefore, the coalescence effect in the case of the three-wave interaction differs from the analogous phenomenon in the case of the degenerate interaction by the presence of an amplitude threshold (for l = const) or, in view of the similarity properties, a threshold in the pulse length (for A = const) for fixed group-wave velocities.

In conclusion, we note the fact that, in the case of the interaction of parametrically coupled waves in a nonlinear equilibrium medium, the velocity of propagation of the pulse front of one of the waves may substantially exceed the linear group velocity of this wave. This effect is similar to the propagation of a pulse with velocity greater than the velocity of light in a nonlinear active medium, [15] and is explained by the intensive generation of the field in this pulse in the region of overlap of all the wave fields.

The authors are indebted to A. V. Gaponov for his interest in this work and for useful discussions.

1)In contrast to the well-known parametric "trapping" of pulses and beams of coupled waves by the pump field, [11] where absolute instability is possible only in the linear approximation, the present effect is characterized by an absolute nonlinear instability (instability "in the large") for fields of all resonance-coupled waves.

2)We note that, in the case of explosive instability, phase synchronization ensures that the initial distribution Φ (0, r) rapidly goes over to $\Phi = 0$.

3)We note that the explosion time t_∞ for $\xi < 1$ is seen from (12) to be greater than the time necessary to transport the resulting field A_2 out of the region occupied by $A_1(\Delta t = l/v)$ in the absence of wave interaction.

4)After changing the variables to $x_n = x/l$, $t_n = (v_0/l)t$, $v_n = v/v_0$, we have $A_n = (l/v_0)A$ where l, v_0 are the characteristic length and velocity of the pulse, and the set of equations for the new dimensionless variables is a system which repeats (5) and which we have solved numerically. Each individually found solution gives for different l, v_0 a set of similar solutions.

5)The double-hump shape of the second harmonic pulse for $t \to t_\infty$ is explained by the fact that it is formed as a result of competition between linear loss and nonlinear amplification, depending on the steepness of its trailing edge.

6)Our calculations suggest that $A_{crit} \in (1; 1 \cdot 15)$.

7)This solution is also the exact solution of (14) corresponding to the mode regime of propagation [10].

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