Dynamics of the formation and interaction of Langmuir solitons and strong turbulence

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We study a model of strong Langmuir turbulence as a "gas" of Langmuir solitons—bunches of oscillations which are localized in regions of a reduced plasma density and which occur through the action of the pressure of a high-frequency field. Using an electronic computer we have investigated in the one-dimensional case the formation of solitons from broad Langmuir wave packets as well as elemenatry processes involving solitons—the fusion of solitons in binary collisions, the interaction of a soliton with sound pulses, and the breaking up of a soliton by the sound produced in the fusion of two solitons. We construct a kinetic equation for a gas of solitons. We find its solution, which is in satisfactory agreement with the results of a numerical simulation of one-dimensional Langmuir turbulence.

1. INTRODUCTION

At the moment, turbulence excited in a plasma by beams of electrons or of electromagnetic waves is the subject of an intensive study both by analytical methods and by numerical experiments on electronic computers. The revelation of its basic properties enables us to predict the efficiency of the absorption of the energy of the beams and the form of the distribution functions of the plasma particles—key questions in the problem of the use of powerful beams of light or of relativistic electrons for heating a plasma.

It is impossible to use perturbation theory methods to study this turbulence, inasmuch as in the region where such a theory can be applied, $w/nT < (\Delta k r_D)^2$, there are no sufficiently intense channels to dissipate the energy of the Langmuir turbulence excited by the beams into heat [1,2] (w/nT is the ratio of the energy density of the oscillations to the thermal energy of the plasma, Δk is the average spread in wave vectors, and $r_D^2 = 2T_e/4\pi ne^2$).

Numerical experiments with Langmuir turbulence show that the plasma is strongly heated under conditions when $w/nT > (k_0 r_D)^2$ (k_0 is a characteristic wave vector of the oscillations excited by the beams). In contrast to the conclusions of the theory of weak turbulence, the energy of the oscillations is transferred from long-wavelength to short-wavelength ranges. [3-5]

Vedenov and Rudakov [6] showed in 1964 that when $w/nT > (\Delta k r_D)^2$ Langmuir turbulence is unstable under spatial modulation of w.[1)] It was noted in later studies [7] that self-modulation can proceed up to the formation of mutually separate bunches of Langmuir oscillations—the solitons. This effect is demonstrated in Fig. 1 of the present paper.

In [8], and later in [9], the hypothesis was formulated that strong Langmuir turbulence is a "gas" of solitons which can fuse into one another or break up under collisions. We gave in [9] on the basis of the soliton turbulence hypothesis an explanation of recent numberical experiments [4,5] in which it was observed that the spectral density of the energy of one-dimensional Langmuir turbulence, excited either by a beam of relativistic electrons [4] or by electromagnetic waves, is proportional to k^{-2} as a result of a parametric instability [5] from the pumping region ω_p/c up to the absorption region ω_p/v_{Te} while the turbulence energy flux at short wavelengths leads, owing to Landau damping, to the formation of the energetic part of the distribution function. Such a spectrum is obtained if we assume that due to collisions and break-up of solitons an energy equipartition law is established over sets of states in a fixed volume of solitons with equal amplitudes.

It is clear that only a complete corroboration of the reality of soliton turbulence and of the distribution law will enable us to predict the properties of turbulent heating in the three-dimensional case, where it seems very unrealistic to hope for computer simulation of practical situations.

If the soliton structure of turbulence does occur, the traditional approach to an analysis of turbulence in the k-representation is unnatural. One of us (Rudakov) has suggested to construct the theory of strong Langmuir turbulence by analogy of gas-kinetic theory. One must first of all study the elementary processes of the interaction of solitons with one another and with particles. After that one must construct and solve a kinetic equation for the soliton distribution function. One can check the solutions obtained against a one-dimensional numerical experiment. In this way even in the one-dimensional

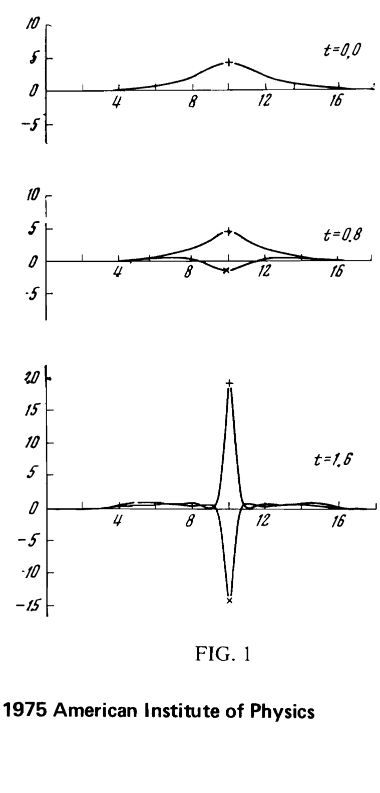

FIG. 1

case the numerical solution of the appropriate set of equations with a subsequent analysis of the results is the basic method of investigation. Even though the interactions between solitons and particles can be calculated analytically, [8,10] the soliton interactions cannot be evaluated analytically in the general case.

The present paper is devoted to a computer calculation of the basic properties of the interaction dynamics of one-dimensional solitons.

2. EQUATIONS. CONSERVATION LAWS

We can describe the one-dimensional electrical field $E \exp(-i\omega_p t)$ of the Langmuir oscillations in the hydrodynamic approximation by the following relatively simple set of equations: [7,8]

$$-2i\frac{\partial E}{\partial t}+\frac{3}{2}\omega_p r_D^2\frac{\partial^2 E}{\partial x^2}-\omega_p\frac{\delta n}{n_0}E=0,$$

$$\frac{\partial^2}{\partial t^2}\delta n-c_s^2\frac{\partial^2}{\partial x^2}\delta n=\frac{\partial^2}{\partial x^2}\frac{|E|^2}{8\pi M}. \tag{1}$$

This equation for the complex amplitude $E(x, t)$ is obtained by averaging over times much larger than $1/\omega_p$ (see [7]); δn is the low-frequency deviation of the plasma density from its average value n_0, which occurs through the action of the pressure force of the Langmuir waves—$(\partial/\partial x)(|E|^2/8\pi)$. This effect turns out to be larger by a factor $(r_D \partial \ln E/\partial x)^2$ than the other non-linear terms in the equation for the electrons which we have dropped here. It is convenient to introduce in Eqs. (1) dimensionless variables through the definitions:

$$x\to\frac{3}{2}\omega_p\frac{r_D^2}{c_s}x,\quad t\to\frac{3}{2}\omega_p\frac{r_D^2}{c_s^2}t,\quad \frac{\delta n}{n_0}\to\frac{3}{2}\frac{c_s^2}{v_T^2}\delta n,$$

In that case

$$|E|^2\to\frac{16\pi}{3}Mn_0\frac{c_s^4}{v_T^2}|E|^2. \tag{2}$$

$$-2i\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial x^2}-\delta nE=0,$$

$$\frac{\partial^2}{\partial t^2}\delta n-\frac{\partial^2}{\partial x^2}\delta n=\frac{\partial^2}{\partial x^2}|E|^2. \tag{3}$$

This set of equations has three integrals of motion, the physical meaning of which is, respectively, the number of quanta (the adiabatic invariant I_1), the momentum, and the energy of the oscillations, less $\omega_p I_1$:

$$I_1=2\int|E|^2dx,$$

$$I_2=\int\left[i\left(E^*\frac{\partial E}{\partial x}-E\frac{\partial E^*}{\partial x}\right)+v\delta n\right]dx,$$

$$I_3=\int\left[\delta n|E|^2+\left|\frac{\partial E}{\partial x}\right|^2+\frac{(\delta n)^2}{2}+\frac{v^2}{2}\right]dx, \tag{4}$$

where

$$\partial v/\partial x=-\partial\delta n/\partial t.$$

In the limiting case when $|E|^2 \ll 1$, $k^2 \ll 1$, corresponding to the physical conditions

$$\frac{E^2}{8\pi nT}\ll\frac{m}{M},\quad\left(r_D\frac{\partial\ln E}{\partial x}\right)^2\ll\frac{m}{M},$$

we get a completely integrable set of equations [11] which describes a localized Langmuir field as a set of conserved solitons. Soliton is the name for a packet of Langmuir waves with a self-consistent form, i.e., a solution of the set of Eqs. (2) and (3) for the case when $k^2 < 1$:

$$E(x,t)=E_0\,\mathrm{ch}^{-1}\frac{E_0}{\sqrt{2}}(x-kt)e^{-i\Omega t+ikx},\quad 2\Omega=k^2-\frac{E_0^2}{2(1-k^2)} \tag{5}$$

The group velocity k and the amplitude E_0 determine the soliton. The integral I_3 is negative and a minimum for a stationary solition, $I_3 = -E_0^3\sqrt{2}/3$. From energetic considerations it is therefore clear that when we take dissipative effects (interactions with particles, [8,10], nonlinear interaction with plane waves with $k' \gg k$, [8] and the emission of sound; see the present paper, Fig. 1) into account any Langmuir wave must tend to be aggregated into a soliton, while the solitons tend to stop, $k \to 0$, and fuse, increasing $|I_3|$. It is interesting to note that in the limiting case when $1 \gg E_0 \gg k^2$ the prohibition of the fusion of solitons follows immediately from an analysis of the conservation laws. Indeed, when two identical solitons with E_0 fuse, we obtain a soliton with amplitude $2E_0$, as follows from the integral I_1, if we substitute into it E for the soliton (5). However, the integral I_3 for the soliton with $2E_0$, which will be negative will have an absolute magnitude four times larger than the original value of I_3 for the two solitons with E_0. The excess energy, equal to $-6I_3(E_0)$, can nowhere be dumped when small amplitude solitons interact, as all motions in that case proceed subsonically and the emitted sound energy is small.

3. RESULTS OF THE CALCULATIONS

To show the basic properties of the formation and interaction of solitons we performed three series of calculations using Eqs. (2) and (3) with symmetric boundary conditions at the end of the section. We looked especially at the case $|E|^2 \gg 1$ as this corresponds just to the amplitude range of practical interest where in real and numerical experiments the energy transfer takes place. The group velocity of the solitons does not differ greatly from the sound velocity (in most of the cases studied k = 0.5), as the influence of kinetic effects on the deceleration of the soliton when it interacts with ions which have a velocity close to the soliton velocity is small in the range $(T_i/T_e)^{1/2} < k < 1$.

a) Development of the two-stream oscillating instability, formation of solitons. In this series of calculations we gave as the initial condition a broad packet of the Langmuir field $|E|^2$ with a single maximum. We varied the width l of the packet, its amplitude, and k (advance of the phase over the length l). We have given in Fig. 1 the quantities $|E|^2$ and δn at different times (in all figures we have indicated the curves for $|E|^2$ by a + sign and those for δn by a × sign). At the initial moment we gave an arbitrary packet of Langmuir oscillations at rest with

$$|E|^2=[2/(1+0{,}1(x-10)^2)]^2,$$

which is not consistent with the density ($\delta n = 0$). It is clear that the self-modulation effect leads to the accumulation of practically all the energy of the initial Langmuir packet into a single soliton. The excess of the initial energy I_3 over the I_3 for the soliton is expended in the form of sound wave energy. The condition for the two-stream oscillating instability,

$$w/nT>(r_D\partial\ln E/\partial x)^2,$$

or, $I_3 < 0$, which amounts to the same, was amply satisfied. Even if initially in the packet $|E|^2 = -\delta n$ and the group velocity were larger than the sound velocity, $k > 1$, even then would we obtain a stationary soliton with k = 0 and the momentum would be carried away by the sound.

b) Collision of two solitons. In this series of calculations we studied the collision of two solitons. We varied their amplitudes and velocities. For small amplitudes,

$|E|^2 < 1$, the solitons pass through one another, but disperse with a smaller velocity. This loss in energy I_3 is caused by the emission of sound when they interact. Such calculations gave, of course, a picture which is completely symmetric in time, in the framework of the adiabatic approximation, $k^2 \ll |E|^2 \ll 1$. When $|E|^2 \gtrsim 1$ (Fig. 2) the solitons merge and the energy which is released is emitted as sound. The fusion process speeds up sharply as soon as the solitons start to overlap. The fast appearing excess of h.f. pressure squeezes the plasma. In the sound trains, moving away from the soliton, there is therefore always in front a compression wave. We show in Fig. 3 the interaction between two solitons moving with velocities, close to that of sound ($k = 0.8$). For such velocities we have in fact a sound pulse, which is only slightly altered under the influence of the h.f. pressure, and sound pulses should pass through one another as they are described by the linear Eq. (3).

We show in Fig. 4 the collision of two solitons, one of which is at rest. It is clear that the moving soliton starts to pass through the one at rest, provided their amplitudes differ by at least a factor two.

If the amplitude of the soliton at rest is less than the amplitude of the moving one, they fuse even when their amplitudes differ greatly. This fact may be a consequence of the fact that the soliton with the smaller amplitude is absorbed by the larger one, while it follows from Eqs. (4) and (5) that is width then decreases as $(1 - k^2)^{-1}$, and solitons with similar scales can already fuse.

We note, moreover, that the greater the difference in amplitude of the interacting solitons, the shorter the wavelength of the emitted sound.

c) Interaction of solitons with sound. The energy conservation laws allow the process of the break-up of solitons when they interact with sound and when they absorb it. We show in Figs. 5 and 6 the interaction of a soliton at rest with a traveling sound pulse of given form and polarity.

It is a priori possible that a narrow positive sound pulse ($\delta n_s > 0$) can break up a soliton at rest, as it can

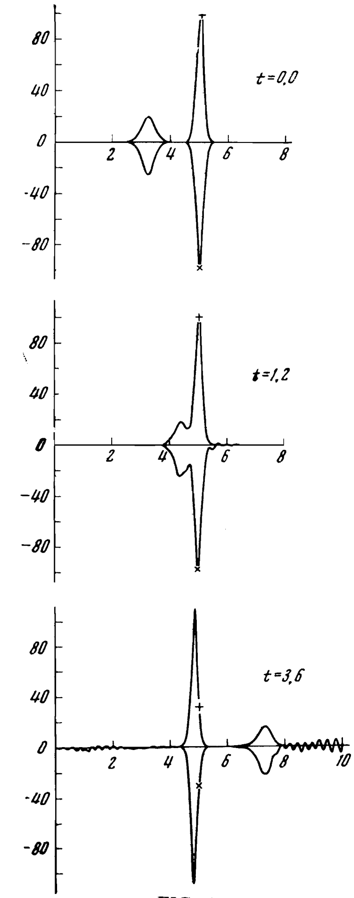

FIG. 4

destroy the potential well of the soliton so that the two solitons which are formed can then disperse. However, it is clear from Fig. 5 that the process proceeds differently. For any magnitude and width of the sound pulse the soliton contracts until its potential well becomes higher and narrower than the given sound pulse, and after that the sound pulse passes freely through the soliton.

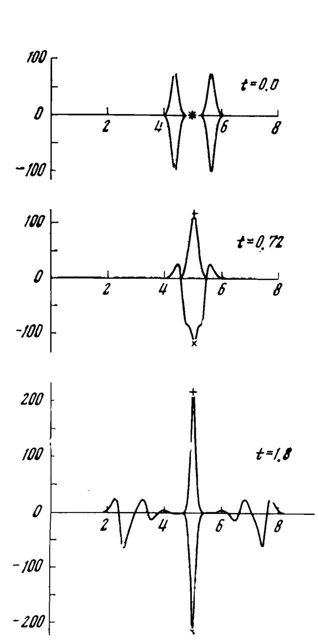

FIG. 2

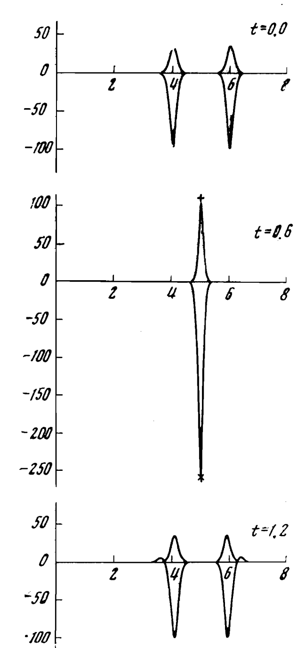

FIG. 3

The interaction process of a negative polarity sound pulse with a soliton at rest (Fig. 6) proceeds differently. When it approaches the soliton the Langmuir field "flows" into the density well, and there arises a state of two colliding solitons, one of which is at rest. These two quasi-solitons then fuse into a single moving one, and the sound is partially reflected, and partially passes through the soliton, trapping part of the Langmuir energy.

We show in Figs. 7 and 8 three-soliton interactions when the sound train originating as the result of the fusion of two identical solitons interacts with the third soliton which is at rest. In Fig. 7 the third soliton is broken up into two by the sound pulse when it reaches it, while in Fig. 8 the amplitude of the third soliton is too large and its break-up does not take place. Numerically we established that the break-up of the third soliton occurs when its amplitude is less than one half the amplitude of the solitons which produce the sound train.

4. DISCUSSION

On the basis of the numerical results considered above and of the regularities which we have established we can, with all due reservations and merely as an illustration of the method developed here, construct a balance in the

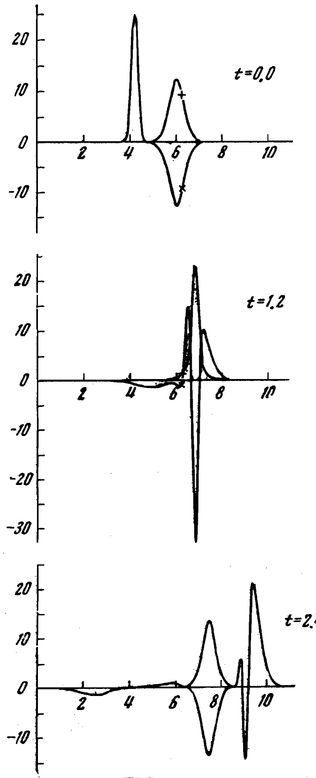

FIG. 5

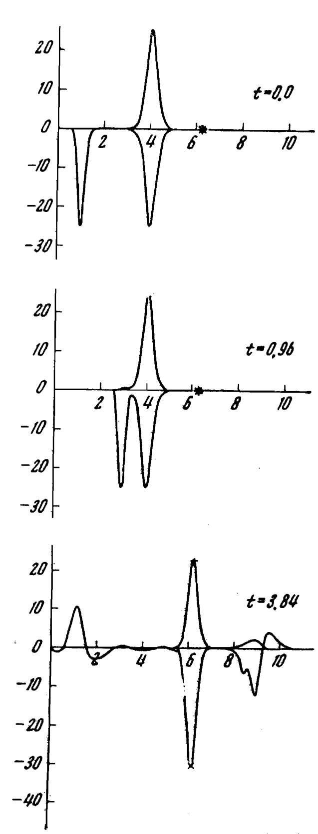

FIG. 6

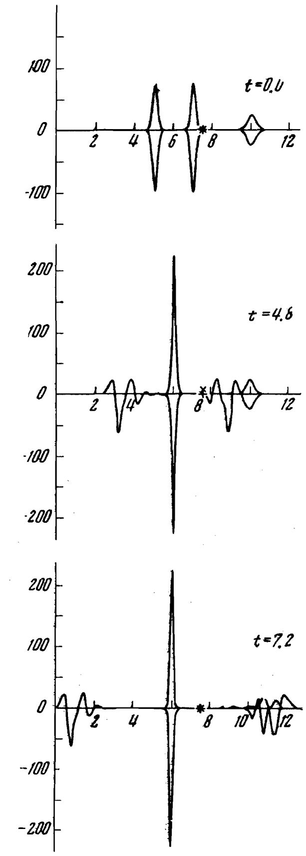

FIG. 7

number of solitons and from that derive qualitative conclusions about the form of the amplitude distribution function of the solitons. To do that we simplify the problem, by postulating that there are only solitons with amplitudes 2^nE_0. We use the following conclusions from the analysis of the calculations: only solitons with the same amplitude can fuse together, and the sound which is then produced can break-up solitons of half the amplitude. It follows from energetic considerations (from the integral I_3) that for each fusion reaction there can be 64 fission reactions. If we bear in mind that a soliton is broken up only by a negative polarity pulse, and that they constitute about half of the pulses, there can not be more than 32 such break-up reactions.

Let $N(E_0)$ be the number of solitons with a given amplitude per unit length and let these solitons have velocities in the range from v_{Ti} to c_S, independent of E_0. In agreement with the remarks made above from the state $N(E_0)$ there will disappear per unit time $c_SN^2(E_0)$ solitons due to fusion and 16 $c_SN^2(E_0)$ due to break-up caused by sound which was produced in the fusion of solitons with amplitude $2E_0$. Into that state there will appear per unit time $\frac{1}{2}c_SN^2(E_0/2)$ solitons as a result of the fusion of solitons with amplitude $E_0/2$ and 32 $c_SN^2(4E_0)$ solitons as a result of the break-up of solitons with amplitude $2E_0$ by the action of sound obtained in the fusion of solitons of amplitude $4E_0$. We thus have the following balance of the density per unit length $N(E_0)$ of solitons of amplitude E_0:

$$\frac{\partial N(E_0)}{\partial t}=\frac{1}{2}c_sN^2\left(\frac{E_0}{2}\right)+32c_sN^2(4E_0)-c_sN^2(E_0)-16c_sN^2(2E_0). \qquad (6)$$

This equation has a stationary solution $N \approx E_0^{-5/4}$. However, we have not taken into account in this equation the outflow of solitons of amplitude E_0 due to break-up by sound produced in the fusion of solitons of amplitude $8E_0$, and so on. A similar analysis shows that this increases the slope to $N \approx E_0^{-m}$, $m > \frac{3}{2}$.

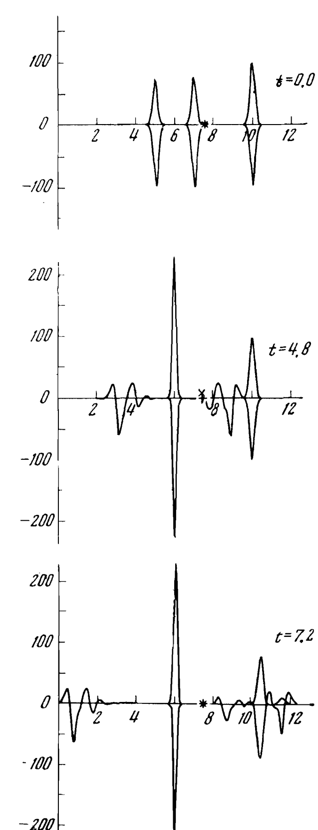

FIG. 8

For such a soliton distribution function the spectral density of the turbulence energy in wave number space will be of the form

$$\int E_{k'}^2\,dk' \sim \sum_{E_0} N(E_0)\int E^2(E_0)\,dx \sim \sum_{E_0} N(E_0)\int \frac{dk}{\operatorname{ch}^2(k'/E_0)}$$
$$\sim \int N(k')\,dk' \qquad (7)$$

(the Fourier transform of the field of a soliton with $E_0 \gg k$, $E_{k'} = \cosh^{-1}(k'/E_0)$, see [8]), that is,

$$E_{k'}^2 \approx (k')^{-m}, \quad m > {}^3/_2.$$

We have obtained here a flatter wave number dependence of the energy than in numerical experiments,[4,5] where $m \sim 2$, which is not unreasonable for such a coarse analysis.

Such an, at least qualitative, agreement shows that the soliton structure of Langmuir turbulence is completely realistic in the one-dimensional case.

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