Quantum electromagnetic waves in cylindrical conductors

É. N. Bogachek and G. A. Gogadze

Physico-technical Institute of Low Temperatures, Ukrainian Academy of Sciences
(Submitted December 26, 1973)
Zh. Eksp. Teor. Fiz. 67, 621–626 (August 1974)

Undamped quantum electromagnetic waves in cylindrical conductors are considered. It is shown that the phase velocities of the waves change as a function of the applied magnetic field flux. For thin-walled cylinders, the period is equal to hc/e.

The energy quantization of the electron is important in cylindrical conductors of small cross section. In a magnetic field parallel to the axis of the cylinder, the spectrum of quantum states is modified in such a way that, along with the de Haas–van Alphen effect, there arise effects of oscillations of the various thermodynamic and kinetic quantities with a period (in the flux) $\Phi_0 = hc/e$, which have been called [1-3] "oscillating phenomena of the flux-quantization type.[1)] Thanks to the quantization, the propagation of weakly damped electromagnetic waves similar to quantum waves in the volume case, also becomes possible [4,5]. The purpose of the present study is to investigate the corresponding phenomenon. We shall show that as a consequence of collisionless Landau damping in cylindrical conductors (hollow and solid), there exist quantum waves whose phase velocities change periodically with change in the magnetic field flux Φ, while for hollow, thin-walled cylinders, the period is equal to Φ_0.

1. The identical nature of the quantized spectrum for thin-walled cylinders in longitudinal magnetic and vector-potential fields has been proven previously (this case can be realized in principle with the help of a long solenoid placed inside a cavity); the spectrum takes the form

$$E_{mn}(p_z)=\varepsilon_{mn}^{(1)}+p_z^2/2m^*,$$
$$\varepsilon_{mn}^{(1)}=\frac{\hbar^2}{2m^*}\left\{\frac{(m+\eta)^2}{R^2}+\frac{\pi^2n^2}{d^2}\right\}, \qquad (1)$$

where $\eta=\Phi/\Phi_0$, R is the radius of the cylinder, d the thickness of its walls ($d \ll R$), p_z the component of the quasimomentum along the axis of symmetry, and m^* the effective mass of the electron. As is seen from (1), a change in the flux by an amount $\Delta\Phi=\Phi_0$ is equivalent to the replacement of the discrete number m by m + 1. The value of the density of states does not change on such a substitution. This means that the density of states oscillates on variation of the flux with a period Φ_0.

Quantization of the motion of electrons in a solid cylindrical sample placed in a longitudinal magnetic field produces a more complicated oscillation picture. In a weak field that satisfies the condition $r_H \gg R$ (r_H is the cyclotron radius), we obtain the following expression for the spectrum:

$$E_{mn}(p_z)=\varepsilon_{mn}^{(2)}+p_z^2/2m^*,$$
$$\varepsilon_{mn}^{(2)}=\frac{\hbar^2}{2m^*R^2}\left\{\gamma_{mn}^2+2\eta m+\frac{1}{3}\eta^2\left[1+\frac{2(m^2-1)}{\gamma_{mn}^2}\right]\right\}, \qquad (2)$$

where γ_{mn} are the zeroes of the Bessel function of order m. Analysis has shown [2] that the spectrum (2) leads to the appearance of two oscillations that differ in period: oscillations of the de Haas–van Alphen type (associated with the "volume" electrons), whose period is equal to $(\Delta\Phi)_1 \sim (\Phi_0^2/\Phi)(\hbar/p_F R)^3$ in order of magnitude, and oscillations that are strictly periodic in Φ with period $(\Delta\Phi)_2=\Phi_0$ (oscillations of the flux-quantization type). The surface states of electrons with the largest values of the magnetic quantum number are responsible for the latter oscillations. Thus, in the calculation of the thermodynamic quantities of the solid cylinder, the contribution of states with small values of the magnetic quantum number (the paths of the carriers pass near the center of the conductor) is effectively separated from that of states with $m \approx \gamma_{mn}$, which correspond to the magnetic surface levels (their spectrum is described by equation (4) of [2]).

As has already been noted, the reason for the existence of quantum electromagnetic waves is connected with the absorption of energy due to Landau damping. The dispersion equation, which determines the spectrum of the quantum waves, is obtained under the assumption of an infinite free path of conduction electrons and specular reflection from the boundary of the sample. Moreover, we limited ourselves to consideration of a metal model with one type of carrier.

Quantum waves in a cylindrical conductor are acoustic plasma oscillations similar in nature to the longitudinal quantum waves propagated in a bulk metal along a constant magnetic field.[2)] In the motion of an acoustic plasmon the number of electrons from one group of carriers increases in some parts of space and simultaneously the number of electrons from another group of carriers decreases, so that the condition of electroneutrality is not violated. However, whereas in the bulk metal the electrons are split into separate groups by a strong magnetic field, formation of groups is brought about in the case considered by size quantization of the motion of the electrons in the cross section of the conductor. Here the magnetic field plays the role of an external parameter, with the help of which we can change the phase velocities of the plasma modes.

2. To derive the dispersion law of the electromagnetic waves, we assume that the radius of the cylinder is so small that all the quantities—the current, the electric field E, and so on, depend on a single space coordinate z directed along the axis and, generally speaking, on the time t. Then the equation of continuity takes the form

$$\frac{\partial I}{\partial z}+\frac{\partial Q}{\partial t}=0, \qquad (3)$$

where I(z,t) and Q(z,t) are the linear current density and the charge density, respectively, at the point z and time t. The value of the field **E** can be found in the quasistatic approximation from the condition [3)]

$$\operatorname{div}\mathbf{E}=4\pi\tilde{Q}, \qquad (4)$$

where we have for the hollow thin-walled cylinder

$$\tilde{Q}=(2\pi\rho)^{-1}Q(z,t)\,\delta(\rho-R)$$

(ρ is the distance measured from the axis). After a series of transformations,[4] we get the following dispersion equation:

$$\sigma(\omega, q)=i\omega/q^2\beta(q)S, \tag{5}$$

where q is the wave vector, $\beta(q)=2I_0(qR)K_0(qR)$ (I_0, K_0 are Bessel functions of imaginary argument), $S=2\pi Rd$.

For the solid cylinder, we represent Q in the form $Q=\tilde{Q}(z, t)\delta(\rho)$, where $\delta(\rho)$ is the two-dimensional δ function in the (x,y) plane. The dispersion equation preserves the form (5), but $\beta(q)=2\ln(1/qR)$, $S=\pi R^2$.

We now proceed to find the explicit expression for σ. In the classical treatment we start out from the kinetic equation with a vanishing collision integral. Solving the equation, we obtain (in the case of specular reflection) the following for the zz-component of the conductivity:

$$\sigma_{zz}\equiv\sigma=\sigma'+i\sigma'',$$

$$\sigma'=\frac{e^2m^{*2}}{2\pi\hbar^3\omega}u^3\Theta(v_F-u), \tag{6}$$

$$\sigma''=\frac{3\omega}{8\pi}\left(\frac{\omega_p}{\omega}\right)^2\left\{\left(\frac{u}{v_F}\right)^3\ln\left|\frac{1+u/v_F}{1-u/v_F}\right|-2\left(\frac{u}{v_F}\right)^2\right\},$$

where $u=\omega/q$ is the phase velocity of the wave; $\Theta(x)=1$, $x>0$; $\Theta(x)=0$, $x<0$; $\omega_p=(4\pi ne^2/m^*)^{1/2}$ is the plasma frequency. In the derivation of (6), it was assumed that the current and field depend only on z, which is justified if $\delta_0\gg R$ (δ_0 is the skin-layer thickness). Substituting (6) in (5), we obtain the dispersion equation which determines the dependence of ω on q. The quantity σ, which determines the damping, differs from zero in the case $u<v_F$. For $u>v_F$, the dispersion equation (5) takes the form

$$F\left(\frac{u}{v_F}\right)=\frac{8\pi}{3S}\left(\frac{v_F}{\omega_p}\right)^2\beta^{-1}(q),$$
$$F(\alpha)=\alpha\ln\left|\frac{\alpha+1}{\alpha-1}\right|-2. \tag{7}$$

The asymptotic form of $F(\alpha)$ for $\alpha\gg1$ is $F\sim2/3\alpha^2$. For example, in the case of a solid cylinder, we get the dispersion law:[7,8]

$$\frac{\omega}{q}=\frac{1}{\sqrt{2}}\omega_pR\ln^{1/2}\frac{1}{qR}. \tag{8}$$

For arbitrary α, Eq. (7) can be most simply analyzed graphically (see Fig. 1).

For $u<v_F$, Eq. (5) also has a solution, but the damping turns out to be very strong and wave propagation is practically impossible. In Fig. 1, this branch of the function F is shown by the dashed line.

We note that, although the dispersion equation was obtained in zero magnetic field, it is clear that it also holds in weak fields ($r_H\gg R$).

3. We now go over to the quantum case. We shall start out from the equation for the density matrix. Solving it by the method of Kubo [9] with account of spatial dispersion, we obtain the following expression for the longitudinal conductivity:

$$\sigma(\omega,q)=\frac{2}{S}\sum_{a,a'}\frac{f_a-f_{a'}}{E_{a'}-E_a}\frac{1}{\delta+i(\omega_{a'a}-\omega)}\cdot$$
$$\times\int\int d\rho\,dz\,j_z^{(aa')}(\rho,z)e^{-iqz}\int d\rho'\,j_z^{(a'a)}(\rho',0). \tag{9}$$

Summation is carried out over all the quantum numbers $a\in(m, n, p_z)$, which characterize the state of the electron in the cylindrical sample, f_a is the Fermi distribution function, $\delta\to+0$. In the derivation of (9), it was

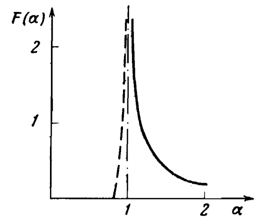

FIG. 1

assumed that waves whose length $\lambda\gg R$ are considered.

Proceeding to the treatment of a thin-walled cylinder in a magnetic field, we use the spectrum (1). From the laws of conservation of energy and the z component of the quasimomentum, we obtain a condition that is necessary for absorption of a quantum of the field:

$$\varepsilon_{mn}^{(1)}-\varepsilon_{m'n'}^{(1)}+\hbar\omega_q=\frac{p_z}{m^*}\hbar q+\frac{\hbar^2q^2}{2m^*}. \tag{10}$$

We require that the distance between the levels be large, so that the condition

$$\varepsilon_{mn}^{(1)}-\varepsilon_{m'n'}^{(1)}>\hbar qv_F \tag{11}$$

will be satisfied. The inequality (11) is equivalent to $qR<1$; here (10) holds only in the absence of transitions ($n=n'$, $m=m'$). The quasimomentum of the electrons which take part in the absorption is equal to $P_z=m^*\omega/q-\hbar q/2$. The energy dissipation due to Landau damping is associated with the real part of the conductivity. By calculating the matrix elements of the current density operator j_z in the absence of transitions, and carrying out integration over p_z, we obtain the following expression for $\mathrm{Re}\,\sigma$ $(T=0)$:

$$\mathrm{Re}\,\sigma(\omega,q)=\frac{e^2m^*\omega}{\hbar^2q^32\pi Rd}\sum_{m,n}\{\Theta[\zeta-E_{mn}(P_z)]-\Theta[\zeta-E_{mn}(P_z)-\hbar\omega]\} \tag{12}$$

(ζ is the chemical potential of the metal). The sum in (12) differs from zero when the inequalities $\zeta>E_{mn}(P_z)$, $\zeta<E_{mn}(P_z)+\hbar\omega$ are satisfied.

At $T=0$, a contribution to the absorption is made only by electrons with energy ζ, for which, in accord with (1), there is a set of allowed values $p_z=p_z^{mn}(H)$. When p_z^{mn} coincides with P_z at some field value, there is a giant oscillation.[10] The dependence of $\mathrm{Re}\,\sigma(\omega,q)$ on the magnetic-field flux is a set of narrow and high rectangular maxima. The strong absorption makes propagation of electromagnetic waves inside these intervals impossible. On the other hand, if the width of the peaks $\omega R\Phi_0/v_F$ is small in comparison with their separation distance $\Delta\Phi=\Phi_0$, then the existence of quantum waves outside the given intervals is possible. The nondissipative part of the conductivity has the form

$$\mathrm{Im}\,\sigma=-\frac{e^2m^*\omega}{2\pi^2Rd\hbar^2q^3}\sum_{m,n}\ln\left|\frac{(v_{mn}+\hbar q/2m^*)^2-u^2}{(v_{mn}-\hbar q/2m^*)^2-u^2}\right|, \tag{13}$$

where $v_{mn}=[2m^{*-1}(\zeta-\epsilon_{mn}^{(1)})]^{1/2}$, $u=\omega/q$. Assuming for simplicity that the inequality $\hbar q/2m^*\ll|v_{mn}-u|$ is satisfied, and expanding (13) in a power series in $\hbar q/2m^*$, we obtain for $\mathrm{Im}\,\sigma$

$$\mathrm{Im}\,\sigma=-\frac{e^2\omega}{2\pi^2Rd\hbar q^2}G(u),$$
$$G(u)=\sum_{mn}\left\{\frac{1}{v_{mn}-u}+\frac{1}{v_{mn}+u}\right\}. \tag{14}$$

Substituting (14) in (5), we get the relation

$$G(u)=-\pi\hbar/e^2\beta(q). \tag{15}$$

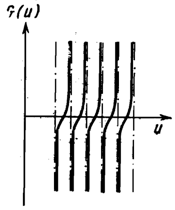

FIG. 2

4. Equation (15) represents the dispersion equation of longitudinal quantum waves in a hollow cylindrical conductor. For a thin-walled cylinder ($d \ll R$), the interval of change of the quantum number n is not large. In the limiting quantum case ($n=1$), the plasma modes are numbered by means of a single quantum number m only, and the analysis of Eq. (15) is entirely analogous to what is contained in [4].

Figure 2 shows a graph of the function G(u), whose points of intersection with the horizontal line $G=-\pi\hbar/e^2\beta(q)$ give the roots of Eq. (15). When the flux Φ through the cross section of the cylinder changes by $\hbar c/e$, the spectrum of the allowed electron velocities v_{mn} repeats itself. It is natural that in this case, the spectrum of the phase velocities of the quantum waves also repeats itself. Therefore not only the dispersion law of the quantum waves is of experimental interest, but also the periodic dependence of their phase velocities on the magnetic field.

For the existence of quantum waves, it is necessary that their velocities be far removed from the spike centers—by distances that are much greater than the region of diffuseness of the absorption peaks. Taking into account that the final width of the giant spike $\hbar q/m^*$ is smeared out by temperature and by collisions, we arrive at the following criteria for wave propagation without significant attenuation:

$$\frac{u_{mn}}{R} \gg \omega, \quad \frac{\hbar v_F}{R} \Big/ \zeta \gg \frac{\nu}{\omega}\left(\frac{v_F}{u_{mn}}\right)^{-1}, \quad \frac{\hbar u_{mn}}{R} \gg kT, \tag{16}$$

where ν is the collision frequency and u_{mn} is the velocity of the acoustic mode which lies between the nearest values of v_{mn}. These conditions can be satisfied in the case of semimetals.

Undamped quantum electromagnetic waves also exist in a solid cylindrical conductor. However, the general picture here turns out to be more complicated because of the structure of the spectrum (2). In particular, the periodic dependence of the phase velocities of the waves on the flux Φ with period Φ_0 is lacking.[5]

In conclusion, we discuss the possibility of experimental observation of the quantum waves. Let the investigated sample (the "whisker") be placed in a resonator in which an electromagnetic field of fixed frequency Ω is excited. If at some H the condition $\Omega = u_{mn}(H)k_s$ is satisfied, where $k_s = 2\pi s/l$ is the wave vector of the standing waves in the sample (l is the length of the whisker, s an integer), we should observe resonant pumping of the energy of the field of the resonator into the energy of the quantum electromagnetic waves. Inasmuch as $\Omega \sim c/L$, where L is the size of the resonator, observation of resonances with small values of s is possible upon satisfaction of the relation $l/L \sim u_{mn}/c \sim v_F/c \ll 1$. Evidently, this relation is not optimal from the point of view of the coupling of the field of the quantum wave with the resonator. At $l \sim L$, waves will be excited with large values of s.

The authors express their gratitude to I. O. Kulik, É. A. Kaner and V. D. Fil' for discussions.

1) The effects considered are not macroscopic quantum phenomena in the usual sense of the word, since they exist in systems with disturbed long-range order. The amplitude of the oscillations tends to zero upon an increase in the radius of the cylinder. Nevertheless, it is important that the phenomena considered can take place in systems with sufficiently large dimensions ($L \sim 10^{-3}$-10^{-4} cm) which in practical respects can be considered "macroscopic."

2) Plasma waves of the acoustic type in a metal with widely different mass values for the electrons and holes were first considered by Pines and Schrieffer. [6] Transverse quantum waves and their coupling with the longitudinal in metals with a single type of carrier were studied by Kaner, Skobov and Lyubimov. [5]

3) Equations (3), (4) do not contain any retardation, which, for waves with phase velocities of the order of the Fermi velocity, is inconsequential.

4) We follow the approach that Kulik used [7] in studying retarded plasma waves in thin superconducting filaments and films. We note that the relation (5) can be obtained in a different way. By solving Maxwell's equations inside a cylindrical conductor and outside it, and joining the solutions on the boundaries (the tangential components of the electric and magnetic fields E_z and H_φ are continuous), we get the dispersion equation, whence, in the limit $c \to \infty$ (c is the velocity of light) and $qR \ll 1$, we get (5).

5) In kinetics, we can distinguish the contribution of states of the "whispering gallery" type, including the diffuseness of scattering at the boundary. [3] The specifics of the present problem lie in the fact that the total spreading out of the spectrum of the "volume" electrons would lead to strong attenuation of the quantum waves.

1. I. O. Kulik, ZhETF Pis. Red. **11**, 407 (1970) [JETP Lett. **11**, 275 (1970)].
2. É. N. Bogachek and G. A. Gogadze, Zh. Eksp. Teor. Fiz. **63**, 1839 (1972) [Sov. Phys.-JETP **36**, 973 (1973)].
3. É. N. Bogachek and G. A. Gogadze, ZhETF Pis. Red. **17**, 164 (1973) [JETP Lett. **17**, 118 (1973)].
4. S. L. Ginzburg, O. V. Konstantinov and V. I. Perel', Fiz. Tverd. Tela **9**, 2139 (1967) [Sov. Phys.-Solid State **9**, 1684 (1968)]; O. V. Konstantinov and V. I. Perel', Zh. Eksp. Teor. Fiz. **53**, 2034 (1967) [Sov. Phys.-JETP **26**, 1151 (1968)].
5. É. A. Kaner and V. G. Skobov, Phys. Stat. Sol. **22**, 333 (1967); É. A. Kaner, O. I. Lyubimov, and V. G. Skobov, Zh. Eksp. Teor. Fiz. **58**, 730 (1970) [Sov. Phys.-JETP **31**, 391 (1970)].
6. D. Pines and J. Schrieffer, Phys. Rev. **124**, 1387 (1961).
7. I. O. Kulik, Zh. Eksp. Teor. Fiz. **65**, 2016 (1973) [Sov. Phys.-JETP **38**, 1008 (1975)].
8. R. A. Ferrell, Phys. Rev. Lett. **13**, 330 (1964).
9. R. Kubo, J. Phys. Soc. Japan **12**, 570 (1957).
10. V. I. Gurevich, V. G. Skobov and Yu. A. Firsov, Zh. Eksp. Teor. Fiz. **40**, 786 (1961) [Sov. Phys.-JETP **13**, 552 (1961)].

Translated by R. T. Beyer
71