Probabilistic description of random fields satisfying simplest equations of the hydrodynamic type

A. N. Malakhov and A. I. Saichev

Gor'kii State University

(Submitted February 20, 1974)

Zh. Eksp. Teor. Fiz. 67, 940–950 (September 1974)

A kinetic equation is derived for the probability distribution, considered in the hydrodynamic approximation, of the velocity field and beam density of noninteracting particles. Various statistical properties of the beam are investigated by means of the derived equation. The meaning of the kinetic-equation solutions is discussed in the region in which caustics are formed and the beam is split into several streams. It is noted that all the results obtained can be interpreted by geometric optics.

INTRODUCTION

The study of the statistical properties, particularly the probability distributions, of random fields satisfying equations of the hydrodynamic type is of great interest. It is stimulated by the ever increasing number of problems that arise in the theory of turbulence, in optics and in many other branches of physics (see, e.g., [1-6]). However, the probability distributions of these fields have not been investigated so far with sufficient rigor, owing to the strong nonlinearity of the initial equations. Nonetheless, in some cases it becomes possible to obtain exact kinetic equations of the probability densities of the fields or, in other words, it becomes possible to sum an infinite coupled chain of equations for the moments of the considered fields.

In this paper, using as an example the derivation of the kinetic equation for the probability distribution of the components of the velocity vector $v_\alpha(\mathbf{x}, t)$, of the tensor $u_{\beta\gamma} = \partial v_\beta/\partial x_\gamma$, and the density $\rho(\mathbf{x}, t)$ of the beam of noninteracting particles, considered in the hydrodynamic approximation, we investigate the statistical properties of nonlinear waves that satisfy the simplest equations of the hydrodynamic type. The obtained equations enable us also to investigate the properties of the random simple waves, at which one arrives in the study of waves in a plasma, of gravitational waves, etc. (see, e.g., [3-7]). They can also be interpreted as the kinetic equations of an optical wave considered in the geometrical-optics approximation, since the geometrical-optics equations coincide with the equations of a beam of noninteracting particles in the hydrodynamic approximation.

We analyze the solutions of the derived equations in the particular case of a one-dimensional beam. We explain their meaning in a multiple-stream beam. We obtain the probable density distribution of the beam of noninteracting particles propagating in the absence of external forces or, in the language of geometrical optics, the probability distribution of the intensity of a light wave behind a one-dimensional phase screen.

1. KINETIC EQUATION FOR A BEAM OF NONINTERACTING PARTICLES

The velocity field and the density of a beam of noninteracting particles situated in an external randomly-inhomogeneous force field satisfy in the hydrodynamic approximation the equations (see, e.g., [3,8])

$$\frac{\partial v_\alpha}{\partial t} + v_\beta \frac{\partial v_\alpha}{\partial x_\beta} = F_\alpha, \tag{1.1}$$

$$\frac{\partial \rho}{\partial t} + v_\beta \frac{\partial \rho}{\partial x_\beta} + \rho \frac{\partial v_\beta}{\partial x_\beta} = 0,$$

where v_α are the components of the vector velocity field, ρ is the density of the beam, F_α are the components of the random vector force field, $\alpha, \beta, \gamma, \delta = 1, 2, 3$. We supplement (1.1) with the nine equations for the component of the tensor $u_{\alpha\gamma} = \partial v_\alpha/\partial x_\gamma$, differentiating the equation for v_α with respect to x_γ:

$$\frac{\partial u_{\alpha\gamma}}{\partial t} + v_\beta \frac{\partial u_{\alpha\gamma}}{\partial x_\beta} + u_{\beta\gamma} u_{\alpha\beta} = \frac{\partial F_\alpha}{\partial x_\gamma}. \tag{1.2}$$

2. Our problem is to find a closed equation for the one-point probability density $W[v_\alpha, u_{\beta\gamma}, \rho; x_\delta, t]$. To this end we consider the mean value of the arbitrary function $\varphi[v_\alpha, u_{\beta\gamma}, \rho]$ and write down its derivative with respect to time, using (1.1) and (1.2). As a result we obtain

$$\frac{\partial\langle\varphi\rangle}{\partial t} + \frac{\partial}{\partial x_\beta}\langle v_\beta\varphi\rangle - \langle u_{\beta\beta}\varphi\rangle + \left\langle \frac{\partial\varphi}{\partial\rho}\rho u_{\beta\beta}\right\rangle + \left\langle \frac{\partial\varphi}{\partial u_{\alpha\gamma}} u_{\beta\gamma}u_{\alpha\beta}\right\rangle = \left\langle \frac{\partial\varphi}{\partial u_{\alpha\gamma}}\frac{\partial F_\alpha}{\partial x_\gamma}\right\rangle + \left\langle F_\alpha \frac{\partial\varphi}{\partial v_\alpha}\right\rangle. \tag{1.3}$$

We note that the mean values in the left hand side of this equation turn out to be closed with respect to the sought probability density, i.e., they can be calculated with the aid of this density. At the same time, the terms in the right-hand side are not closed with respect to this density.

We assume now that F_α are the components of a Gaussian field δ-correlated in time having a correlation function

$$\langle F_\alpha(\mathbf{x}, t) F_\beta(\mathbf{x}+\mathbf{s}, t+\tau)\rangle = D_{\alpha\beta}[s]\delta(\tau).$$

We recall that a real random field can be replaced with sufficient accuracy by a δ-correlated field if its correlation time is much shorter than the characteristic times of variation of the beam parameters. In this case the terms in the right-hand side of (1.3) can also be closed by using the Furutsu-Novikov formula [6,9], which leads to

$$\left\langle \frac{\partial\varphi}{\partial v_\alpha}F_\beta\right\rangle = D_{\alpha\beta}\left\langle \frac{\partial^2\varphi}{\partial v_\alpha\,\partial v_\beta}\right\rangle, \qquad \left\langle \frac{\partial\varphi}{\partial u_{\alpha\gamma}}\frac{\partial F_\alpha}{\partial x_\gamma}\right\rangle = B_{\alpha\beta}\left\langle \frac{\partial^2\varphi}{\partial u_{\alpha\gamma}\,\partial u_{\beta\gamma}}\right\rangle, \tag{1.4}$$

where

$$D_{\alpha\beta} = \frac{1}{2}D_{\alpha\beta}[0], \qquad B_{\alpha\beta} = -\frac{1}{2}\frac{\partial^2 D_{\alpha\beta}[s]}{\partial s^2}\bigg|_{s=0}.$$

Substituting (1.4) in (1.3) and taking into account the arbitrariness of the function φ, we can easily obtain from (1.3) the sought kinetic equation for W:

$$\frac{\partial W}{\partial t}+v_\beta\frac{\partial W}{\partial x_\beta}-u_{\beta\beta}W-u_{\beta\beta}\frac{\partial}{\partial\rho}[\rho W]$$
$$-\frac{\partial}{\partial u_{\alpha\gamma}}[u_{\beta\gamma}u_{\alpha\beta}W]=D_{\alpha\beta}\frac{\partial^2 W}{\partial v_\alpha\partial v_\beta}+B_{\alpha\beta}\frac{\partial^2 W}{\partial u_{\alpha\gamma}\partial u_{\beta\gamma}}. \tag{1.5}$$

It is necessary to add to this equation the initial condition

$$W[v_\alpha,\ u_{\beta\gamma},\ \rho;\ x_\delta,\ 0]=W_0[v_\alpha,\ u_{\beta\gamma},\ \rho;\ x_\delta].$$

The obtained kinetic equation describes the evolution of the probability density $W[v_\alpha, u_{\beta\gamma}, \rho; x, t]$ in time and makes it possible, in principle, to determine the changes of such physically-interesting statistical characteristics of the beam as the probability distribution of its density, of the velocity field, of the number of streams in the beam, etc.

3. We now use simple examples and particular cases to explain the meaning and certain properties of the solutions of (1.5). For the time being we note only that if the beam is statistically homogeneous at the initial instant of time and the velocity field does not depend on the beam density and on the tensor $u_{\alpha\gamma}$ at that point of space, i.e., if

$$W_0[v_\alpha,\ u_{\beta\gamma},\ \rho;\ x_\delta]=V_0[v_\alpha]U[u_{\beta\gamma},\ \rho],$$

then W can be represented in the form of the product of two probability distributions:

$$W[v_\alpha,\ u_{\beta\gamma},\ \rho;\ t]=V[v_\alpha;\ t]U[u_{\beta\gamma},\ \rho;\ t], \tag{1.6}$$

which satisfy the individual equations

$$\frac{\partial V}{\partial t}=D_{\alpha\beta}\frac{\partial^2 V}{\partial v_\alpha\partial v_\beta}, \tag{1.7}$$

$$\frac{\partial U}{\partial t}-u_{\beta\beta}U-u_{\beta\beta}\frac{\partial}{\partial\rho}[\rho U]-\frac{\partial}{\partial u_{\alpha\gamma}}[u_{\beta\gamma}u_{\alpha\beta}U]=B_{\alpha\beta}\frac{\partial^2 U}{\partial u_{\alpha\gamma}\partial u_{\beta\gamma}} \tag{1.8}$$

and the initial conditions

$$V[v_\alpha;\ 0]=V_0[v_\alpha],\quad U[u_{\beta\gamma},\ \rho;\ 0]=U_0[u_{\beta\gamma},\ \rho].$$

Thus, it follows from (1.6)—(1.8) that a beam-velocity field that is initially independent of the beam density and of the tensor $u_{\beta\gamma}$ remains statistically independent of these quantities during all the succeeding instants of time. For example, an initially Gaussian velocity field still remains Gaussian, in spite of the nonlinearity of the initial equations, since its probability distribution satisfies the usual diffusion equation (1.7).

We note that this result pertains only to the single-point probability density of a statistically homogeneous beam, for in this case the probability is not influenced by the terms $v_\beta\partial v_\alpha/\partial x_\beta$, $v_\beta\partial\rho/\partial x_\beta$, and $v_\beta\partial u_{\alpha\gamma}/\partial x_\beta$, which lead to the nonlinearity of the equations that connect the beam density and the tensor $u_{\beta\gamma}$ with the velocity. It can be shown, for example, that even a two-point initially-Gaussian probability density of the velocity field will become more and more non-Gaussian with increasing time as a result of the nonlinearity of the initial equations.

2. THE MEANING OF THE KINETIC EQUATION FOR SINGLE-STREAM AND MULTIPLE-STREAM BEAMS

1. We confine ourselves for simplicity to the study of the statistical properties of only a one-dimensional beam moving under the influence of a force F(x, t) along the x axis. In this case, Eq. (1.5) goes over into the simpler equation for the probability density $w = w[v, u, \rho; x, t]$:

$$\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial x}-uw-u\frac{\partial}{\partial\rho}[\rho w]-\frac{\partial}{\partial u}[u^2w]=D\frac{\partial^2 w}{\partial u^2}+B\frac{\partial^2 w}{\partial u^2} \tag{2.1}$$

with the initial condition

$$w[v,\ u,\ \rho;\ x,\ 0]=w_0[v,\ u,\ \rho;\ x]. \tag{2.2}$$

There is no known solution of (2.1) in the general case. It is easy, however, to go over from (2.1) to an easily-solved equation for the average particle density in phase space (x, v):

$$R(v,x;t)=\int_{-\infty}^{\infty}du\int_0^{\infty}d\rho w[v,u,\rho;x,t].$$

Multiplying Eq. (2.1) by ρ and integrating it with respect to u and ρ, we arrive at the equation

$$\frac{\partial R}{\partial t}+v\frac{\partial R}{\partial x}=D\frac{\partial^2 R}{\partial v^2}. \tag{2.3}$$

We note that the solution of this equation can be interpreted as the probability distribution of the velocities and coordinates of one particle at the instant t, if at the instant t = 0 its value was

$$R_0(v,x)=\int_{-\infty}^{\infty}du\int_0^{\infty}d\rho\,\rho w_0[v,u,\rho;x].$$

Equation (2.3) itself is then the well known Einstein-Fokker-Planck equation, which describes the diffusion of particles in a field of random forces (see, e.g., [6,8]).

Another particular case of (2.1) is the integro-differential equation obtained in [10] for the one-dimensional probability density of the beam velocity field v(x, t). It follows from (2.1) if the latter is integrated term by term with respect to ρ and u, and if simple transformations are made.

If the beam propagates in the absence of external forces ($D = B \equiv 0$), i.e., if the velocity of each beam particle remains unchanged, then (2.1) can easily be solved by the method of characteristics:

$$w[v,u,\rho;x,t]=w_0\left[v,\frac{u}{1-ut},\frac{\rho}{|1-ut|};x-vt\right]\frac{1}{(1-ut)^4}. \tag{2.4}$$

2. Let us analyze the result by assuming the density $\rho_0(x)$ and the velocity $v_0(x)$ of the beam particles at the initial instant of time to be specified functions of x. We assume also the beam to be initially single-stream. In this case the velocities of particles situated in physically infinitesimally small intervals dx are practically indistinguishable from one another, and the functions $v_0(x)$ and $\rho_0(x)$ are single-valued. By the same token, we consider the initial condition

$$w_0=\delta[v-v_0(x)]\delta[u-u_0(x)]\delta[\rho-\rho_0(x)],$$

and consequently

$$w=\delta[v-v_0(x-vt)]\delta\left[\frac{u}{1-ut}-u_0(x-vt)\right]$$
$$\times\delta\left[\frac{\rho}{|1-ut|}-\rho_0(x-vt)\right]\frac{1}{(1-ut)^4}. \tag{2.5}$$

Let the particle velocities $v_0(x)$ vary sufficiently smoothly along x, i.e., let the modulus of $u_0(x)$ be limited. Then the beam remains single-stream within times $0 \le t \le 1/u_0$, where $u_0 = -\min u_0(x)$ (see, e.g., [8,11]). During this time interval, (2.5) can be represented in the form

$$w=\delta[v-v_1]\delta\left[u-\frac{u_0(x-v_1t)}{1+tu_0(x-v_1t)}\right]\delta\left[\rho-\frac{\rho_0(x-v_1t)}{|1+tu_0(x-v_1t)|}\right],$$

where $v_1 = v_1(x, t)$ is the only root of the equation

$$v - v_0(x - vt) = 0, \qquad (2.6)$$

and is equal to the particle velocity of the single-stream beam at the x at the instant of time t.

The expression obtained is the probability distribution of the velocity, of the velocity gradient, and of the density of a determined single-stream beam of noninteracting particles.

The solutions (2.4) and (2.5) acquire entirely different meanings in time intervals in which some particles of the beam overtake the others and the beam becomes multiple-stream. Assume that a given instant of time t and at a given point x the beam consists of N streams, i.e., the velocity v(x, t) assumes the N values $v_1(x, t) \dots v_N(x, t)$ of the roots of Eq. (2.6). Then (2.5) can be rewritten in the following equivalent form:

$$w = \sum_{n=1}^{N} \delta(v - v_n)\, \delta\left[u - \frac{u_0(x - v_n t)}{1 + t u_0(x - v_n t)}\right] \delta\left[\rho - \frac{\rho_0(x - v_n t)}{|1 + t u_0(x - v_n t)|}\right]. \qquad (2.7)$$

It is easily seen that here w no longer has the meaning of the probability distribution of a beam, but is the sum of the probability distributions of each of the N streams of the beam. It is obvious that the normalization of w is equal to the sum of the normalizations of the probability distributions of all the fluxes, i.e., to N.

3. Let now $\rho_0(x)$, $v_0(x)$, and $u_0(x)$ be random functions with specified statistical properties. Averaging (2.5) over the random initial conditions, we arrive at (2.4). Therefore the meaning of the function w specified by formula (2.4) is determined entirely by the meaning of w for the determined beam (2.5). For example, so long as a beam with unity probability is single-stream, the quantity w expressed by formula (2.4) has, just as in the determined case, the meaning of the probability distribution of the beam parameters. In a multiple-stream beam, on the other hand, w can be represented in analogy with expression (2.7) in the following manner:

$$w = \sum_{N=1}^{\infty} p(N; x, t) \sum_{n=1}^{N} w_n[v, u, \rho; x, t]. \qquad (2.8)$$

Here p(N: x, t) is the probability that the beam at the point x and at the instant t consists of N streams, and $w_n(v, u, \rho; x, t \mid N)$ is the probability distribution of the parameters of the n-th stream in the N-stream beam. We note that $p(N; x, t)$ and $w_n[v, u, \rho; x, t \mid N]$ cannot be obtained separately from each other within the framework of this analysis.

Thus, in a multiple-stream beam, w is not the probability distribution. As follows from (2.8), the normalized value of w is equal to the average number of streams of the beam at a given instant of time t and at the given point x:

$$\int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du \int_{0}^{\infty} d\rho\, w[v, u, \rho; x, t] = \sum_{N=1}^{\infty} N p(N; x, t) = \langle N(x, t) \rangle. \qquad (2.9)$$

4. It follows from (2.8) that the mean value of any function of the beam parameters, calculated with the aid of w, is in fact the weighted sum of the mean values in each of the streams of the multiple-stream beam:

$$\langle f(v, u, \rho) \rangle = \sum_{N=1}^{\infty} p(N; x, t) \sum_{n=1}^{N} \langle f(v, u, \rho) \rangle_{n,N}. \qquad (2.10)$$

Here

$$\langle f(v, u, \rho) \rangle_{n,N} = \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du \int_{0}^{\infty} d\rho\, f(v, u, \rho)\, w_n[v, u, \rho; x, t \mid N]$$

is the mean value of the function $f(v, u, \rho)$ in the n-th stream of the N-stream beam.

It is seen from (2.8) and (2.10) that in the region where the beam is multiple-stream the mean values obtained with the aid of w do not always have the same meaning as for a single-stream beam. Thus, whereas the average beam density $\langle \rho \rangle$, the average flux $\langle \rho v \rangle$, and the average kinetic-energy density $\langle \rho v^2 \rangle/2$ of the beam, which are equal to the sums of the corresponding mean values for each of the streams, retain their meaning also in the multiple-stream beam, the mean value $\langle \ln \rho \rangle$, for example, calculated with the aid of w for a multiple-stream beam, no longer has the meaning of the average logarithm of the total density, and is equal to the sum of the average logarithms of the densities of each of the streams—a quantity that has no clear-cut physical meaning.

3. AVERAGE NUMBER OF BEAM STREAMS

1. An important physical characteristic of a beam of noninteracting particles is the average number of its streams. Let us calculate this average value, integrating (2.4) with respect to v, u, ρ and making the change of variable $z = u/(1 - ut)$ under the integral sign. As a result we get

$$\langle N(x, t) \rangle = \iint_{-\infty}^{\infty} w_0[v, z; x - vt]\, |1 + zt|\, dv\, dz,$$

where

$$w_0[v, u; x] = \int_0^{\infty} w_0[v, u, \rho; x]\, d\rho$$

is the simultaneous single-point distribution of v and u at the initial instant of time. In the case when this distribution does not depend on x, the expression for the average number of streams becomes even simpler:

$$\langle N(x, t) \rangle = \int_{-\infty}^{\infty} w_0[u]\, |1 + ut|\, du. \qquad (3.1)$$

Here

$$w_0[u] = \int_{-\infty}^{\infty} w_0[v, u]\, dv$$

is the probability distribution of the function $u_0(x)$.

It follows from (3.1) that although at large time intervals the number of streams is described by the asymptotic formula

$$\langle N(x, t) \rangle = \gamma_u t, \qquad (3.2)$$

where

$$\gamma_u = \int_{-\infty}^{\infty} |u|\, w_0[u]\, du$$

characterizes the average spread of the velocity gradient, which exists at the initial instant of time. Obviously, the larger γ_u, the more rapidly are spillovers produced and the larger, according to (3.2), the rate of growth of the number of streams. To better understand the proportionality to t, we break up the beam at the initial instant of time into parts of length l_0 equal to the characteristic scale of variation of the velocity field $v_0(x)$. It is obvious that after a sufficiently long time each of these parts of the beam will spread out and occupy an interval of length $L \sim \gamma_v t$, where γ_v is the average velocity spread of the beam particles. It is obvious here that N parts of the beam will overlap at each point. Their number is approximately equal to

$$N \sim L/l_0 \sim (\gamma_v/l_0)\, t.$$

Since $\gamma_v/l_0 \sim \gamma_u$, this formula explains qualitatively the relation (3.2).

2. So long as the average number of streams is not very large, it is possible sometimes to neglect the fact that the beam is multiple-stream and ascribe to w the meaning of the ordering probability density in a single-stream beam.

Let us calculate by way of example the average number of streams, assuming $w_0[u]$ to be a Gaussian distribution. In this case we have on the basis of (3.1)

$$\langle N(x,t)\rangle = \sqrt{\frac{2}{\pi}}\int_0^{1/\sqrt{2}\sigma t}\exp\left\{-\frac{z^2}{2}\right\}dz + \sigma t\sqrt{\frac{2}{\pi}}\exp\left\{-\frac{1}{2\sigma^2 t^2}\right\}. \tag{3.3}$$

The quantity $\sigma = (\langle u_0^2\rangle)^{1/2}$ has here the same meaning and the same order of magnitude as γ_u. At $\sigma t = 1$, i.e., over times during which the velocity fluctuations of the beam particles lead to large changes in the beam density, we have $\langle N\rangle \approx 1.17$ and in the solution of many problems the beams can be regarded as single-stream, while the function w can be regarded as the probability density. On the other hand, (3.3) goes over into (3.2) at $\sigma t \gg 1$, as it should,

$$\langle N(x,\ t)\rangle = \sigma\sqrt{2/\pi}\,t = \gamma_u t.$$

4. BEAM-PARTICLE DENSITY DISTRIBUTION

1. Let us find the probability distribution of beam particles moving with constant velocity, assuming for the sake of argument that at the initial instant of time the particle density is known and equal to $\rho_0(x)$. Then the simultaneous one-point distribution of v, u, and ρ, which takes at t = 0 the form

$$w[v,\ u,\ \rho;\ x,\ 0] = w_0[v,\ u;\ x]\delta[\rho-\rho_0(x)],$$

becomes at $t > 0$, according to (2.4),

$$w = w_0\left[v,\frac{u}{1-ut};x-vt\right]\delta\left[\frac{\rho}{|1-ut|}-\rho_0(x-vt)\right]. \tag{4.1}$$

integrating this expression with respect to v and u, we obtain the sought probability density $G[\rho;\ x,\ t]$:

$$\begin{aligned}G = \frac{1}{\rho^3 t}\int_{-\infty}^{\infty}\rho_0^2(x-vt)\left\{w_0\left[v,\frac{1}{t}\left(\frac{\rho_0(x-vt)}{\rho}-1\right);x-vt\right]\right.\\ \left.+w_0\left[v,-\frac{1}{t}\left(\frac{\rho_0(x-vt)}{\rho}+1\right);x-vt\right]\right\}dv.\end{aligned} \tag{4.2}$$

As already mentioned, in a single-stream beam G has the meaning of the probability distribution of the density, and in a multiple-stream beam it is equal to the weighted sum of the corresponding probability distributions.

2. Let us consider a particular and rather interesting case, in which the beam density ρ_0 is constant along x at the initial instant of time, and the velocity field is statistically homogeneous, i.e., $w_0[v,\ u;\ x] \equiv w_0[v,\ u]$. In this case (4.2) becomes

$$G[\rho;t] = \frac{1}{\rho t}\left(\frac{\rho_0}{\rho}\right)^2\left\{w_0\left[\frac{1}{t}\left(\frac{\rho_0}{\rho}-1\right)\right]+w_0\left[-\frac{1}{t}\left(\frac{\rho_0}{\rho}+1\right)\right]\right\}, \tag{4.3}$$

where $w_0[u]$, as before, is the distribution function of $u_0(x)$.

We note that expression (4.3) as applied to the propagation of light waves behind a phase screen (see Sec. 5) was obtained earlier in [12,13]. In these papers, however, they did not make clear enough the meaning of this distribution, which is connected with the multiple-stream character of the beam or, in the language of light waves, with the fact that at a large distance behind a phase screen several quasimonochromatic waves arrive at each point at different angles.

3. Let us investigate in greater detail the distribution (4.3). Using this distribution, we can easily show for example, that

$$\langle\rho(x,t)\rangle = \int_0^{\infty}\rho G[\rho;t]d\rho = \rho_0\int_{-\infty}^{\infty}w_0[u]du = \rho_0,$$

i.e., the average density of a statistically homogeneous beam, having the same physical meaning in the single-stream and multiple-stream beams, is conserved, as expected, and does not depend on the number of streams in the beam.

Let us ascertain the behavior of $G[\rho;\ t]$ as $\rho \to \infty$. From (4.3) we get

$$G[\rho;t] = \frac{2}{\rho^3}\frac{\rho_0^2}{t}w_0\left[-\frac{1}{t}\right]\quad(\rho\to\infty). \tag{4.4}$$

We assume now that at the initial instant of time the smoothness at which the particle velocity varies along x is limited by the condition $w_0[u] \equiv 0$ at $u < -u_0 < 0$. Then, over times $t < 1/u_0$, it follows from (4.4) that $G[\rho;\ t] \equiv 0$ as $\rho \to \infty$. The exact formula (4.3) shows that this equality is valid for all $\rho > \rho^* = \rho_0/(1-u_0t)$. This means that over these time intervals the particle density in the beam remains finite everywhere. We note here that the condition $t < 1/u_0$ is none other than the condition that the beam be single-stream, and $t_c = 1/u_0$ is the time during which caustics are produced in a beam that is initially single-stream and the beam becomes multiple-stream. Thus, for arbitrary t, the beam density does not exceed $\rho^* = \rho^*(t)$.

Over times $t > t_c$, when the beam ceases to be single-stream, the value of $G[\rho;\ t]$ decreases like $1/\rho^3$ as $\rho \to \infty$, in accordance with (4.4), i.e., so slowly that $\langle\rho^n\rangle = \infty$ at $n \geq 2$. This result is a consequence of the singularities of the beam density on the caustics that appear when spillovers are produced (see, e.g., [3,8]).

So slow a decrease of the probability distribution of the particle density as $\rho \to \infty$ can be easily explained by assuming that (x, t) is an ergodic function in x, for which

$$\langle\rho^n(x,t)\rangle = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\rho^n(x,t)\,dx. \tag{4.5}$$

As is well known (see, e.g., [8]), the realizations of the beam density of the noninteracting particles in the hydrodynamic approximation have near the caustics singularities of the type $\rho \sim 1/\sqrt{x}$. Therefore, over times at which the probability of the existence of caustics is not equal to zero, the integral (4.5) diverges and $\langle\rho^n\rangle = \infty$, so long as $n \geq 2$. Thus, $G[\rho;\ t]$ decreases as $\rho \to \infty$ like $1/\rho^3$ because the realizations of the density have singularities of the type $\rho \sim 1/\sqrt{x}$ near the caustics.

4. Let us continue the discussion of the distribution (4.3). For a single-stream beam, when the second term in the right-hand side of (4.3) vanishes identically, $G[\rho;\ t]$ takes the form

$$G[\rho;t] = \frac{\rho_0^2}{t\rho^3}w_0\left[\frac{1}{t}\left(\frac{\rho_0}{\rho}-1\right)\right]. \tag{4.6}$$

Let us clarify how the intensity variance $\delta = \langle(\rho-\rho_0)^2\rangle\rho_0^2$ tends to infinity as $t \to t_c$ in the simplest case when

$$w_0[u] = \begin{cases} t_c/2; & |u|<1/t_c,\\ 0; & |u|\geqslant 1/t_c.\end{cases} \tag{4.7}$$

Calculations show that

$$\delta=\frac{t_c}{2t}\ln\left(\frac{t_c+t}{t_c-t}\right), \tag{4.8}$$

i.e., when t tends to the time t_c during which the caustics are produced, the variance of the density tends to infinity like $-\ln(t_c-t)$. On the other hand, if we consider the quantity $\chi=\ln(\rho/\rho_0)$, then we can show with the aid of (4.6) and (4.7) that the mean values $\langle\chi^n\rangle$ for arbitrary n and $t\to t_c$ remain finite. Thus, for example, by calculating $\langle\chi\rangle$ and $\langle\chi^2\rangle$ with the aid of (4.6) and (4.7) we obtain

$$\langle\chi\rangle=\frac{t_c}{4t}\left\{\left(1+\frac{t}{t_c}\right)^2\left[1-\ln\left(1+\frac{t}{t_c}\right)\right]-\left(1-\frac{t}{t_c}\right)\left[\frac{1}{2}-\ln\left(1-\frac{t}{t_c}\right)\right]\right\}, \tag{4.9}$$

$$\langle\chi^2\rangle=\langle\chi\rangle+\frac{t_c}{4t}\left[\left(1+\frac{t}{t_c}\right)^2\ln\left(1+\frac{t}{t_c}\right)-\left(1-\frac{t}{t_c}\right)^2\ln\left(1-\frac{t}{t_c}\right)^2\right].$$

Estimating their value at $t=t_c$, we find that $\langle\chi\rangle\approx 0.19$ and $\langle\chi^2\rangle\approx 0.29$.

5. We note in conclusion that the results have a geometrical interpretation. Thus, by virtue of the aforementioned analogy we can interpret w in the single-stream case as the probability distributions of the propagation angles v, of the curvature of the wave front u, and of the intensity ρ of the light wave at a point with spatial coordinates (x, t) in the small-angle approximation of two-dimensional geometrical optics. In particular, expression (4.2) describes the probability distribution of the intensity of the light wave at the points (x, t) at a distance t from a one-dimensional phase screen placed in the (x, 0) plane, on which a light beam is perpendicularly incident with intensity $\rho_0(x)$. The statistical properties of the phase screen are described by the probability distribution $w_0[v, u; x]$.

A more complete and consistent investigation of the propagation of the wave in the geometrical-optical approximation with the aid of the equations obtained in this paper will be reported elsewhere. We note for the time being that expression (4.6) can be regarded as the distribution of the intensity behind a statistically homogeneous one-dimensional phase screen at distances $t < t_c$ from the screen, in which there are still no caustics and the geometrical-optics approximation is still valid.

6. Usually the statistical properties of a phase screen are so specified (see, e.g. [14]) that $w_0[u]$ is a Gaussian distribution and caustics are produced at arbitrary $t>0$. In other words, immediately behind the phase screen, as follows from (4.6), we have $\langle\rho^2\rangle=\infty$. This result is incorrect because the geometrical-optics approximation is violated in the vicinity of the caustics and in the calculation of $\langle\rho^2\rangle$ it is necessary to take the diffraction into account.

However, as seen from (4.9), unlike the moments of the intensity, the moments of the level of the light wave are critical to the appearance of caustics. It can therefore be assumed that so long as $\langle N\rangle\approx 1$ the light wave is practically single-valued and the probability density (4.6) describes adequately the evolution of the moments of the level with increasing distance from the phase screen.

As already mentioned, in the concrete case of a Gaussian distribution $w_0[u]$ at $\sigma t=1$, i.e., in the region of strong fluctuations of the intensity and of the developed caustics, we have $\langle N\rangle\approx 1.17$, which is close enough to unity, so that we can expect at these distances that the probability density

$$g[\chi;t]=\frac{1}{(2\pi\sigma^2t^2)^{1/2}}\exp\left\{-2\chi-\frac{1}{2\sigma^2t^2}(e^{-\chi}-1)^2\right\},$$

obtained with the aid of (4.6) makes it possible to determine sufficiently correctly the values of the moments of the level of the light wave behind the phase screen.

[1] A. S. Monin and A. M. Yaglom, Statisticheskaya gidromekhanika (Statistical Hydromechanics), Parts 1, 2, Nauka (1965, 1967).

[2] V. N. Tsytovich, Nelineĭnye effekty v plazme (Nonlinear Effects in Plasma), Nauka (1967).

[3] B. B. Kadomtsev, and V. I. Karpman, Usp. Fiz. Nauk **103**, 193 (1971) [Sov. Phys.-Uspekhi **14**, 40 (1971)].

[4] V. I. Karpman, Nelineĭnye volny v dispergiruyushchikh sredakh (Nonlinear Waves in Dispersive Media), Nauka (1973).

[5] A. S. Monin, Prik. Mat. Mekh. **31**, 6 (1967).

[6] V. I. Klyatskin and V. I. Tatarskiĭ, Izv. vuzov, Radiofizika **15**, 1433 (1972).

[7] V. P. Kuznetsov, Akust. Zh. **15**, 554 (1969) [Sov. Phys.-Acoust. **15**, 480 (1970)].

[8] Ya. B. Zel'dovich and A. D. Myshkis, Elementy matematicheskoi fiziki (Elements of Mathematical Physics), Nauka (1973).

[9] K. Furutsu, J. Res. NBS **D67**, No. 3, 305 (1963).

[10] A. N. Malakhov and A. I. Saichev, Izv. vuzov, Radiofizika **17**, 699 (1974).

[11] R. Courant, Partial Differential Equations, Wiley, 1962.

[12] E. E. Salpeter, Astrophys J., **147**, 433, 1967.

[13] V. A. Alimov and L. M. Erukhimov, Izv. vuzov, Radiofizika **16**, 1540 (1973).

[14] V. V. Tamoĭkin and A. A. Fraĭman, Izv. vuzov, Radiofizika **14**, 1427 (1971).

Translated by J. G. Adashko
107