Nonlinear cyclotron resonance in a plasma

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It is shown that the interaction of charged particles with the field of a traveling monochromatic plane wave of small but finite amplitude under resonance conditions at half-integer cyclotron frequency harmonics $\omega(k)=k_z v_z + \frac{1}{2} n\omega_B (n=+1, \pm 3, ...)$ results in the appearance of wave damping.

1. INTRODUCTION

It is shown in the present paper that electromagnetic waves of small but finite amplitude can be absorbed effectively by resonant particles of a plasma in a magnetic field for which the condition

$$\omega(\mathbf{k})=k_z v_z+{}^1/_2 n\omega_B, \quad n=\pm 1, \pm 3, \ldots, \tag{1.1}$$

is satisfied, where k is the wave vector, $\omega(k)$ the frequency of the wave, ω_B the cyclotron frequency of the particle, and k_z and v_z the projections of the wave vector and the particle velocity on the direction of the external magnetic field $\mathbf{B}_0$.

The origin of this resonance, which can be called resonance at half-integer harmonics of the cyclotron frequency, can be understood from the following considerations. In the absence of an electromagnetic wave, a charged particle moving in a magnetic field along a helix can be regarded as an oscillator with the eigenfrequency ω_B. The Lorentz force exerted on the particle by the field of the wave oscillates, in the approximation linear in the field amplitude, with frequency $\omega(k) - k_z v_z$, and with account of the finiteness of the Larmor radius of the particle, with frequency $\omega(k) - k_z v_z - s\omega_B$ (s is an integer).

Linear cyclotron resonance occurs upon coincidence of these frequencies with ω_B. With account of the small (in comparison with the wavelength) but finite displacement of the particles in the wave field, which oscillates with the frequencies $\omega(k) - k_z v_z - l\omega_B$, $l = 0, \pm 1, \ldots$, the Lorentz force will contain a term that is quadratic in the field amplitude and that oscillates with the frequencies $2\omega(k) - 2k_z v_z - (l + s)\omega_B$. Nonlinear cyclotron resonance takes place upon coincidence of this frequency with ω_B.

The present paper is devoted to the theory of nonlinear cyclotron resonance. The motion of a single charged particle in the field of a plane traveling monochromatic wave is considered by the method of averaging under the resonance conditions (1.1). The width of the region of particle capture on the phase plane ($\bar{v}_z$, ψ_n) where ψ_n is the resonance phase, and the frequency of oscillations of the captured particles Ω_{tr} are found. The damping decrement of the wave is found under conditions when capture of the resonant particles by the wave field does not occur. The value of the damping decrement turns out to be equal, in order of magnitude, to

$$\gamma\sim\frac{v_{\mathscr{E}}^2}{v_T^2}\omega, \quad |n|=1, \tag{1.2}$$

$$\gamma\sim\frac{v_{\mathscr{E}}^2}{v_T^2}\left(\frac{kv_T}{\omega_B}\right)^{2|n|-3}\omega, \quad n=\pm 3, \pm 5, \ldots, \quad \frac{kv_\perp}{\omega_B}\ll 1, \tag{1.3}$$

where $v_{\mathscr{E}} = e\mathscr{E}/m\omega$ is the velocity acquired by the particle in the wave field and v_T is the characteristic thermal velocity. (In obtaining the estimates (1.2) and (1.3), it was assumed that $k_z \sim k_\perp$ and $v_T \ll \omega/k$.)

A comparison of (1.2) and (1.3) shows that the resonances at the higher odd harmonics ($|n| \geq 3$) lead to very weak damping. The damping of the field at $|n| = 1$ may prove important for comparatively small fields, and the resonance $|n| = 1$ can be used to heat the ions in a plasma to high temperatures.

2. MOTION OF THE CHARGED PARTICLE

The equations of motion of a nonrelativistic charged particle have the form

$$m\frac{d\mathbf{v}}{dt}=e\mathbf{E}+\frac{e}{c}[\mathbf{v},\mathbf{B}+\mathbf{B}_0], \quad \mathbf{v}=\frac{d\mathbf{r}}{dt}, \tag{2.1*}$$

where E and B are the electric and magnetic fields of the wave,

$$\mathbf{E}=\mathrm{Re}\,\mathscr{E}\mathbf{e}e^{i(\mathbf{kr}-\omega t+\alpha)}, \quad \mathbf{B}=\mathrm{Re}\frac{c}{\omega}[\mathbf{k},\mathbf{E}], \tag{2.2}$$

E is the amplitude, α the initial phase, and ~~e the~~ polarization vector of the wave. For weak absorption, the amplitude of the wave E and the phase α change slowly with time. In this case, the polarization vector can be chosen in the form $\mathbf{e} = (e_x, ie_y, e_z)$; e_x, e_y, e_z are real numbers.[2]

If the amplitude E and the phase α of the wave are constant, there exist three exact integrals of motion of

$$\mathbf{v}-\omega_B\left[\mathbf{r},\frac{\mathbf{B}_0}{B_0}\right]-\frac{\mathbf{k}}{2\omega}v^2-\frac{e\mathscr{E}}{m\omega}\mathrm{Re}\,i\mathbf{e}e^{i(\mathbf{kr}-\omega t+\alpha)}=\mathbf{C}, \tag{2.3}$$

Eqs. (2.1): where C is a constant of integration. (The integrals (2.3) were obtained in the relativistic case by Woolley.[3]) It is not possible to carry out further integration of Eqs. (2.1) and we must turn to approximate methods to study the character of motion of the particle.

We choose a set of coordinates with the z axis parallel to the external magnetic field $\mathbf{B}_0$, and the x axis lying in the plane of the vectors k and $\mathbf{B}_0$.

Making the following substitution of variables in Eqs. (2.1):

$$v_x=v_\perp\cos\theta, \quad x=\xi-\frac{v_\perp}{\omega_B}\sin\theta,$$
$$v_y=v_\perp\sin\theta, \quad y=\eta+\frac{v_\perp}{\omega_B}\cos\theta, \tag{2.4}$$

where θ is the aximuthal angle in velocity space and ξ and η are the transverse coordinates of the Larmor center, we get

$$\frac{d\xi}{dt}=-\frac{e\mathscr{E}}{m\omega_B}\sum_s R_\xi^{(s)}\sin(\Phi_s+\alpha), \quad \frac{d\eta}{dt}=-\frac{e\mathscr{E}}{m\omega_B}\sum_s R_\eta^{(s)}\cos(\Phi_s+\alpha),$$

$$\frac{dv_z}{dt}=\frac{e\mathscr{E}}{m}\sum_s R_{v_z}^{(s)}\cos(\Phi_s+\alpha), \quad \frac{dv_\perp}{dt}=\frac{e\mathscr{E}}{m}\sum_s R_{v_\perp}^{(s)}\cos(\Phi_s+\alpha),$$

$$\frac{d\theta}{dt}=-\omega_B+\frac{1}{v_\perp}\frac{e\mathscr{E}}{m}\sum_s R_\theta^{(s)}\sin(\Phi_s+\alpha).$$

Here

$$
\begin{aligned}
R_\xi^{(s)} &= e_y\left(1-\frac{s\omega_B}{\omega}-\frac{k_z v_z}{\omega}\right)J_s(a),\\
R_\eta^{(s)} &= \left[e_x\left(1-\frac{k_z v_z}{\omega}\right)+e_z\frac{k_x v_z}{\omega}\right]J_s(a)-e_y\frac{k_x v_\perp}{\omega}J_s'(a),\\
R_{v_z}^{(s)} &= \left[e_z\left(1-\frac{s\omega_B}{\omega}\right)+e_x\frac{k_z}{k_x}\frac{s\omega_B}{\omega}\right]J_s(a)-e_y\frac{k_z v_\perp}{\omega}J_s'(a).\\
R_{v_\perp}^{(s)} &= \left[e_x\left(1-\frac{k_z v_z}{\omega}\right)+e_z\frac{k_x v_z}{\omega}\right]\frac{s}{a}J_s(a)-e_y\left(1-\frac{k_z v_z}{\omega}\right)J_s'(a),\\
R_\theta^{(s)} &= \left[e_x\left(1-\frac{k_z v_z}{\omega}\right)+e_z\frac{k_x v_z}{\omega}\right]J_s'(a)\\
&\quad -e_y\left(1-\frac{k_z v_z}{\omega}\right)\frac{s}{a}J_s(a)+e_y\frac{k_x v_\perp}{\omega}J_s(a),
\end{aligned}
\tag{2.6}
$$

$J_s(a)$ is a Bessel function, $a = k_x v_\perp/\omega_B$, and $\Phi_s = k_z z - \omega t - s\theta + k_x\xi$.

We note that the equations for the quantities ξ, v_z, $v_\perp$ and θ do not contain η. Therefore the equation for η is given no further consideration.

We shall assume that the amplitude of the wave is small:

$$
c\frac{\mathscr{E}}{B_0}\ll\frac{|\omega_B|}{k},\qquad c\frac{\mathscr{E}}{B_0}\ll v_\perp. \tag{2.7}
$$

Then the set of equations (2.5) contains rapidly changing phase θ and $k_z z - \omega t$, and can be investigated by the method of averaging.[1] To study Cherenkov and cyclotron resonances, it suffices to restrict ourselves to an approximation that is linear in the amplitude.[4,5] The study of the resonances (1.1) requires consideration of the next approximation, which is quadratic in the field amplitude. For this purpose, we make the following substitution of variables in Eqs. (2.5):

$$
v_z=\bar v_z+\tilde v_z,\quad v_\perp=\bar v_\perp+\tilde v_\perp,\quad \theta=\bar\theta+\tilde\theta,\quad \xi=\bar\xi+\tilde\xi,\quad z=\bar z+\tilde z, \tag{2.8}
$$

where

$$
\begin{aligned}
\tilde v_z &= \frac{e\mathscr{E}}{m}\sum_l\frac{R_{v_z}^{(l)}(\bar v_\perp)}{l\omega_B-\omega+k_z\bar v_z}\sin(\bar\Phi_l+\alpha),\\
\tilde v_\perp &= \frac{e\mathscr{E}}{m}\sum_l\frac{R_{v_\perp}^{(l)}(\bar v_z,\bar v_\perp)}{l\omega_B-\omega+k_z\bar v_z}\sin(\bar\Phi_l+\alpha),\\
\tilde\theta &= -\frac{1}{\bar v_\perp}\frac{e\mathscr{E}}{m}\sum_l\frac{R_\theta^{(l)}(\bar v_z,\bar v_\perp)}{l\omega_B-\omega+k_z\bar v_z}\cos(\bar\Phi_l+\alpha),\\
\tilde\xi &= \frac{e\mathscr{E}}{m\omega_B}\sum_l\frac{R_\xi^{(l)}(\bar v_z,\bar v_\perp)}{l\omega_B-\omega+k_z\bar v_z}\cos(\bar\Phi_l+\alpha),\\
\tilde z &= -\frac{e\mathscr{E}}{m}\sum_l\frac{R_{v_z}^{(l)}(\bar v_\perp)}{(l\omega_B-\omega+k_z\bar v_z)^2}\cos(\bar\Phi_l+\alpha).
\end{aligned}
\tag{2.9}
$$

As a result of this substitution, we get the following equations for the mean values $\bar\xi$, $\bar v_z$, $\bar v_\perp$, $\bar\theta$ under the resonance conditions (1.1):

$$
\begin{aligned}
\frac{\partial\bar\xi}{dt} &= \left(c\frac{\mathscr{E}}{B_0}\right)^2\frac{k}{\omega_B}F_\xi(\bar v_z,\bar v_\perp)\cos(\psi_n+2\alpha),\\
\frac{d\bar v_z}{dt} &= \left(c\frac{\mathscr{E}}{B_0}\right)^2 kF_{v_z}(\bar v_z,\bar v_\perp)\sin(\psi_n+2\alpha),\\
\frac{d\bar v_\perp}{dt} &= \left(c\frac{\mathscr{E}}{B_0}\right)^2 kF_{v_\perp}(\bar v_z,\bar v_\perp)\sin(\psi_n+2\alpha),\\
\frac{d\bar\theta}{dt} &= -\omega_B+\frac{1}{\bar v_\perp}\left(c\frac{\mathscr{E}}{B_0}\right)^2 kF_{1\theta}(\bar v_z,\bar v_\perp)+\frac{1}{\bar v_\perp}\left(c\frac{\mathscr{E}}{B_0}\right)^2 kF_{2\theta}(\bar v_z,\bar v_\perp)\cos(\psi_n+2\alpha),
\end{aligned}
\tag{2.10}
$$

where $k = \sqrt{k_x^2 + k_z^2}$ and ψ_n is the resonance phase:

$$
\psi_n=2k_z\bar z-2\omega t-n\bar\theta+2k_x\bar\xi. \tag{2.11}
$$

The quantities $F_\alpha(\bar v_z, \bar v_\perp)$ are determined by the expressions

$$
\begin{aligned}
F_\xi(\bar v_z,\bar v_\perp) &= \frac12\sum_s\frac{\omega_B}{s\omega_B-\omega+k_z\bar v_z}\left\{\frac{k_x}{k}\left[-R_\xi^{(s)}R_\xi^{(n-s)}\right.\right.\\
&\quad\left.+R_{v_\perp}^{(s)}\frac{\partial}{\partial\bar a}R_\xi^{(n-s)}-\frac{n-s}{\bar a}R_\theta^{(s)}R_\xi^{(n-s)}\right]\\
&\quad\left.+\frac{\omega_B}{s\omega_B-\omega+k_z\bar v_z}\frac{k_z}{k}2e_y\left(1-\frac{n\omega_B}{2\omega}-\frac{k_z\bar v_z}{\omega}\right)J_{n-s}(\bar a)R_{v_z}^{(s)}\right\},\\
F_{v_z}(\bar v_z,\bar v_\perp) &= \frac12\sum_s\frac{\omega_B}{s\omega_B-\omega+k_z\bar v_z}\left\{\frac{k_x}{k}\left[-R_\xi^{(s)}R_{v_z}^{(n-s)}+R_{v_\perp}^{(s)}\frac{\partial}{\partial\bar a}R_{v_z}^{(n-s)}\right.\right.\\
&\quad\left.\left.-\frac{n-s}{\bar a}R_\theta^{(s)}R_{v_z}^{(n-s)}\right]+\frac{\omega_B}{s\omega_B-\omega+k_z\bar v_z}\frac{k_z}{k}R_{v_z}^{(s)}R_{v_z}^{(n-s)}\right\},\\
F_{v_\perp}(\bar v_z,\bar v_\perp) &= \frac12\sum_s\frac{\omega_B}{s\omega_B-\omega+k_z\bar v_z}\left\{\frac{k_x}{k}\left[-R_\xi^{(s)}R_{v_\perp}^{(n-s)}+R_{v_\perp}^{(s)}\frac{\partial}{\partial\bar a}R_{v_\perp}^{(n-s)}\right.\right.\\
&\quad\left.\left.-\frac{n-s}{\bar a}R_\theta^{(s)}R_{v_\perp}^{(n-s)}\right]+\frac{\omega_B}{k}R_{v_z}^{(s)}\frac{\partial}{\partial\bar v_z}R_{v_\perp}^{(n-s)}+\frac{\omega_B}{s\omega_B-\omega+k_z\bar v_z}\frac{k_z}{k}R_{v_z}^{(s)}R_{v_\perp}^{(n-s)}\right\},\\
F_{1\theta}(\bar v_z,\bar v_\perp) &= \frac12\sum_s\frac{\omega_B}{s\omega_B-\omega+k_z\bar v_z}\left\{\frac{k_x}{k}\left[R_\theta^{(s)}R_\xi^{(s)}-\frac{1}{\bar a}R_\theta^{(s)}R_{v_\perp}^{(s)}\right.\right.\\
&\quad\left.+R_{v_\perp}^{(s)}\frac{\partial}{\partial\bar a}R_\theta^{(s)}+\frac{s}{\bar a}R_\theta^{(s)2}\right]+\frac{\omega_B}{k}R_{v_z}^{(s)}\frac{\partial}{\partial\bar v_z}R_\theta^{(s)}\\
&\quad\left.-\frac{\omega_B}{s\omega_B-\omega+k_z\bar v_z}\frac{k_z}{k}R_{v_z}^{(s)}R_\theta^{(s)}\right\},\\
F_{2\theta}(\bar v_z,\bar v_\perp) &= \frac12\sum_s\frac{\omega_B}{s\omega_B-\omega+k_z\bar v_z}\left\{\frac{k_x}{k}\left[R_\xi^{(s)}R_\theta^{(n-s)}\right.\right.\\
&\quad\left.+\frac{1}{\bar a}R_{v_\perp}^{(s)}R_\theta^{(n-s)}-R_{v_\perp}^{(s)}\frac{\partial}{\partial\bar a}R_\theta^{(n-s)}+\frac{n-s}{\bar a}R_\theta^{(s)}R_\theta^{(n-s)}\right]\\
&\quad\left.-\frac{\omega_B}{k}R_{v_z}^{(s)}\frac{\partial}{\partial\bar v_z}R_\theta^{(n-s)}-\frac{\omega_B}{s\omega_B-\omega+k_z\bar v_z}\frac{k_z}{k}R_{v_z}^{(s)}R_\theta^{(n-s)}\right\}.
\end{aligned}
\tag{2.12}
$$

The functions $R_\alpha^{(s)}$ and $R_\alpha^{(n-s)}$ in these expressions depend on the averaged variables $\bar v_z$, $\bar v_\perp$.

It is also convenient to introduce the equation for the resonance phase

$$
\begin{aligned}
\frac{d\psi_n}{dt} &= n\omega_B-2\omega+2k_z\bar v_z-\frac{n}{\bar v_\perp}\left(c\frac{\mathscr{E}}{B_0}\right)^2 kF_{1\theta}\\
&\quad+\left(c\frac{\mathscr{E}}{B_0}\right)^2 k\left(\frac{2k_x}{\omega_B}F_\xi-\frac{n}{\bar v_\perp}F_{2\theta}\right)\cos(\psi_n+2\alpha).
\end{aligned}
\tag{2.13}
$$

Equations (2.10), (2.13) describe the nonlinear motion of a charged particle in the field of a traveling monochromatic wave under the resonance conditions (1.1) on the phase plane $(\bar v_z, \psi_n)$ or $(\bar v_\perp, \psi_n)$.

At small values $a \ll 1$ ($k_z \sim k_x$) we get the following order-of-magnitude estimates:

$$
\begin{aligned}
&F_\xi\sim a^{|n|},\quad F_{v_z}\sim a^{|n|},\quad F_{v_\perp}\sim 1\ (|n|=1),\quad F_{v_\perp}\sim a^{|n|-3}\ (|n|\geqslant 3),\\
&F_{1\theta}\sim 1,\quad F_{2\theta}\sim 1\ (|n|=1),\quad F_{2\theta}\sim a^{|n|-3}\ (|n|\geqslant 3).
\end{aligned}
\tag{2.14}
$$

These estimates show that for $a \ll 1$ the greatest change is experienced by the quantities $\bar v_\perp$ and $\bar\theta$.

The singular points $\bar v_{zc}$, $\bar v_{\perp c}$, ψ_{nc} on the phase planes are determined from the equations

$$
\begin{aligned}
&\sin(\psi_{nc}+2\alpha)=0,\quad \psi_{nc}=l\pi-2\alpha,\quad l=0,\ \pm1,\ \ldots,\\
&n\omega_B-2\omega+2k_z v_{zc}-\frac{n}{\bar v_{\perp c}}\left(c\frac{\mathscr{E}}{B_0}\right)^2 kF_{1\theta}(\bar v_{zc},\bar v_{\perp c})\\
&\quad+\left(c\frac{\mathscr{E}}{B_0}\right)^2 k\left[\frac{2k_x}{\omega_B}F_\xi(\bar v_{zc},\bar v_{\perp c})-\frac{n}{\bar v_{\perp c}}F_{2\theta}(\bar v_{zc},\bar v_{\perp c})\right](-1)^l=0.
\end{aligned}
\tag{2.15}
$$

(Moreover, the existence of singular points determined by the equation $F_{vz}(\bar v_{zc}, \bar v_{\perp c}) = 0$ is possible. These will not be discussed here.)

For a wave propagating at an angle to the magnetic field that is not close to $\pi/2$, we get the following integral of motion by using Eqs. (2.10) and (2.13):

$$\left(c\frac{\mathscr{E}}{B_0}\right)^2\frac{k}{k_z}F_{v_z}(\bar{v}_z,\bar{v}_\perp)\cos(\psi_n+2\alpha)=C_2-\left(\bar{v}_z-\frac{2\omega-n\omega_B}{2k_z}\right)^2. \quad (2.16)$$

It follows from Eqs. (2.15) and the integral (2.16) that the motion of the particle on the phase plane $(\bar{v}_z, \psi_n)$ can be infinite or finite. In the latter case, the particle is captured by the field of the wave. The region of capture in the variable $\bar{v}_z$ is of the order of

$$\Delta v_z\sim c\frac{\mathscr{E}}{B_0}\left[\frac{k}{k_z}F_{v_z}(\bar{v}_{zc},\bar{v}_{\perp c})\right]^{1/2}, \quad (2.17)$$

and the oscillation frequency of the captured particles near the centers is equal to

$$\Omega_{tr}^2=2\left(c\frac{\mathscr{E}}{B_0}\right)^2 kk_zF_{v_z}(\bar{v}_{zc},\bar{v}_{\perp c}). \quad (2.18)$$

In the derivation of the integral (2.16), we used the facts that $d\bar{v}_z/dt \propto \mathscr{E}^2$ and $d\bar{v}_\perp/dt \propto \mathscr{E}^2$, while $d\psi_n/dt \sim k_z\Delta v_z \propto \mathscr{E}$. In the case of small $a \ll 1$, the expressions for the integral (2.16) and frequency Ω_{tr} (2.18) are valid only for not very strong fields, namely,

$$|a|^{|n|+2}\gg\left(c\frac{\mathscr{E}}{B_0}\frac{k}{\omega_B}\right)^2, \quad k_z\sim k_x. \quad (2.19)$$

Since the next approximation, which contains the resonance (1.1), is proportional to $\mathscr{E}^4$, the perturbation of the considered trajectories in the $(\bar{v}_z, \psi_n)$ plane by the higher approximation is proportional to $\mathscr{E}^2$.

In concluding this section, we note that as a result of averaging of the exact integrals (2.3), we can obtain three more approximate integrals of motion of the particle under the resonance conditions (1.1).

3. DAMPING OF THE WAVES

We now determine the damping of the waves due to their absorption by resonant particles. If we consider the problem of the evolution of the initial perturbation in the absence of extraneous currents, which maintain the field, then the capture of resonant particles by the field of the oscillations leads to saturation of the absorption and to a sharp decrease in the damping decrement (cf. with[6,7]). Therefore, we shall consider the case of strong damping, when capture of the particles by the field of the wave is unimportant, i.e., the wave is damped before the particles perform an oscillation in the "potential" well of this wave. For this it is necessary that the damping decrement γ be large in comparison with Ω_{tr}. This situation is analogous to the case in which we can use the linear theory of damping for cyclotron resonances $\omega \approx k_zv_z + n\omega_B$. However, in resonance at the half-integer harmonics ω_B the damping decrement is proportional to $\mathscr{E}^2$, while Ω_{tr} is proportional to $\mathscr{E}$; therefore the condition $\gamma \gg \Omega_{tr}$ is satisfied only at rather strong fields. We recall that for resonance $\omega \approx k_zv_z + n\omega_B$, the quantity $\Omega_{tr} \propto \sqrt{\mathscr{E}}$ and γ does not depend on $\mathscr{E}$; therefore, capture is absent only for weak fields.

If the criterion $\gamma \gg \Omega_{tr}$ is not satisfied, but the Coulomb collisions remove the particle from the resonance region Δv_z before it completes an oscillation in the "potential" well of the wave, i.e., if

$$\Omega_{tr}\ll\nu_{eff}=\nu_c(v_z/\Delta v_z)^2, \quad (3.1)$$

then the effect of capture can also be neglected (cf. with[8]). (Here ν_c is the frequency of collisions between the ions if the resonant particles are ions, or between ions and electrons, and between electrons if the resonant particles are electrons.) In this case, "Maxwellization" of the distribution function takes place in the resonance region. Inasmuch as the quantities $\Omega_{tr} \propto \mathscr{E}$ and $\Delta v_z \propto \mathscr{E}$ are small, then, in contrast with linear cyclotron resonances, when $\Omega_{tr} \propto \sqrt{\mathscr{E}}$ and $\Delta v_z \propto \sqrt{\mathscr{E}}$, the role of Coulomb collisions turns out to be more significant in the nonlinear cyclotron resonance.

In the following, we shall investigate the damping of a wave when capture of the resonant particles is absent because the inequality $\gamma \gg \Omega_{tr}$ or condition (3.1) is satisfied. We find the change in the kinetic energy absorbed by the resonant particles in a unit volume of plasma:

$$\frac{dT}{dt}=\frac{mn_0}{2}\frac{1}{\lambda}\int_0^\lambda dl\int d\mathbf{v}(v_z^2+v_\perp^2)\frac{\partial f_0}{\partial t}, \quad (3.2)$$

where λ is the wave length, l the distance in the direction of the wave vector $\mathbf{k}(2\pi\lambda^{-1}dl = d(\mathbf{k}\cdot\mathbf{r}))$, f_0 is the equilibrium distribution function, which, at the initial instant of time, is equal to $f_0|_{t=0} = f_0(v_{z0}, v_{\perp 0})$ (v_{z0} and $v_{\perp 0}$ are the initial values of the longitudinal and transverse velocities of the particles). The term connected with the initial perturbation of the distribution function has been omitted from (3.2). The contribution of this term to the change in the kinetic energy as $t \to \infty$ turns out to be small as a consequence of the effect of "mixing" (cf. with[6]).

Taking into account that

$$\frac{\partial f_0}{\partial t}=\frac{\partial f_0}{\partial v_{z0}}\frac{\partial v_{z0}}{\partial t}+\frac{\partial f_0}{\partial v_{\perp 0}}\frac{\partial v_{\perp 0}}{\partial t}$$
$$=-\frac{e\mathscr{E}}{m}\sum_n\left[\frac{\partial f_0}{\partial v_{z0}}R_{v_z}^{(n)}(v_{\perp 0})+\frac{\partial f_0}{\partial v_{\perp 0}}R_{v_\perp}^{(n)}(v_{z0},v_{\perp 0})\right]\cos(\Phi_{n0}+\alpha)$$

and transforming to the initial variables in the cylindrical system of coordinates (2.4), and also using the z component of the integral (2.3), we get

$$\frac{dT}{dt}=-e\mathscr{E}n_0\frac{1}{2\pi}\int_0^{2\pi}d(k_x\xi_0+k_zz_0)\int v_{\perp 0}\,dv_{\perp 0}\,dv_{z0}\,d\theta_0$$
$$\times\left[\frac{\omega}{k_z}v_z(t)+\frac{e\mathscr{E}}{mk_z}e_z\sum_s J_s(a)\sin(\Phi_s+\alpha)\right] \quad (3.3)$$
$$\times\sum_n\left[\frac{\partial f_0}{\partial v_{z0}}R_{v_z}^{(n)}(v_{\perp 0})+\frac{\partial f_0}{\partial v_{\perp 0}}R_{v_\perp}^{(n)}(v_{z0},v_{\perp 0})\right]\cos(\Phi_{n0}+\alpha).$$

In order to obtain the change in the kinetic energy under the resonance conditions (1.1), we transform in (3.3) to the averaged variables. (We note that the Jacobian of this transformation is equal to unity with accuracy to terms of order $\mathscr{E}^2$.)

For the quantity $\bar{v}_z(t)$, we find from (2.10):

$$\bar{v}_z(t)=\bar{v}_{z0}-\left(c\frac{\mathscr{E}}{B_0}\right)^2kF_{v_z}(\bar{v}_{z0},\bar{v}_{\perp 0})$$
$$\times\frac{\cos[\psi_{n0}+(2k_z\bar{v}_{z0}-2\omega+n\omega_B)t+2\alpha]-\cos(\psi_{n0}+2\alpha)}{2k_z\bar{v}_{z0}-2\omega+n\omega_B}$$

Since the component in this expression which describes the change in the longitudinal velocity of the particle under the resonance conditions (1.1) is proportional to $\mathscr{E}^2$, it turns out to be sufficient in converting to the averaged variables to limit oneself to the substitution (2.8) and (2.9). Carrying out the integration over the variables $k_x\bar{\xi}_0 + k_z\bar{z}_0$, $\bar{\theta}_0$ and over the variable v_{z0}, and using the relation

$$\lim_{t\to\infty}\frac{\sin(2k_z\bar{v}_{z0}-2\omega+n\omega_B)t}{2k_z\bar{v}_{z0}-2\omega+n\omega_B}=\frac{\pi}{2k_z}\delta\left(\bar{v}_{z0}-\frac{2\omega-n\omega_B}{2k_z}\right),$$

we obtain the following asymptotic (as $t \to \infty$) expression for the change in the kinetic energy:

$$\frac{dT}{dt}=-\frac{\pi}{8}\frac{e^2}{m^2}\frac{\omega_p{}^2\omega}{v_T{}^2\omega_B{}^4}\frac{k^2}{k_z{}^2}J\mathscr{E}^4, \tag{3.4}$$

where $\omega_p^2 = 4\pi/e^2n_0/m$ and

$$J=v_T{}^2\int\limits_0^\infty \bar{v}_{\perp 0}\,d\bar{v}_{\perp 0}F_{v_z}(\bar{v}_{z0},\bar{v}_{\perp 0})\left\{\frac{\partial f_0}{\partial\bar{v}_{\perp 0}}F_{v_\perp}(\bar{v}_{z0},\bar{v}_{\perp 0})\right.$$
$$+\frac{\partial f_0}{\partial\bar{v}_{z0}}F_{v_z}(\bar{v}_{z0},\bar{v}_{\perp 0})+\frac{1}{2}\frac{\omega_B}{k}\sum_s\frac{\omega_B}{s\omega_B-\omega+k_z\bar{v}_{z0}}\left[\frac{\partial^2 f_0}{\partial\bar{v}_{\perp 0}{}^2}R_{v_\perp}^{(s)}(\bar{v}_{z0},\bar{v}_{\perp 0})\right. \tag{3.5}$$
$$\left.+\frac{\partial^2 f_0}{\partial\bar{v}_{z0}\partial\bar{v}_{\perp 0}}R_{v_z}^{(s)}(\bar{v}_{\perp 0})\right]R_{v_\perp}^{(n-s)}(\bar{v}_{z0},\bar{v}_{\perp 0})+\frac{1}{2}\frac{\omega_B}{k}\sum_s\frac{\omega_B}{s\omega_B-\omega+k_z\bar{v}_{z0}}\cdot$$
$$\left.\times\left[\frac{\partial^2 f_0}{\partial\bar{v}_{z0}{}^2}R_v{}^{(s)}(\bar{v}_{\perp 0})+\frac{\partial^2 f_0}{\partial\bar{v}_{z0}\partial\bar{v}_{\perp 0}}R_{v_\perp}^{(s)}(\bar{v}_{z0},\bar{v}_{\perp 0})\right]R_{v_z}^{(n-s)}(\bar{v}_{\perp 0})\right\}\Bigg|_{\bar{v}_{z0}=\frac{2\omega-\omega n_B}{2k_z}}$$

We make use of the law of conservation of energy

$$dW/dt+dT/dt=0, \tag{3.6}$$

where W is the mean value of the energy density of the electric and magnetic fields in the plasma and the oscillations energy of the plasma particles in the field of the wave:

$$W=\frac{1}{16\pi}\frac{\partial}{\partial\omega}(\omega e_i^*\Lambda_{ij}e_j)\mathscr{E}^2(t),\quad \Lambda_{ij}=\frac{c^2k^2}{\omega^2}\left(\frac{k_ik_j}{k^2}-\delta_{ij}\right)+\varepsilon_{ij}' \tag{3.7}$$

ϵ_{ij}' is the Hermitian part of the dielectric tensor of the plasma. Then Eq. (3.6) can be written in the form

$$d\mathscr{E}^2/dt=-2\gamma\mathscr{E}^4/\mathscr{E}^2(0), \tag{3.8}$$

where

$$\gamma=-\pi^2\frac{v_{\mathscr{E}}{}^2}{v_T{}^2}\frac{\omega_p{}^2\omega^3}{\omega_B{}^4}\frac{k^2}{k_z{}^2}J\left[\frac{\partial}{\partial\omega}(\omega e_i^*\Lambda_{ij}e_j)\right]^{-1}, \tag{3.9}$$

$v_E = e\mathscr{E}(0)/m\omega$ is the oscillation amplitude of the particle in the field of the wave at $t = 0$. The value of γ characterizes the rate of change of the wave amplitude. Integrating (3.8), we find the damping law of the amplitude of the wave field:

$$\mathscr{E}^2(t)=\mathscr{E}^2(0)/(1+2\gamma t). \tag{3.10}$$

It then follows that for $\gamma t \gg 1$, the quantity $\mathscr{E}$ falls off as $t^{-1/2}$.

4. DISCUSSION OF RESULTS

We estimate the value of the damping decrement for nonlinear electron cyclotron resonances. Assuming that $\omega_{pe} \lesssim \omega_{Be}$, and that the phase velocity of the wave is of the order of the speed of light $\omega \sim kc$, we get for $\omega \approx \omega_{Be}/2$

$$\gamma\sim\frac{v_{\mathscr{E}}{}^2}{v_{Te}{}^2}\frac{\omega_{pe}{}^2}{\omega_{Be}{}^2}\omega\exp(-z^2),\quad z=\frac{2\omega-\omega_{Be}}{2\sqrt{2}\,k_zv_{Te}}. \tag{4.1}$$

It follows from this expression that the width of the absorption resonance is of the order of $\Delta\omega \sim c^{-1}v_{Te}\omega_{Be}$.

For resonances $|n| \geq 3$, the damping decrement of waves with a phase velocity of the order of c is very small:

$$\gamma\sim\frac{v_{\mathscr{E}}{}^2}{v_{Te}{}^2}\frac{\omega_{pe}{}^2}{\omega_{Be}{}^2}\omega\left(\frac{v_{Te}}{c}\right)^{2|n|-3}. \tag{4.2}$$

We note that account of relativistic effects in the equations of motion (2.1) (or in the kinetic equation) should also lead to weak nonlinear damping, consideration of which might possibly change the result (4.2). For slow waves $(ck/\omega \gg 1)$, when the frequency ω of the wave is close to the frequency of the upper or lower hybrid resonance ω_+ or ω_-, where

$$\omega_\pm{}^2={}^1/{}_2(\omega_{pe}{}^2+\omega_{Be}{}^2)\pm{}^1/{}_2[(\omega_{pe}{}^2+\omega_{Be}{}^2)^2-4\omega_{pe}{}^2\omega_{Be}{}^2k_z{}^2/k^2]^{1/2}, \tag{4.3}$$

the damping falls off sharply. In this case, the value of the damping decrement (4.2) increases by a factor of $(ck/\omega)^{2|n|-3}$.

In the low-frequency region, for Alfven or fast magnetoacoustic waves with frequency $\omega \approx \omega_{Bi}/2 \sim kv_A$ (v_A is the Alfven velocity), the damping decrement is equal in order of magnitude to

$$\gamma\sim v_{\mathscr{E}}{}^2\omega/v_{Ti}{}^2. \tag{4.4}$$

For higher resonances ($\omega = {}^1/{}_2 n\omega_{Bi}$, $|n| \geq 3$) the damping decrement is reduced by a factor of $(v_A/v_{Ti})^{2|n|-3}$ in comparison with (4.4).

If the amplitude of the wave is maintained by a stationary external source, then in the absence of collisions, capture of the resonant particles by the field of the wave takes place, and absorption saturation sets in. However, if sufficiently frequent Coulomb collisions prevent capture of the particles, then upon satisfaction of the condition (3.1), the distribution function will be Maxwellian in the resonance region. The wave energy absorbed by the resonant particles, will be transferred to the nonresonant particles of the plasma, i.e., heating of the plasma will occur. The change in the temperature is then determined by the condition

$$\frac{3}{2}\frac{d}{dt}nT=2\gamma\mathscr{E}^2. \tag{4.5}$$

The absorption coefficient of the wave $\kappa = \mathrm{Im}\,k$, which characterizes the spatial decay of the wave amplitude, is expressed in terms of the quantity γ with the help of the condition $\kappa = \gamma dk/d\omega$.

As an example, let us consider the absorption of high-frequency electromagnetic waves with frequency $\omega \approx \omega_{Be}/2$ in a plasma. If $n_0 \sim 5\times10^{12}\ \mathrm{cm}^{-3}$, $B_0 \sim 5$ kG, $T_e \sim 10$ eV and $\mathscr{E} \sim 1$ kV/cm, we then get $v_{\mathscr{E}} \sim 3\times10^7$ cm/sec, $v_{Te} \sim 10^8$ cm/sec and $\gamma/\omega \sim 0.1$, and the wave is attenuated at a distance $l \sim c/\gamma \sim 5$ cm. This example shows that the nonlinear electron resonance $\omega \approx \omega_{Be}/2$ can easily be detected in a gas-discharge plasma. Using this resonance for the heating of a plasma in big thermonuclear installations ($n_0 \sim 5\times10^{14}\ \mathrm{cm}^{-3}$, $B_0 \sim 50$ kG, $T_e \sim 1$ keV, $\lambda = 2\pi c/\omega \sim 5$ mm) requires the use of very high powers: even at $\mathscr{E} \sim 100$ kV/cm, we get $\gamma/\omega \sim 4\times10^{-4}$.

The nonlinear ion cyclotron resonance $\omega \approx \omega_{Bi}/2$ on Alfvén and fast magnetoacoustic waves in a plasma with thermonuclear parameters ($n_0 \sim 10^{14}\ \mathrm{cm}^{-3}$, $B_0 \sim 40$ kG, $T_i \sim 1$ keV) leads to very strong damping of these waves even for relatively small values of the amplitude; for example, for $B \sim 100$ G ($\mathscr{E} \sim 1$ kV/cm), we obtain $\gamma/\omega \sim 10^{-2}$. The excitation of waves with frequency $\omega \approx \omega_{Bi}/2$ in a plasma of large dimensions can be more effective than that of waves with frequency $\omega = \omega_{Bi}$ or $\omega = 2\omega_{Bi}$, since the former have a large wavelength; a damping mechanism that leads to heating of the ion component of the plasma exists for them, as for waves with frequencies ω_{Bi} and $2\omega_{Bi}$.

The condition for applicability of the developed theory (3.1) and (2.9) are satisfied for the numerical examples given.

In conclusion, we note the possibility of nonlinear cyclotron excitation of waves in a plasma. If the particles of the plasma have an anisotropic velocity distribution, for example, because of the presence of a helical beam of electrons, then the quantity γ, which is determined by Eq. (3.9), can become negative. In this case, an "explosive" instability arises; in accord with (3.10), the amplitude of the oscillations grows to infinity within a finite time $\Delta t = 1/2\gamma$. However, for sufficiently large values of the amplitude, the growth of this instability should be restricted to the higher nonlinear terms.

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