Long-range part of a localized Langmuir perturbation in an electron plasma

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It is demonstrated that a static field arises near a localized Langmuir perturbation and that the strength of the field decreases only in accordance with a power law ($E \sim 1/x^2$) wtih increasing distance from the perturbation region. The region in which this field exists is of the order of L^2/r_D, where L is the dimension of the Langmuir perturbation and r_D is the Debye radius.

We investigate here the structure of the field of a localized Langmuir perturbation in a uniform electron plasma.

In the linear approximation, the electric field of the Langmuir perturbation can be represented in the form

$$E(x, t) = \mathscr{E}(x, t) e^{-i\omega_p t} + \text{c.c.}, \tag{1}$$

where the function $\mathscr{E}(x, t)$ satisfies the equation (see, e.g.,[1]):

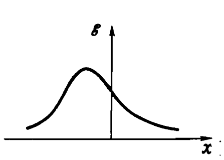
$$\frac{\partial \mathscr{E}}{\partial t} = i \frac{3 v_T^2}{4\omega_p} \frac{\partial^2 \mathscr{E}}{\partial x^2} \tag{2}$$

($v_T = (2T/m)^{1/2}$ is the thermal velocity of the electron, and ω_p is the electron plasma frequency.

We consider below perturbations that can be characterized by a single spatial scale L and decrease sufficiently rapidly (exponentially) at infinity (see the figure). The scale L, naturally, is assumed to be large in comparison with the Debye radius: $L \to v_T/\omega_p$.

Owing to the smallness of the group velocity of the Langmuir oscillations, the perturbation remains in the initial region of space for a long time $\tau \sim \omega_p L^2/v_T^2$ (τ is large not only in comparison with the Langmuir period $2\pi/\omega_p$, but also in comparison with the time of flight of the electron through the perturbation L/v_T). In other words, in the linear approximation it turns out that for a long time the presence of a perturbation manifests itself in any way in the region $|x| \gg L$.

It will be shown in the paper that when account was taken of effects quadratic in the amplitude $\mathscr{E}$, the situation is radically altered: a quasistatic electric field is produced around the perturbation and decreases with increasing distance from the perturbation region only in power-law fashion. The reason for the appearance of this long-range part of the electric field is the following. In the region where the perturbation is localized, the electrons that pass through are acted upon by a high-frequency pressure force (proportional to $\partial |\mathscr{E}|^2/\partial x$), which distorts their distribution function. The resultant distortions[1)] are transported with thermal velocity over larger distances (compared with L) and lead to the appearance in the region $|x| \gg L$ of an electric field whose established value becomes such as to ensure quasineutrality of the plasma.



Principal attention will be paid to finding the field at not too large values of $|x|$, $|x| \lesssim v_T\tau$ (but $|x| \gg L$!). In this region, the distortions of the distribution function are quasistatic: the time of flight of the electron from the region of localization of the Langmuir perturbation to the point x is small in comparison with the time τ of alteration of the Langmuir perturbation. It is precisely this circumstance which enables us to write down an expression for the long-range electric field in a simple and universal form.

The initial equations are the kinetic equation for the electron distribution function f and the Poisson equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f}{\partial v} = 0,$$
$$\frac{\partial E}{\partial x} = 4\pi e\left(n_0 - \int f\,dv\right), \tag{3}$$

where E is the electric field, n_0 is the density of the neutralizing ion background, and e and m are the charge and mass of the electron. In accordance with the foregoing, the problem consists of finding the distribution function in second order in $\mathscr{E}$. We use the method of successive approximation. We put

$$E = E_1 + E_2 + \ldots,$$
$$f = f_0 + f_1 + f_2 + \ldots$$

The unperturbed distribution function f_0 will be regarded as Maxwellian

$$f_0 = n(m/2\pi T)^{1/2} \exp(-mv^2/2T);$$

E_1 coincides with the function defined by (1). Accordingly we have in the linear approximation

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} = \frac{e}{m}\frac{\partial f_0}{\partial v}[\mathscr{E}(x, t)e^{-i\omega_p t} + \text{c.c.}], \tag{4}$$

whence

$$f_1 = \frac{e}{m}\frac{\partial f_0}{\partial v}\int_{-\infty}^{t} \mathscr{E}[x - v(t'-t), t'] \exp\{-i\omega_p t'\} dt' + \text{c.c.} \tag{5}$$

Recognizing that $\mathscr{E}$ varies slowly in comparison with the rapidly oscillating exponential, we can write the following iteration series for f_1

$$f_1(x, v, t) = \frac{e}{m}\frac{\partial f_0}{\partial v} e^{-i\omega_p t}\left[\frac{i}{\omega_p}\mathscr{E} + \frac{1}{\omega_p^2}\frac{d\mathscr{E}}{dt} - \frac{i}{\omega_p^3}\frac{d^2\mathscr{E}}{dt^2} - \frac{1}{\omega_p^4}\frac{d^3\mathscr{E}}{dt^3} + \ldots\right] + \text{c.c.},$$

where $d/dt = \partial/\partial t + v\partial/\partial x$. It follows from (5), in addition, that f_1 is exponentially small at large distances ($|x| \gg L$).

The second-order quantities satisfy the system of equations

$$\frac{df_2}{dt}-\frac{e}{m}E_2\frac{\partial f_0}{\partial v}=\frac{e}{m}E_1\frac{\partial f_1}{\partial v}=\frac{e^2}{m^2\omega_p^2}\frac{\partial}{\partial v}\left\{\frac{\partial f_0}{\partial v}\left[\frac{d|\mathscr{E}|^2}{dt}\right.\right.$$
$$\left.\left.-\frac{i}{\omega_p}\frac{d}{dt}\left(\mathscr{E}^*\frac{d\mathscr{E}}{dt}-\mathscr{E}\frac{d\mathscr{E}^*}{dt}\right)-\frac{1}{\omega_p^2}\left(\frac{d^3|\mathscr{E}|^2}{dt^3}-3\frac{d}{dt}\left|\frac{d\mathscr{E}}{dt}\right|^2\right)\right]\right\}, \quad (6)$$
$$\frac{dE_2}{dx}=-4\pi e\int f_2\,dv. \quad (7)$$

When writing down the right-hand side of (6) we have retained only terms that vary slowly with time; the terms containing the factors $\exp\{\pm 2i\omega_p t\}$ make an exponentially small contribution to the function f_2 at large distances, and can therefore be omitted.

The system (6) and (7) has one common property. If the right-hand side of (6) contains a certain function in the form of a time derivative[2)], $dF(x, v, t)/dt$, with

$$\int F\,dv=0, \quad (8)$$

then this system has a solution $f_2 = F$, $E_2 = 0$, Since actually the expression in the right-hand side of (6) is exponentially small at $x \gg L$, this means that these terms which are total derivatives with respect to time and satisfy the condition (8) make no contribution to the perturbation of the distribution function at large x, and consequently can be omitted when finding the long-range part of the perturbation.

Bearing this general property in mind, we can represent Eq. (6) in the form

$$\frac{df_2}{dt}-\frac{e}{m}E_2\frac{\partial f_0}{\partial v}=\frac{e^2}{m^2\omega_p^2}\frac{\partial f_0}{\partial v}\left\{\frac{\partial|\mathscr{E}|^2}{\partial x}-\frac{2i}{\omega_p}\left(\frac{\partial\mathscr{E}^*}{\partial x}\frac{\partial\mathscr{E}}{\partial t}\right.\right.$$
$$\left.\left.-\frac{\partial\mathscr{E}}{\partial x}\frac{\partial\mathscr{E}^*}{\partial t}\right)-\frac{1}{\omega_p^2}\left(v\frac{d}{dt}\frac{\partial^2|\mathscr{E}|^2}{\partial x^2}-3\frac{\partial}{\partial x}\left|\frac{d\mathscr{E}}{dt}\right|^2\right)\right\}. \quad (9)$$

Taking Eq. (2) into account and confining ourselves in the curly brackets of the right-hand side of (9) to terms of zeroth and first orders[3)] in the parameter $(v_T/\omega_p L)^2$, we can reduce this equation to the form

$$\frac{\partial f_2}{\partial t}+v\frac{\partial f_2}{\partial x}-\frac{e}{m}E_2\frac{\partial f_0}{\partial v}=\frac{e^2}{m^2\omega_p^2}\frac{\partial f_0}{\partial v}\left\{\frac{\partial|\mathscr{E}|^2}{\partial x}+\frac{3T}{m\omega_p^2}\frac{\partial}{\partial x}\left|\frac{\partial\mathscr{E}}{\partial x}\right|^2\right.$$
$$\left.-\frac{v^2}{\omega_p^2}\left(\frac{\partial^3|\mathscr{E}|^2}{\partial x^3}-3\frac{\partial}{\partial x}\left|\frac{\partial\mathscr{E}}{\partial x}\right|^2\right)\right\}=\frac{1}{m}\frac{\partial U}{\partial x}\frac{\partial f_0}{\partial v}, \quad (10)$$

where U stands for the potential of the high-frequency force (which, naturally, differs from zero only in the region of the packet).

As already noted, the long-range part of the perturbation is precisely due to the presence of this high-frequency force. It must be recognized, however, that in the region of the packet the particle is acted upon also by an electric field that is determined from the condition of quasineutrality of the plasma and cancels out partially the high-frequency force. It is therefore necessary to find first the effective potential

$$U_{eff}=U-e\varphi,\ \partial\varphi/\partial x=-E_2,$$

which acts on the particle in the region of the packet. To this end we find first φ. When solving this part of the problem we can neglect the time derivative in (10), after which f_2 in the region of the packet can be easily determined:

$$f_2=-T^{-1}f_0(U-e\varphi).$$

Bearing in mind that the dimension of the packet is large in comparison with the Debye radius, we can obtain φ from (7) by successive approximations, putting in the zeroth approximation

$$\varphi=\frac{1}{ne}\int Uf_0\,dv,$$

and then substituting the obtained expression in the left-hand side of (7). The result is the following expression for the effective potential acting on the particle in the region of the packet

$$U_{eff}=-\frac{e^2}{m\omega_p^2}\left[3\frac{T/m-v^2}{\omega_p^2}\left|\frac{\partial\mathscr{E}}{\partial x}\right|^2+\frac{v^2}{\omega_p^2}\frac{\partial^2|\mathscr{E}|^2}{\partial x^2}\right], \quad (11)$$

where only the principal terms in the parameter $v_T/\omega_p L$ have been retained.

The problem reduces now to finding, at large distances $|x| \gg L$, the electron-density perturbation δn due to the action of the potential U_{eff} on the electrons in the region of the Langmuir oscillation. The electric field can then be easily determined from the quasineutrality condition

$$E_2=\frac{T}{en}\frac{\partial}{\partial x}\delta n. \quad (12)$$

The distribution-function change connected with the action of the effective potential can be obtained by integrating Eq. (10) along the trajectories:

$$f_2=\frac{\partial f_0}{\partial v}\int_0^t\frac{\partial}{\partial x}U_{eff}[x+v(t'-t),v,t']\,dt'=-\frac{1}{v}\frac{\partial f_0}{\partial v}U_{eff}(x-vt,v,0)$$
$$-\frac{1}{v^2}\frac{\partial f_0}{\partial v}\int_{-\infty}^{x}\frac{\partial}{\partial t}U_{eff}\left(y,v,t+\frac{y-x}{v}\right)dy.$$

For the sake of argument, we consider the perturbation at $x > 0$, and take into account the exponential smallness of U_{eff} in the region $x \gg L$ of interest to us.

The first term is connected with allowance for the initial conditions. It can be verified that it makes only a small contribution to the perturbation of the electric field in the region $x \lesssim v_T t$, and will therefore be omitted from now on. The density perturbation, accordingly, will be

$$\delta n=\int f_2\,dv=\frac{m}{T}\int_0^\infty\frac{dv}{v}f_0(v)\int_{x-vt}^{\infty}dy\frac{\partial}{\partial t}U_{eff}\left(y,v,t+\frac{y-x}{v}\right). \quad (13)$$

We have replaced the upper limit in the integral with respect to dy by ∞, bearing in mind that $|x| \gg L$.

As is obvious beforehand, and as follows also formally from (13), the density perturbation at $|x| \gg L$ can be connected only with the nonstationary character of the effective potential. It is all the more surprising that the long-range electric field is static. To verify this, we interchange the order of integration in (13) and change over from the variable v to the variable $t' = (y - x)/v$

$$\delta n=-\frac{m}{T}\int_{-\infty}^{\infty}dy\int_{-t}^{0}\frac{dt'}{t'}f_0\left(\frac{x-y}{t'}\right)\frac{\partial}{\partial t}U_{eff}\left(y,\frac{y-x}{t'},t+t'\right). \quad (14)$$

Recognizing that, according to Eq. (2),

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}U_{eff}(y,v,t)\,dy=0,$$

we can represent (14) in the form

$$\delta n=-\frac{m}{T}\int_{-\infty}^{\infty}dy\int_{-t}^{0}\frac{dt'}{t'}\left[f_0\left(\frac{x-y}{t'}\right)\frac{\partial}{\partial t}U_{eff}\left(y,\frac{y-x}{t'},t+t'\right)\right.$$
$$\left.-f_0\left(\frac{x}{t'}\right)\frac{\partial}{\partial t}U_{eff}\left(y,\frac{-x}{t'},t+t'\right)\right],$$

i.e.,

$$\delta n=\frac{e^2}{T\omega_p^3}\int\limits_{-\infty}^{\infty}dy\int\limits_{-t}^{0}\frac{dt'}{t'}\left\{\frac{3T}{m\omega_p^2}\left[f_0\left(\frac{x-y}{t'}\right)-f_0\left(\frac{x}{t'}\right)\right]\frac{\partial}{\partial t}\left|\frac{\partial\mathscr{E}}{\partial y}\right|^2\right.$$
$$\left.+\frac{1}{\omega_p^2}\left[\frac{(x-y)^2}{t'^2}f_0\left(\frac{x-y}{t'}\right)-\frac{x^2}{t'^2}f_0\left(\frac{x}{t'}\right)\right]\left(\frac{\partial^2|\mathscr{E}|^2}{\partial y^2}-3\left|\frac{\partial\mathscr{E}}{\partial y}\right|^2\right)\right\}.$$

Bearing in mind that $\mathscr{E}$ differs significantly from zero only at $y \lesssim L \ll x$, and that the function f_0 is exponentially small at $t' < x/v_T$, we can expand in powers of y/t' in the square brackets. As a result we obtain in the first nonvanishing approximation

$$\delta n=-\frac{3e^2n}{m^2\omega_p^4}\left(\frac{m}{2\pi T}\right)^{1/2}\frac{1}{x}\frac{\partial}{\partial t}\int\limits_{-\infty}^{\infty}y\left|\frac{\partial\mathscr{E}}{\partial y}\right|^2dy.$$

Accordingly, the long-range part of the electric field is

$$E_2=\frac{3eT}{m^2\omega_p^4}\left(\frac{m}{2\pi T}\right)^{1/2}\frac{1}{x^2}\frac{\partial}{\partial t}\int\limits_{-\infty}^{\infty}y\left|\frac{\partial\mathscr{E}}{\partial y}\right|^2dy=\frac{Q}{x^2}. \qquad (15)$$

The quantity Q, which has the dimension of charge, can be called the effective charge of the Langmuir oscillation. Using Eq. (2), it is easy to verify that it does not depend on the time, i.e., the long-range part of the electric field turns out to be not merely quasistatic, but completely static. With the aid of (2) we can express the effective charge in terms of the characteristics of the perturbation at the initial instant of time:

$$Q=\frac{9ieT^2}{2m^3\omega_p^5}\left(\frac{m}{2\pi T}\right)^{1/2}\int\limits_{-\infty}^{\infty}\left(\mathscr{E}_0^*\frac{\partial^3\mathscr{E}_0}{\partial y^3}-\mathscr{E}_0\frac{\partial^3\mathscr{E}_0^*}{\partial y^3}\right)dy,$$

$\mathscr{E}_0=\mathscr{E}(y, 0)$. In the region of large x, $x \gg \tau v_T$, formula (15) no longer holds. The spatial dependence of the field ceases to be universal in this case and depends on the details of the change of the form of the perturbation with time, but the field here is very weak.

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1) Within the framework changes exponentially weakly ($L \ll v_T/\omega_p$!) of the linear approximation, the distribution function of the electrons as they pass through the perturbation region, so that the "long-range" part of the perturbation does not appear in the linear approximation.

2) We recall that we define the total derivative with respect to time as $\partial/\partial t + v\,\partial/\partial x$.

3) We cannot confine ourselves only to the zeroth-order term since, as will be shown below, it is almost completely cancelled out by the quasistatic electric field.

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[1] V. I. Karpman, Nelineinye volny v dispergiruyushchikh sredakh (Nonlinear Waves in Dispersing Media), Nauka (1973), p. 125.

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