Instability of Poiseuille flow of a nematic fluid

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The flow of an anisotropic liquid along a cylindrical capillary is considered, and it is shown that laminar flow of the nematic fluid with a certain anisotropic viscosity becomes unstable with respect to infinitesimal perturbations at Reynolds numbers exceeding a certain critical value.

1. Recently,[1] the turbulization of laminar flow of a nematic liquid between parallel plates has been observed experimentally for certain sufficiently small values of the Reynolds number R. As is known, the analogous flow of an isotropic liquid is absolutely stable to infinitesimally small perturbations. It has been shown in the work of one of the authors[2] that the instability of Couette flow of an anisotropic liquid is connected with the presence of an additional degree of freedom in the mesophase—the angle of orientation of the director. The instability in the flow of an incompressible nematic arises because of the violation of the equilibrium orientation of the director under the action of the moments of forces existing in the mesophase; the increasing departures of the director from the equilibrium position leads to a disruption of the stationary flow and the development of turbulence. The existence and low value of the turbulence threshold in a nematic liquid depends on the degree of anisotropy of the viscosity of the material. It is of interest to investigate the effect of the symmetry of laminar flow on the development of the instability. The flow of a nematic along a cylindrical capillary is considered in the present research.

The problem of the development, in flow along a tube of a true instability to infinitesimally small perturbations is very complicated. The experimental studies of the flows of isotropic liquids in tubes show that there is no instability leading to an increase in perturbations in time at a given point of space at any value of the number R.[3] In the case of a nematic liquid, a theoretical study can be carried out, in which significant use is made of the specific features of the mesophase and, in particular, the smallness of the parameter $\epsilon = -\alpha_3/\alpha_2$, which represents the ratio of the shear viscosities of the nematic. In the present paper, the critical number R_c, which characterizes the appearance of a true instability in the nematic, is found as a function of ϵ and $R_c(\epsilon)$.

2. We consider the flow of a nematic along the z axis (along the tube). The unperturbed distribution of the velocities v_0 and the angle of deviation of the director θ_0 from the z axis do not depend on the cylindrical coordinates z and φ. This distribution, which depends on the coordinate r, can be determined from the solution of the equations which describe the nematic mesophase.[4]

The corresponding stationary equations, written accurate to θ_0^2, take the form

$$K_{11}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r\theta_0)\right)=\frac{1}{2}\frac{dv_0}{dr}(\gamma_1+\gamma_2(1-2\theta_0^2)),$$
$$-\frac{1}{\rho}\frac{dp_0}{dz}+\frac{1}{r}\frac{d}{dr}\left[r\frac{dv_0}{dr}\left(\frac{\alpha_3+\alpha_4+\alpha_6}{2}-\gamma_2\theta_0^2\right)\right]=0 \tag{1}$$

with the boundary conditions $\theta_0(r_0) = 0$ and $v_0(r_0) = 0$, where r_0 is the radius of the tube, K_{jj} is the modulus of elasticity of the nematic, ρ the density, α_i the viscosity coefficient; $\gamma_1 = \alpha_3 - \alpha_2$; $\gamma_2 = \alpha_6 - \alpha_5 = \alpha_2 + \alpha_3$. The first of Eqs. (1) represent the equation of equilibrium of the director, and the second represents the Navier-Stokes equation for the velocity of Poiseuille flow, where dp_0/dz is the pressure gradient along the z axis.

For $|\epsilon| \ll 1$, we can neglect the quantity θ_0 in the second of Eqs. (1) and write v_0 in the form

$$v_0=v^*\left(1-\frac{r^2}{r_0^2}\right),\quad v^*=-\frac{1}{\rho}\frac{dp_0}{dz}\frac{r_0^2}{2(\alpha_4+\alpha_6)}.$$

Substituting v_0 in the first of Eqs. (1) and introducing the dimensionless parameters $R = \rho r_0 v^*/\gamma_1$ and $\lambda_{jj} = \rho K_{jj}/\gamma_1^2$ and the variable $x = r^2/2r_0^2$, we find that the stationary distribution of the director $u_0(x) = (r/r_0)\theta_0$ is described by the equation

$$\frac{\lambda_{11}}{R}\frac{d^2u_0}{dx^2}=-2\varepsilon-\frac{u_0^2}{x} \tag{2}$$

with the boundary condition $u_0(1/2) = 0$.

If $|\epsilon| \ll (\lambda_{11}/R)^2 \ll 1$, then it follows from (2) that the solution has the approximate form

$$u_0(x)\approx\frac{\varepsilon R}{\lambda_{11}}x\left(\frac{1}{2}-x\right). \tag{3}$$

For $|\epsilon| \gtrsim 1$, we have $|\theta_0| \sim 1$ and the distribution of the director is described by the equation

$$\frac{d^2\theta_0}{dy^2}=\frac{1}{8}\sin 2\theta_0-\frac{R}{\sqrt{2}\lambda_{11}}e^{-3y/2}[1-(1-2\varepsilon)\cos 2\theta_0], \tag{4}$$

where $y = -\ln(r^2/2r_0^2)$, $\ln 2 \le y < \infty$, with the boundary condition $\theta_0(y = \ln 2) = 0$.

The following nonstationary small perturbations are added to the stationary solutions of Eq. (2) under study;

$$v_r'(r,z,t),\ v_z'(r,z,t),\ \theta'(r,z,t),\ p'(r,z,t).$$

The onset of the instability is characterized by the appearance, in the set of equations describing the mesophase, of solutions

$$\Psi(r,z,t)\sim\psi(r)e^{i(kz-\omega t)}$$

with complex frequency $\omega = \omega' + i\omega''$, $\omega'' > 0$.

By linearizing the equations[4] relative to v_r', v_z' and θ', and neglecting the quantity $\theta_0 \sim \epsilon$ in the Navier-Stokes equations in comparison with the characteristic values $(r_0 k)^2 \sim r_0^2\partial^2/\partial z^2$, we obtain the following set of equations for v_r', v_z' and θ':

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r')+\frac{\partial v_z'}{\partial z}=0,$$
$$\frac{\partial v_r'}{\partial t}+v_0\frac{\partial v_r'}{\partial z}=-\frac{1}{\rho}\frac{\partial p'}{\partial r}+\frac{1}{2}(\alpha_2+\alpha_4+\alpha_5)\left(\frac{\partial^2 v_r'}{\partial z^2}+\frac{\partial^2 v_z'}{\partial r\,\partial z}\right)$$
$$+\alpha_4\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r'}{\partial r}\right)+\alpha_2\left(v_0\frac{\partial^2\theta'}{\partial z^2}-\frac{\partial^2\theta'}{\partial t\,\partial z}\right)+\alpha_6\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{dv_0}{dr}\theta'\right),$$

$$\frac{\partial v_z'}{\partial t}+v_r'\frac{\partial v_0}{dr}+v_0\frac{\partial v_z'}{\partial z}=-\frac{1}{\rho}\frac{\partial p'}{\partial z}+(\alpha_4+\alpha_5+\alpha_6)\frac{\partial^2 v_z'}{\partial z^2}$$
$$+\frac{1}{2}(\alpha_4+\alpha_6)\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial v_r'}{\partial z}+\frac{\partial v_z'}{\partial r}\right)\right]+\alpha_5\frac{dv_0}{dr}\frac{\partial\theta'}{\partial z}, \qquad (5)$$
$$K_{11}\left[\frac{\partial^2\theta'}{\partial r^2}+\frac{1}{r}\frac{\partial\theta'}{\partial r}-\frac{1}{r^2}\theta'\right]+K_{33}\frac{\partial^2\theta'}{\partial z^2}-$$
$$-\gamma_1\left(\frac{\partial\theta'}{\partial t}+v_0\frac{\partial\theta'}{\partial z}+2\theta_0\frac{dv_0}{dr}\theta'\right)=\alpha_2\frac{\partial v_r'}{\partial z}.$$

The boundary conditions for the set of Eqs. (5) are of the form

$$v_z'=v_r'=\theta'=0 \qquad (6)$$

at $r = r_0$.

The dispersion equation, which determines the $\omega(k)$ dependence, is obtained by substituting in (6) functions v_z', v_r', and θ', which are combinations of the linearly independent solutions $\psi_j(r)^{\,i(kz-\omega t)}$ of the set (5). The form of the functions $\psi_j(r)$ depends on the relation between the dimensionless parameters λ_{jj}, ϵ, R, r_0k, and $\Omega = \omega/v^*k$. For fixed $\lambda_{jj} \ll 1$ and $|\epsilon| \ll 1$ it is of interest to consider the region of values $1 \gg r_0k \gg |\epsilon|^{1/2}$,R, in which the imaginary part Ω'' of the reduced frequency $\Omega = \Omega' + i\Omega''$ is small. A rather cumbersome analysis shows that in such a region of values of r_0k and $R(\epsilon < 0)$ an asymptotic expansion is possible in terms of the large parameter r_0kR, if $R \gg \lambda_{11}|\epsilon|^{-1/2}$, or (r_0kR/λ_{11}) if $R < \lambda_{11}|\epsilon|^{-1/2}$. The expansion in the parameter r_0kR is equivalent to a situation in which, in the given approximation, the principal contribution to the dispersion equation is made by velocity perturbations determined from the Navier-Stokes equations. In this case, the perturbation θ' makes a small contribution to the Navier-Stokes equation and is expressed in terms of v_r' and v_z' with the help of the last equation of the set (5). The corresponding solution of the dispersion equation shows that $\Omega'' < 0$, i.e., infinitesimally small perturbations in the given region $(R \gg \lambda_{11}|\epsilon|^{-1/2})$ are damped in time.

3. We now consider the range of values $1 \gg kr_0 \gg |\epsilon|^{1/2}$ and $R \ll \lambda_{11}|\epsilon|^{-1/2}$ in which the expansion in $\Lambda = r_0kR/\lambda_{11}$ is valid. In this case, a significant role is played by the elasticity of the liquid crystal in the layer and θ_0 is a smooth function of r. Here the basic contribution to the dispersion equation is made by the perturbations θ', while v_r' and v_z' are small corrections expressed in terms of θ' with the help of the Navier-Stokes equations. Here the equation of motion of the director takes the form

$$x\frac{d^2u}{dx^2}-(a+bx)u+\frac{\varepsilon\Lambda^2}{(r_0k)^2}x\left(x-\frac{1}{2}\right)u=0;$$
$$a=\frac{i}{2}\Lambda(1-\Omega)+\lambda_{33}(kr_0)^2/2\lambda_{11}, \qquad b=-i\Lambda. \qquad (7)$$

The solution $u = (r/r_0)\theta'$ of Eq. (7) should vanish at the point r = 0. This solution, which depends exponentially on quantities that are large in value and proportional to Λ, is given by the expression

$$u\approx Z^{-1/4}(x)\,\mathrm{sh}\,S(x),$$
$$S(x)=xZ^{1/2}\left[1-i\frac{\varepsilon\Lambda}{8(r_0k)^2}\left(1+\frac{3a}{b}\right)\right]-\frac{a}{2\sqrt{b}}\left[1+i\frac{\varepsilon\Lambda}{4(r_0k)^2}\left(1+\frac{3a}{2b}\right)\right]$$
$$\times\left[\ln\left|\frac{Z^{1/2}-b^{1/2}}{Z^{1/2}+b^{1/2}}\right|+i\arg\left(\frac{Z^{1/2}-b^{1/2}}{Z^{1/2}+b^{1/2}}\right)\right], \qquad (8)$$
$$Z(x)=(a+bx)/x, \qquad \lim_{x\to 0}\arg\left(\frac{Z^{1/2}-b^{1/2}}{Z^{1/2}+b^{1/2}}\right)=0.$$

The boundary conditions (6) reduce in this case to the condition $\theta' = 0$ for $r = r_0$, and the dispersion equation, in accord with (8), takes the form

$$S(1/2)=in\pi,\ n=1,2,\ldots \qquad (9)$$

At n = 0, Eq. (9) loses meaning: in this case, it is necessary to seek a solution ψ_j in the form of an expansion in powers of a small parameter Λ. However, as can be seen, such Λ correspond to values of r_0k and R for which there is no instability. In the zeroth approximation, neglecting the equantities ϵ and $(r_0k)^2$ in S(x), we obtain the equation

$$\Omega^{1/2}+\frac{1}{2}(1-\Omega)\left[\ln\left|\frac{\Omega^{1/2}-1}{\Omega^{1/2}+1}\right|+i\arg\left(\frac{\Omega^{1/2}-1}{\Omega^{1/2}+1}\right)\right]=g(1-i), \qquad (10)$$
$$g=2\pi n/\Lambda^{1/2}.$$

Equation (10) has a nontrivial solution $\Omega(g)$ for $g \geq g^*$, while the only real solution $\Omega = \Omega^*$ corresponds to the value $g = g^* \approx 0.472$:

$$\Omega^*=\Omega(g^*)\approx 0.699.$$

In the region of existence of the solution, the relative complex frequency is given by the expression

$$\Omega=\Omega'+i\Omega''\approx\Omega^*-i(g-g^*),\quad g\geq g^*. \qquad (11)$$

It follows from (11) that $\Omega'' < 0$, i.e., infinitesimally small perturbations are damped out.

We take into account the small corrections to S(x), which have the order of magnitude of the quantities ϵ and $(r_0k)^2$. In the region of existence of the solutions of Eq. (9), the inequality

$$\Omega''+\lambda_{33}kr_0/R-0.076\varepsilon R/\lambda_{11}r_0k\leq 0 \qquad (12)$$

should be satisfied in place of the inequality $\Omega'' \leq 0$, while (12) turns into an equality at $g = g^*$.

If $\epsilon > 0$, then it is seen from (12) that at $g = g^*$ the imaginary part of the frequency can vanish for definite values of the quantities $R = R_n$ and $k = k_n$, which are connected by the relation

$$\Lambda_n\equiv k_nr_0R_n/\lambda_{11}=(2\pi n/g^*)^2. \qquad (13)$$

From (12) and (13) we find the numbers R_n and k_n as functions of the parameters of the material and the number of the branch of the excitations:

$$R_n\approx 8.06\pi n\lambda_{11}(\lambda_{33}/\lambda_{11}\varepsilon)^{1/4},\quad r_0k_n\approx 2.22\pi n(\lambda_{11}\varepsilon/\lambda_{33})^{1/4}. \qquad (14)$$

Thus, for each branch n there is a perturbation with wave number k_n and frequency ω_n, which begin to increase when the number R exceeds a value R_n that is characteristic for the given branch. The branch with n = 1 is most unstable and here the critical Reynolds number R_c is the number R_1, which is the smallest of all the numbers R_n with $n \geq 1$.

For R numbers larger than the threshold number $R_c = R_1$, the increasing nonstationary perturbations are characterized by the complex frequency

$$\omega(k)=\omega'(k)+i\omega''(k)\approx v^*\left\{0{,}7k+ik^*\left[0.55\left(\frac{\lambda_{33}\varepsilon}{\lambda_{11}}\right)^{1/2}\frac{\Delta R}{R_c}+0.165\frac{\Delta k}{k^*}\right]\right\};$$
$$\Delta k=k-k^*,\quad \Delta R=R-R_c,\quad k^*=k_cR_c/R, \qquad (15)$$
$$-3.34\left(\frac{\lambda_{33}\varepsilon}{\lambda_{11}}\right)^{1/2}\frac{\Delta R}{R_c}\leq\frac{\Delta k}{k^*}\leq 0. \qquad (16)$$

As is seen from (16), the interval of wave numbers k in which $\omega''(k) > 0$ broadens with increase of ΔR.

Thus the origin of the true instability is connected mathematically with the existence of a singular point $g = g^*$ for the dispersion equation that determines the dependence $\Omega(g)$. The imaginary part of the frequency $\omega''(k)$ has a maximum at the value $k = k^*(R \gtrsim R_c)$

which corresponds to such a point. However, in the vicinity of the maximum, the $\omega''(k)$ dependence is linear and not parabolic, and the dispersion dependence $\omega(k)$ exists only for $k \leq k^*$ at $R > R_c$. Therefore, the packet of perturbations near $R = R_c$ does not have a real propagation velocity $\partial\omega/\partial k$ and does not experience a drift along the current. Perturbations, once they have been produced at a given point in space, increase with time at this point.

4. In conclusion, we note some peculiar features of the onset of a hydrodynamic instability in the flow of an anisotropic liquid. The previously considered[2] Poiseuille flow and the Couette flow of a nematic experience a true instability in contrast with the corresponding flows of an isotropic liquid. In essence, the mesophase, at a definite critical velocity distribution of the laminar flow, becomes unstable to that degree of freedom which characterizes the anisotropy of an incompressible medium (the orientation of the director). The growth of the perturbations of the flow velocity is a consequence of such an orientational instability.

The existence of a critical number R_c depends radically on the anisotropic characteristics of the medium. For example, Couette and Poiseuille flows become unstable only at $\epsilon > 0$. Upon vanishing of the shear viscosity α_3 ($\epsilon = 0$), which characterizes the anisotropy of the "viscous" stress tensor in the medium, the number R_c becomes infinite.

For different types of laminar flow, the dependence of $R_c(\epsilon > 0)$ takes the form $R_c = A\epsilon^{-1/4}$, where the constant $A \sim \lambda_{jj}$ is determined by the geometry of the flow; the constant A for Poiseuille flow is about twice that for Couette flow. If, as experiment shows,[1] the quantity ϵ vanishes at some temperature T_0, then in the vicinity of the point T_0, where $\epsilon \sim (T_0 - T)$, there should be a dependence of the turbulence threshold on the temperature, in the form $R_c \sim (T_0 - T)^{-1/4}$. We note that such a dependence is valid only in a narrow range of temperatures close to T_0, where $\epsilon < (\lambda_{11}/R_c)^2 \sim (\lambda_{11}/A)^4$. For $\epsilon > (\lambda_{11}/A)^4$ the stationary regime has a different character, which can be seen from analysis of the solutions of Eq. (4), the stability of which calls for a special study.

At $R > R_c$, a "secondary flow" is established. The square of the amplitude $|\theta'|^2$ in this new regime can be estimated, in accord with[3]:

$$|\theta'|^2 \sim \frac{\omega''(k^*)}{v^* k^*} \sim \left(\frac{R-R_c}{R_c}\right)\varepsilon^{1/2}.$$

Since the departure of the director from the flow axis in the stationary regime becomes a quantity of the order of $|\theta_0| \sim \epsilon R/\lambda_{11}$, the stationary distribution is appreciably distorted at a very small excess above the threshold R_c: $|\theta'| \sim |\theta_0|$ for $(R - R_c)/R_c \sim \epsilon \ll 1$. Therefore, the onset of regimes corresponding to the n branches of the perturbations of the stationary state $R \sim R_n \sim nR_c$ is of low probability.

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[1]Ch. Gähvillwer, Phys. Rev. Lett. 28, 1554 (1972).
[2]S. A. Pikin, Zh. Eksp. Teor. Fiz. 65, 2495 (1973) [Sov. Phys.-JETP 38, 1246 (1974)].
[3]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics) Fizmatgiz, 1953, Ch. 3 [Addison-Wesley, 1958].
[4]Orsay Liquid Crystal Group, J. Chem. Phys. 51, 816 (1969).

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